DATA ALGORITHM FOR VEHICLE DYNAMICS AND TIRE MODELING

by

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ABSTRACT

Without tires a car would be incomplete, tires are the main support of any car. This thesis is designed to portray the dynamic analysis of a race car and the behavior of its tire via tire modeling and data produced by OptimumG. Results relating the vehicle to its driver and the road are determined accurately by using Hans Pacejka magic formula to determine the longitudinal and lateral forces produced by the tire, its yaw, roll and pitch moments acting on each wheel.

Matlab is the primary software used for all calculations in this thesis work; a 3-D dynamic simulation is conducted implementing Newton’s and Euler’s equation of motion. These equations of motions are very useful in understanding the vehicle's motion.

Analyses are performed to show the behavior and characteristics of the tire and essentially the vehicle using different cases to determine such behavior such as; effect of different types of camber angle applied to its tires, effect of increase and decrease in steering ratio to the vehicle's handling and a skid pad test in the form of with constant angular velocity on both rear wheels. To develop understanding on how tire loads characterize vehicle handling is the main purpose of this thesis.

The form and content of this abstract are approved. I recommend its publication.

Approved: Professor John A. Trapp
DEDICATION

I dedicate this thesis to God almighty for the strength and will to complete this thesis work and conclude my graduate studies at the great University of Colorado Denver. I also dedicate this thesis to my family; Dad, Mom, Aira and Ite, my extended family Sam, Tobechi, and my unborn nephew, and lastly my Uncle Funsho for their constant push to never give up, love, support, beliefs and prayers.

A special dedication to the memory of my older brother Imonitie Ojebuoboh, who I miss everyday, whom without his guidance and advises growing up it would have been more difficult to grow as a student and an individual. He taught me to never give up, and I use that advice to keep me moving and to reach and complete my goals.
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I would also like to thank the members of this thesis committee; Dr. Samuel W.J. Welch and Dr. Peter E. Jenkins who agreed to be on my thesis committee because they believed I could and would complete such a huge task. I would like to thank you to Joe Cullen for granting me the permission to audit his vehicle dynamics course, so I could learn more about vehicle dynamics. Last but not least, I would like to thank the department of Mechanical Engineering at the University of Colorado Denver, College of Engineering and applied Sciences and its Professors and staff for guiding me through this incredible and unbelievable journey.
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1. Introduction

Race cars are fast, smooth on the ground and just fun to watch on the track. That is why engineers make it their priority to design a safe and fast race car to win the race. A race car and its driver are one, a complete body. It is the engineers’ job to make sure it is safe, balanced, steady, stable and of course fast. OptimumG is an international vehicle dynamics consultant group that works with automotive companies and motorsports teams to enhance their understanding of vehicle dynamics through seminars, consulting and software development\textsuperscript{[1]}. This thesis will model a Leman’s race car and run simulations focusing on the forces acting on the tires which are needed to make the car make a turn or a cornering maneuver (lateral forces), accelerate and brake (longitudinal force) while absorbing the load, balancing and stabilizing the car (normal force).

1.1 Background

A tire to a layman is a round inflated tube of rubber that moves the vehicle from point A to point B. To an automotive or mechanical engineer, it is a lot more! There are a lot of details to take into account in a tire. There are several tire manufacturing companies in the world today, such names like; Michelin, Good year, Firestone, Dunlop etc who work non-stop in designing, testing and manufacturing tires for all forms of vehicles. When researching on tire modeling, there is an individual who everyone takes good note from or need to take from at least. Hans B. Pacejka is an individual who developed a mathematical equations to understand the mechanical behavior of pneumatic tires. He designed the widely used Magic formula tire model which will also be used in this thesis work; this Magic formula will be analysed and discussed in full details below.
1.2 Research Approach

The approach of this research is a step by step approach. Understanding of how matlab operates and works is very imperative. There are several analysis completed but before any of which is done, these had to happen:
i) Working with thesis adviser and Professor, John Trapp Ph.D. to understand, debug and edit three dimensional Newton – Euler code developed by Dr. Trapp used to solve typical dynamics problem.
ii) Car data collection from the OptimumG team for a typical Lemans car, this proprietary data is generic and certain values and result can’t be shown in this paper because of its proprietary nature. There are graph representations that will show the behavior of these tires.
iii) Implementing the magic formula for characterizing pneumatic tire behavior from OptimumG test data
iv) Editing and manipulation of 3-D code to run for various input and test for behavior at different cases.
v) Running the code and plotting accurate graphs to show and explain the vehicles behavior

1.3 Research Objective

The main purpose of this paper is to enhance the understanding of tire and model the lateral, longitudinal, normal forces and the moments about their axis such as yaw, pitch and roll by modification of an iterative-computational model for calculating such values in matlab. Analysis to show better understanding of these force and moments would be performed to show how the tire behaves with time. Several other cases will be considered, such cases which will have an effect on the vehicle handling and cornering will include; behavior and comparison of tires with camber angles and without camber angles, skid pad test with constant velocity applied
on the rear wheels and effect of increase or decrease in steering to determine the steer coefficient of the vehicle.

The developed three dimensional Newton–Euler code can be used to study the behavior of any moving vehicle, so long the data is correct, the little changes that would be made would be the constraints, number of bodies and its components to get the correct degree of freedom.
2. Literature Review

The literature review briefly describes certain major terms needed to be known in order to understand this thesis. Vehicle dynamics and its various sub-sections such as cornering, aerodynamics effect, wheelbase and track of the vehicle and handling will be discussed in the literature review.

2.1 Vehicle Dynamics

This is a very broad topic and has been discussed in several papers. The effect of cornering, aerodynamics effect, wheelbase size, steady state handling & steering, and many more are constantly considered when designing a vehicle, these are used to find better ways to improve the quality of any vehicle. To do any research in this understanding of some basic vehicle dynamics is imperative.

2.1.1 Cornering

This is the term used when a vehicle negotiates a turn at a corner or in this case when racing. When cornering occurs, depending on the turn (right or left) the normal force acting on the outer side tires are larger than that of the inner side tires. In order to negotiate this turn a lateral force \( F_y \) is needed, a lateral force is an applied tire force that originates at the center of the tire contact with the road\(^3\) and lies in the horizontal road plane and is perpendicular to the direction in which the wheel is moving towards. The figure 2.1 below illustrates the direction the lateral force acting on each wheel, with velocity direction of the vehicle and turning radius.
Figure 2.1: A vehicle with lateral forces applied to its wheels moving on a track

The lateral force applied produces a slip angle; a slip angle has been defined in several ways, Karnopp\textsuperscript{[3]} defines a slip angle $\alpha$ as the angle between the centerline and the direction of travel or as Milliken\textsuperscript{[4]} describes it as the movement in the direction at an angle to the wheel plane.

The figure 2.2 below shows a wheel and its slip angle.

Figure 2.2: A tire showing slip angle
Maurice Olley[2] who to the world of the automobile industry is an outstanding pioneer and innovator in the ride and handling area of a car develops a model that can describe the relationship between lateral force and slip angle, the equation is derived below:

\[ F_y = a \cdot \alpha + b \cdot \alpha^2 \]  

(1)

Where

\( a = \) weight transfer length of wheelbase in the front section of the car in meters, m  
\( b = \) weight transfer length of wheelbase in the rear section of the car in meter, m  
\( \alpha = \) Slip angle in degrees  
\( F_y = \) Lateral Force in Newton, N

The plot below can show the behavior of a tire with lateral force relating to slip angle, the slope of this graph at origin is the cornering stiffness be shown relating the lateral force of a tire to its slip angle, the plot in figure 2.3 is shown in the next page,

![Plot of Lateral Force versus Slip Angle](image)

**Figure 2.3:** Lateral force versus slip angle plot
\[ F_{y_{\text{max}}} = \mu F_z \]  
\[ \frac{dF_y}{d\alpha} = 0 \text{ at } \alpha = \alpha_{\text{max}} \]  
\[ \frac{dF_y}{d\alpha} = a + 2b\alpha \]  
\[ \frac{dF_y}{d\alpha} = a + 2b\alpha = 0 \]  
\[ \frac{dF_y}{d\alpha} = a + 2b\alpha \]

The initial slope as seen in the graph called cornering stiffness is defined as such:

\[ C_{\alpha} = \frac{dF_y}{d\alpha} \text{ at } \alpha = 0 \]  
\[ \frac{dF_y}{d\alpha} = a(1 - \frac{2\alpha}{2\alpha_{\text{max}}}) = 0 \]  

After integrating, the lateral force is derived

\[ F_y = C_{\alpha}(\alpha - \frac{\alpha}{2\alpha_{\text{max}}}) \]

An experiment performed lead to the following; the cornering stiffness \( C_{\alpha} \) is plotted with the normal force which gives a parabola shape. A similar graph figure 2.4 is drawn to explain the equation below.
Figure 2.4: Cornering stiffness versus normal force[^2]

\( C_m, W_m \) and \( \mu \) characterize the tire and are used to derive \( C_\alpha \) and \( \alpha_{max} \)

\[
\frac{dC_\alpha}{dF_z} = 0 \text{ at } F_z = W_m \tag{10}
\]

\[
\frac{dC_\alpha}{dF_z} = a + 2 \times b \times F_z \Rightarrow a + 2 \times b \times W_m = 0 \tag{11}
\]

\[
b = -\frac{a}{2 \times W_m} \tag{12}
\]

\[
C_\alpha = a(F_z - \frac{F_z^2}{2 \times W_m}) \tag{13}
\]

Now at \( C_\alpha = C_m \) and \( F_z = W_m \) we get the fully defined equation of cornering stiffness

\[
C_m = a(W_m - \frac{W_m^2}{2W_m}) \tag{14}
\]

\[
\Rightarrow a = 2 \frac{C_m}{W_m} \tag{15}
\]

Now that \( a \) has been solved for, the equation (13) above can be redefined as,

\[
C_\alpha = 2 \frac{C_m}{W_m} (F_z - \frac{F_z^2}{2 \times W_m})
\]
In order to get the $a_{max}$, $F_{y_{max}} = \mu F_z = F_y$ at $\alpha = a_{max}$ recalling $F_y = C\alpha (\alpha - \frac{\alpha}{2a_{max}})$

$$\mu F_z = C\alpha a_{max} (1 - \frac{a_{max}}{2a_{max}})$$ becomes the redefined equation and $a_{max}$

$$\Rightarrow a_{max} = \frac{2\mu F_z}{C\alpha}$$ (16)

Where

$F_z$ = Normal Force in Newton, N

$C\alpha$ = Cornering Stiffness in N/deg

$a_{max}$ = Maximum slip angle reached in degrees

$W_m$ = nominal load of vehicle in Newton, N

An expanded model arbitrary lateral force model was developed by Maurice Olley in his appendix A in the Chassis Design text by William F. Milliken and Douglas L. Milliken.

2.1.2 Aerodynamic Effect

Aerodynamics is the study of air motion and its effect on solid matter; it can be studied more in fluid dynamics. Race car designers and manufacturers take into account aerodynamics, the force of drag which refers to the forces which act on a solid object in the direction opposing the positive longitudinal direction, a downforce is also acknowledge and it is the force acting opposite the normal force of the vehicle, it is designed to accomplish stability and cornering in a vehicle. There are other aerodynamics forces to understand, such as; side force (wind), the forces in the x, y and z plane have moments also. The pitching, rolling and yawing moment are also taken into account.

The general formulas for drag for each axis are shown below; the coefficients of the various drags are gotten from result from the wind tunnel testing.
Drag Force \[ = 0.5 \cdot \rho \cdot V^2 \cdot C_D \cdot A_F \]  \hspace{1cm} (17)

Down Force \[ = 0.5 \cdot \rho \cdot V^2 \cdot C_L \cdot A_F \]  \hspace{1cm} (18)

Side Force \[ = 0.5 \cdot \rho \cdot V^2 \cdot C_S \cdot A_F \]  \hspace{1cm} (19)

Where:

\[ \rho = \text{Air density kg/m}^3 \]

\[ V = \text{wind velocity m/s} \]

\[ A_F = \text{frontal area of the vehicle m}^2 \]

\[ C_{D,LS} = \text{Coefficient of drag related to the type of force} \]

The equation for \( A_F \) is as follows;

\[ A_F = 1.6 + 0.00056(m_v - 765) \]  \hspace{1cm} (20)

Where \( m_v \) is the mass of the vehicle. To get the moment, the various drag force multiply the wheel base essentially, but note is taken on the weight distribution of the vehicle.

Pitching Moment \[ = 0.5 \cdot \rho \cdot V^2 \cdot C_P \cdot A_F \cdot \text{Wheelbase} \]  \hspace{1cm} (21)

Rolling Moment \[ = 0.5 \cdot \rho \cdot V^2 \cdot C_R \cdot A_F \cdot \text{Wheelbase} \]  \hspace{1cm} (22)

Yaw Moment \[ = 0.5 \cdot \rho \cdot V^2 \cdot C_Y \cdot A_F \cdot \text{Wheelbase} \]  \hspace{1cm} (23)

The equations above are the typical equations to use when calculating drag forces (N) and moments (Nm), the coefficient of drag are gotten from the wind tunnel testing. These equations are applied in the simulation for the testing of the behavior of the vehicle and the tire in this thesis for accurate results. Figure 2.5 below shows a race car and its force and moment acting on it and Figure 2.6 shows how the aerodynamic forces vary with time for the simulated model used in this thesis at constant velocity.
Figure 2.5: Aerodynamic Forces and Moments on a car\cite{6}

2.1.3 Wheelbase and Track

The wheelbase is the horizontal distance between the center of the front wheel and the center of the rear wheel. A track is the distance between the centreline of two wheels on the
same axle, each on the other side of the vehicle. Things to consider when choosing a wheelbase and track for a race car \(^1\); lateral and longitudinal weight transfer, yaw moment of inertia, aerodynamics, packaging, type of car, type of race track and rules of the race. OptimumG \(^1\) describes some of these considerations in relation to track and wheelbase; weight transfer is a function of track width (lateral) and wheelbase length (longitudinal), a wide track will cause less lateral weight transfer during cornering and a long wheelbase will cause less longitudinal weight transfer in braking and accelerating. An increase in track or wheelbase will cause the masses of the non-suspended components (e.g. rim, tire, hub, upright, brake etc.) to be placed further from the centre of gravity of the car, this will increase the cars yaw moment of inertia. In the case of aerodynamics an increase in track width will cause an increase in drag. Track changes in open wheel race cars will affect the entire flow pattern of the car. Figure 2.7 shows the full track of a race car.

![Figure 2.7: Full Track\(^1\)](image)

Figure 2.8 below shows the wheelbase of a race car, where the sum of a and b gives the wheelbase
2.1.4 Handling

Without steering there will be no handling, steering affects handling effect of the vehicle. Vehicle handling describes how the motion of the vehicle is controlled taking into consideration the vehicle's primary six degrees of freedom and how the effect of external disturbance such as wind on stability of the vehicle can be reduced. Steering is made possible by yaw motion (rotation in the z-axis) reference figure 2.5 above; a 3D body can be rotated about three orthogonal axes, the yaw rotation, pitch rotation and the roll rotation. The equation for rotational matrices and steer angle are defined below:

\[
M_y = \text{yaw rotation} = \begin{pmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix}
\] (24)

\[
M_p = \text{pitch rotation} = \begin{pmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{pmatrix}
\] (25)
The rates of the above matrices are found and derived giving the \( \omega_x \), \( \omega_y \) and \( \omega_z \) as the roll, pitch and yaw rates known as body angular velocities

\[
\begin{pmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{pmatrix} =
\begin{pmatrix}
-sin \theta & 0 & 1 \\
\sin \phi \cos \theta & \cos \phi & 0 \\
\cos \phi \cos \theta & -\sin \phi & 0
\end{pmatrix}
\begin{pmatrix}
\cdot \\
\cdot \\
\cdot
\end{pmatrix}
\]  

(27)

The steering angle for a vehicle is shown as

\[
\delta_f = L/R + \alpha_f - \alpha_r
\]  

(28)

From equation (7) at \( \alpha \neq 0 \) and after integrating \( \alpha \) can be defined as

\[
\alpha = \frac{W \times V^2}{C_a \times g \times R}
\]  

(29)

\( \alpha \) is defined for each of the vehicle, the equation variables are described below:

\( \delta_f \) = steer angle in degrees

\( L \) = wheelbase in m

\( R \) = turning radius in m

Equation (24) can be redefined as

\[
\delta_f = L/R + \frac{W_f \times V^2}{C_{af} \times g \times R} - \frac{W_r \times V^2}{C_{ar} \times g \times R}
\]

And eventually we get,

\[
\delta_f = L/R + K_{uw} \frac{V^2}{gR}
\]
Where $K_{us}$ is the understeer coefficient and is defined below,

$$K_{us} = \left( \frac{W_f}{C_{af}} - \frac{W_r}{C_{ar}} \right)$$

(30)

The $K_{us}$ describes if a vehicle is at oversteer, neutral steer or understeer. While neutral steer and understeer is still considered stable, oversteer tends to be unstable. A skid pad test would be performed to see the handling behavior of the vehicle by adjusting the steer angle and the other test will be the effect of wind force on the lateral side of the vehicle and paying attention to lateral velocity of the vehicle. Figure 2.9 shows a physical interpretation of this understeer coefficient values. In this figure, understeer coefficient $K_{us}$ is explained; an increase in speed when cornering a vehicle with no change in steering angle is shown for three cases to explain handling. When the vehicle maintains a constant radius while cornering with increase in speed with constant steering angle, there is a neutral steer, when the vehicle has an increase in the radius while cornering there is an understeer, when there is a decrease in that radius there is an oversteer. Understeer occurs when the front wheels of the car lose traction before the rear wheels. The car is difficult to turn and pushes toward the outside of a turn. Oversteer is the opposite condition. The rear tires lose traction before the front tires. Hence, the rear of the car is loose. It slides toward the outside of the turn, and the car feels like it is going to spin out. A skid pad test is performed and results are gotten based on the understeer coefficient analogy.
Figure 2.9: Curvature Response of different values of understeer coefficient [7]

More description of the characteristics of the different understeer coefficient value is described in the sections next.

2.1.4.1 Neutral Steer

The side force acts at the center of gravity, the cause of neutral steer while cornering; turning radius R is remains constant, no increase or decrease. Steer angle $\delta_f$ is independent of forward speed V. The understeer coefficient is equal to zero, $K_{us} = 0$.

2.1.4.2 Oversteer

The side force acts at the center of gravity, the cause of oversteer while cornering; turning radius R decreases. Steer angle $\delta_f$ decrease with increase in forward speed V. The understeer coefficient is less than zero, $K_{us} < 0$. 

16
2.1.4.3 Understeer

The side force acts at the center of gravity, the cause of oversteer while cornering; turning radius $R$ increases. Steer angle $\delta_j$ increases with increase in forward speed $V$. The understeer coefficient is greater than zero, $K_{us} > 0$.

2.2 Pneumatic Tire

In the case of this thesis, understanding of a pneumatic tire is needed and shown below. Wong\textsuperscript{[10]} describes the purpose of a pneumatic tire as functioning to support the vehicles weight on the ground, cushioning the vehicle over surface with an irregularity (which is really saying absorbing shock from the ground contact), provide sufficient traction for braking and accelerating, and also proving steering control and direction stability. These descriptions clearly explain the main purposes of a tire on a vehicle. Now, these are just sayings, as engineers we back our words with mathematical calculations and accurate results. This leads to the next chapter of equations of motion and tire modeling.

3. Newton – Euler

3.1 Equations of Motion

There are two main equations of motion; Newton’s law of translational motion for a system of particles as a whole, governed by the Newton’s second law,

$$\sum_{i=1}^{N} m\ddot{\mathbf{v}} = \sum_{i=1}^{N} F_i$$  \hspace{1cm} (31)

Where $m$ is mass, $\ddot{\mathbf{v}}$ is acceleration, and $F_i$ is force\textsuperscript{[8]}. The forces acting on the body which could be applied (gravity force, spring and dashpot forces etc) or constrained forces acting on the joints of the body (ball and socket joints, point planes etc). Euler’s law of rotational motion for a system of particles is defined as,

$$J_0\dot{\omega} = \sum_{i=1}^{N} \mathbf{r}_{Bi} \times F_i$$  \hspace{1cm} (32)
Where \( J_0 \) is moment of inertia in \( kg - m^2 \), \( \dot{\omega} \) is angular velocity in rad/sec, and \( r_{Bi} \) is rotation about point B of the particle acting on the \( i^{th} \) particle [8]. Nikravesh [9] describes the steps taken to solve all dynamics problems. There are usually 7 steps:

1. A sketch of the rigid body with a center of mass position vector, with only one world frame and several body frames depending on number of rigid body.

2. Free body diagrams for each rigid body showing applied forces and reaction/constraint forces. Applied forces are given forces such as; gravity [m/s\(^2\)], springs forces [N/m] etc. Reaction/ constraint forces are unknown magnitude that has to be found that has a known direction.

3. Write the Newton’s and Euler’s equations of motion for each of the rigid bodies.

4. Geometry of the systems; use primary variables from step 1 to formulate the geometric constraint between each pair of bodies, one joint at a time (body coordinate formulation) and then write the primary variable sin term of a reduced generalized coordinates (joint coordinate formulations).

5. Number of unknown variables must equal the number of equations to make this system of equations solvable. List all the unknowns present.

6. Reduce the system of ordinary differential equations and geometric constraint equations to a smaller set if possible and solve.

7. Examine the solution and ensure it makes sense and it is accurate.

Equation (31) and (32) above are the simple general forms of the N-E equation, for a three dimensional body, the equations are more complex. Account of the forces acting in all directions, in the longitudinal (x), lateral (y) and normal (z) axis as well as the moments along
the roll (x), pitch (y), and yaw (z) axis is taken into effect. The steps are very essential and useful in any dynamics analysis. Applying the Euler method, the N-E can be re-written as:

\[
m(v_i^{n+1} - v_i^n) \frac{dt}{dt} = F_i^n + F_i^{n+1}
\]  
(33)

\[
m(v_2^{n+1} - v_2^n) \frac{dt}{dt} = F_2^n + F_2^{n+1}
\]  
(34)

\[
m(v_3^{n+1} - v_3^n) \frac{dt}{dt} = F_3^n + F_3^{n+1}
\]  
(35)

\[
I_1 (w_1^{n+1} - w_1^n) = (I_2 - I_3)w_2^n \times w_2^n = M_1^n + M_1^{n+1}
\]  
(36)

\[
I_2 (w_2^{n+1} - w_2^n) = (I_3 - I_1)w_3^n \times w_3^n = M_2^n + M_2^{n+1}
\]  
(37)

\[
I_3 (w_3^{n+1} - w_3^n) = (I_1 - I_2)w_1^n \times w_1^n = M_3^n + M_3^{n+1}
\]  
(38)

The 1, 2 and 3 represent x, y and z respectively, the variables in the equation are defined as;

\[m\] = mass of the object in kg

\[M\] = moment acting on each axis in Nm

\[I\] = moment of inertia in \(kg \cdot m^2\)

\[\omega\] = angular velocity in rad/sec

In the 3-D equation above the \(n\) denotes old time explicit evaluation (explicit methods calculate the state of a system at a later time from the state of the system at the current time) for applied forces and moments term, i.e. spring, dashpots, aerodynamics forces etc. The \(n+1\) term denotes new implicit evaluation (implicit methods find a solution by solving an equation involving both the current state of the system and the later one) for reaction/constraint forces and moments.
The N-E equation is used in this next section to describe a light link, distance between two or more solid objects.

### 3.1.1 Light Link

In the 3-D simulation performed in this thesis, the light links represent the frame and the suspension of the vehicle, the matlab function builds linearized geometric constraint equations for a weightless link between the two bodies. The corresponding constraint force is also loaded into the coefficient array. The figure 3.1 below shows a representation of a light link between two bodies B and C.

![Diagram of Light Link](image)

**Figure 3.1:** Light link between two bodies

For a typical light link body calculation between two bodies in space, the N-E steps above are followed then the equation is added to the simulation for each body connection in the vehicle, the length of the light link is,
\[ L = |\overrightarrow{r_p} - \overrightarrow{r_Q}| \]  
(39)

\[ \overrightarrow{r_p} = \overrightarrow{r_B} + \overrightarrow{r_{PB}} \]  
(40)

\[ \overrightarrow{r_Q} = \overrightarrow{r_C} + \overrightarrow{r_{QC}} \]  
(41)

There is a normal force \( s \) acting on the surface of both body on point \( \overrightarrow{P} \) and \( \overrightarrow{Q} \), the magnitude of the normal force is then dotted to the unit normal which lies on the plane to get the normal force vector. The equation is added to the Newton section of the equation and a moment from the force is added to the Euler part.

\[ F_{\text{body}1} = s(\overrightarrow{r_p} - \overrightarrow{r_Q}) / L \]  
(42)

\[ F_{\text{body}2} = s(\overrightarrow{r_Q} - \overrightarrow{r_p}) / L \]  
(43)

\[ M_{\text{body}1} = \overrightarrow{r_{QC}} \otimes s(\overrightarrow{r_p} - \overrightarrow{r_Q}) / L \]  
(44)

\[ M_{\text{body}2} = \overrightarrow{r_{PB}} \otimes s(\overrightarrow{r_Q} - \overrightarrow{r_p}) / L \]  
(45)

### 3.1.2 Spring and Damper

The suspension in the vehicle makes use of torsional springs and dashpots. Suspension systems serve a dual purpose, the main purpose of the suspension is stability and handling of the vehicle and it connects a vehicle to its wheels and allows relative motion between the two\(^{11}\) as well as isolating the road noise, bumps, and vibrations, etc from the driver. The spring force and dashpot are calculated with the equation shown below;

\[ F_{\text{spring}} = \pm kx_{\text{spring}} \]  
(46)

Where,
$x_{\text{spring}}$ = the displacement vector – the distance and direction the spring is deformed from its equilibrium length.

$F_{\text{spring}}$ = the resulting force vector – the magnitude and direction of the restoring force the spring exerts

$k$ = the rate, spring constant or force constant of the spring, a constant that depends on the spring's material and construction.

A dashpot on the other hand is a mechanical device, a damper which resists motion via viscous friction.\cite{12} The resulting force is proportional to the velocity, but acts in the opposite direction,\cite{13} slowing the motion and absorbing energy. It is commonly used in conjunction with a spring (which acts to resist displacement).

\[ F_{\text{dashpot}} = \pm c \frac{dx}{dt} \] (47)

Where,

\( c \) = damping coefficient in N-s/m.

To apply the equation of both the spring and dashpot to the N-E 3-D code, it is very similar to that of the light link equation, except for some force application changes, the equations are shown below. The matlab function implements a linear spring or damper element by loading the applied force and moment terms to the right hand side of the Newton-Euler equations to model a spring and a dashpot; again a point p on body 1 is connected to point q on body 2. Take note that there are four wheels, so essentially there will be 5 bodies, the main car body and the spring and dashpot connect eachs each light link body which is then connected to the wheel.

\[ L = |\overrightarrow{r_p} - \overrightarrow{r_q}| \] (48)
\[ \underline{r}_p = \underline{r}_B + \underline{r}_{PB} \]  \hspace{1.5cm} (49)

\[ \underline{r}_Q = \underline{r}_C + \underline{r}_{QC} \]  \hspace{1.5cm} (50)

The force is again dotted with its unit normal, the equation is added to the Newton section of the equation and a moment of from the force is added to the Euler part. Equation (46) is re-written as shown for forces acting on a spring and dashpot.

\[ F_{spring} = \pm k(l - l_0) \]

\[ F_{spring} = \pm k(l - l_0) n \]  \hspace{1.5cm} (51)

\[ F_{spring} = \pm k(l - l_0)(\underline{r}_Q - \underline{r}_p) / L \]

\[ F_{spring} = \pm k[(\underline{r}_Q - \underline{r}_p) - l_0](\underline{r}_Q - \underline{r}_p) / L \]

The dashpot becomes,

\[ F_{dashpot} = \pm c[(\underline{r}_Q - \underline{r}_p) - l_0](\underline{r}_Q - \underline{r}_p) / L \]  \hspace{1.5cm} (52)

Both equation (48) and (49) are added to the Force section of the N-E 3-D code and for the moment equation added to the Euler section, there is

\[ M_{spring} = \underline{r}_{QC} \otimes F_{spring} \]  \hspace{1.5cm} (53)

\[ M_{dashpot} = \underline{r}_{PB} \otimes F_{dashpot} \]  \hspace{1.5cm} (54)

The graph below shows the forces of both the spring and dashpot against time, this is gotten from the simulation result. Since a four wheel vehicle is in question, there are four plots.
and these plots are shown below. The simulation result is gotten while the vehicle is negotiating a turn makes a 360 degree turn, forming a circle; the front left wheel creates more normal force, because when making a turn, the outside wheels have more normal forces, more will be explained in the tire modeling section of this thesis. The dashpot force in the front wheels behaves a lot different compared to the rear wheels, a higher amount is calculated and seen on the graph. The Force of the spring has opposite values from the outer wheels compared to the inner wheels; this is due to cornering of the vehicle. The behavior is seen to be similar on each side of the car, just different values because of the wheel negotiating a turn. The black color shows the dashpot forces and the red color shows the spring forces as seen in figure 3.2 below,

**Figure 3.2:** Spring and Dashpot forces with time, cornering exist

The figure above describes the spring and dashpot forces as the vehicle is moving in real time, the initial fluctuation shows transient behavior then steady state. The behavior of the springs of the vehicle is shown the spring forces increase with velocity in the figure below.
Figure 3.3: Spring forces with velocity

3.1.3 Ball and Socket

A ball and socket joint between two bodies or between a body and the world. This function generates 3 constraint equations and 3 force reactions. This is used in the matlab code to calculate forces in the joints and its function is used to connect part of the suspension system to the car body e.g. the rocker. The reaction forces are the forces on body 1 in the world frame coordinates e1, e2, and e3 directions at the point with body frame coordinates p1, p2, p3. There are equal but opposite forces on body 2 at the point with body 2 coordinates q1, q2 and q3. The bodies could be and in general will be subject to (i) other constraints and (ii) applied forces and moments\[8\]. The equation is basically the N-E 3-D equation where it calculates the forces from the two bodies. Equation (31) and (32) above shows these equations

\[
\sum_{i=1}^{N} m \ddot{\mathbf{v}} = \sum_{i=1}^{N} \mathbf{F}_i
\]

\[
J_0 \ddot{\mathbf{\omega}} = \sum_{i=1}^{N} \mathbf{r}_{Bi} \times \mathbf{F}_i
\]

An illustration of two bodies connected by ball and socket joint is shown below, a body P connected to a body P (point mass)
The distance between body P and body Q is \( S = S_1 \bar{b}_1 + S_2 \bar{b}_2 + S_3 \bar{b}_3 \), there is an equal and opposite force \( F \) holding both bodies together, this is in the world frame \( \bar{F} = F_1 \bar{e}_1 + F_2 \bar{e}_2 + F_3 \bar{e}_3 \) which is added with its moment to the N-E 3-D equation. Note the conversion from body frame to world frame has to be calculated.

### 3.1.4 Wheel on ground

A wheel matlab function connects the body (a tire) and the ground surface in the world frame. This function builds the appropriate constraint equations (using five extra variables) between a wheel and the ground. The function generates 6, 7, or 8 constraint equations - five (because of the extra variables and 1, 2, or 3 additional constraints for normal contact, drive direction no slip, or transverse direction no slip. This set of constraint equations generates the three constraint forces (depending upon the case) at the wheel ground contact point on the wheel. This is a very essential part of this thesis, since the idea is to model a tire and understand its behavior via many different test, constraints and disturbances. Figure 3.5 below shows one wheel trying to make contact on a surface which is the ground. There are four wheels in this race car and each of them will have to be in contact with the ground and this function in matlab makes
this possible and a simulation where the vehicle can drive in a straight road or attempt cornering is made possible.

Figure 3.5: Wheel on a flat surface

There are certain steps to follow to create the wheel function, this is shown below;

1. Get the surface ground equation as \( F_{\text{ground}}(x, y, z) = y - x_d = 0 \) for a flat surface, where \( x_d \) is the ground height value.

2. A point on the road for body ground \( Q \) with dimensions \( \overrightarrow{r}_Q = xe_1 + ye_2 + ze_3 \) is established and also the point on the wheel that would be in contact with this point is \( \overrightarrow{r}_p = s_1\overrightarrow{b}_1 + s_2\overrightarrow{b}_2 + s_3\overrightarrow{b}_3 \).

3. Equation of a circle is used, equation is \( f(s_1, s_2) = s_1^2 + s_2^2 - R^2 = 0 \), \( R \) is the radius and \( s_1 = x - h \) and \( s_2 = y - k \) both \( x \) and \( h \) are points in the middle of the circle. A derivation of the function is
calculated $\vec{V}_f = \frac{\partial f}{\partial s_1} \vec{b}_1 + \frac{\partial f}{\partial s_2} \vec{b}_2$ this is used to get the unit vector which is

$$\frac{\vec{V}_f}{|\vec{V}_f|} = \frac{2s_1\vec{b}_1 + 2s_2\vec{b}_2}{\sqrt{(2s_1)^2 + (2s_2)^2}} = \vec{n}$$

4. The force is dotted to the unit vector and added to the N-E 3-D equation and subsequently the moment is defined also.

This is used to calculate any form of wheel on any type of ground, bumpy or smooth as the case may be.

4. **Tire Modeling**

Tire modeling is a method of developing various forces and moments acting on a tire in order to drive and maneuver the vehicle. A great understanding of a tire and its relation between the road, driver, and vehicle is imperative in order to model a tire. Hans B. Pacejka[^14] an expert in vehicle dynamics with more focus on tire modeling provides an opportunity to better understand the behavior of pneumatic tire and its impact on vehicle dynamics.

An empirical method to calculate these forces and moments acting on a tire is called Magic Formula. This formula has been used since the 1980’s, Pacejka[^15] has developed models in both steady and non-steady transient states to improve tire behavior. A simulation designed with this Magic Formula is very useful to be able to see and understand that behavior of the tire and make it easier to locate potential problems that can be fixed. The Magic Formula has several components that make the formula, these components can be used to calculate several forces and moments acting on a tire, that is why it is very widely used in the automotive industry.

The formula which holds for given values of vertical load and camber angle will be broken down to longitudinal force (pure and combined longitudinal slip), lateral force (pure and combined side slip) and aligning torque (pure and combined side slip), some of the other
moments which are not taken into account in this thesis is the overturning moment and the roll resistance moment. Test data provided by OptimumG do not include the coefficients values needed to analyze these moments, although the simulation, if data is produced can run and describe the moments.

The Magic Formula is capable of producing propitious characteristics that closely match measured curves for the side, acceleration and braking force. The behavior of this formula is typically seen on a graph; a curve is produced passing through the origin, reaching a maximum point like amplitude and then tends to a horizontal asymptote[15].

The Magic Formula is shown below;

\[
y = D \sin[C \arctan\{Bx - E(Bx - \arctan Bx)\}]
\]  

(55)

Figure 4.1: SAE tire axis system [16]
\[ Y(X) = y(x) + S_y \]  \hfill (56)

\[ x_0 = X + S_H \]  \hfill (57)

Where,

\( Y \) = Output variable \( F_x, F_y \) and \( M_z \)

\( X \) = Input variable \( \tan \alpha \) or \( \kappa \)

\( B \) = stiffness factor

\( D \) = peak value of the curve

\( C \) = shape factor (determines the shape of the peak)

\( E \) = Curvature Factor

\( S_H \) and \( S_y \) = shifting values; they shift the curve horizontally and vertically.

An idea of how the graph should look like is shown in the graph below. This is a typical lateral force – slope angle graph a data will produce

**Figure 4.2:** Magic Formula graphical representation
The factors of the magic formula have equations and these are shown below, reference the appendix MatLab code to read how these values gotten from the equation influence the main force and moment results gotten from the tires. Other influences of the main output force and moment include camber angle, normal force, a normalized normal force and parameter values gotten from tire testing. The simple form of the formula is shown below

\[ D = \mu F_z \]  

\[ C = \frac{2}{\pi} \sin^{-1}\left(\frac{y_z}{D}\right) \]  

\[ E = \frac{Bx_0 - \tan\left[\frac{\pi}{2}\right]}{Bx_0 - \tan^{-1}(Bx_0)} \]  

\[ BCD = p_1 \sin\left[2\tan^{-1}(F_z/p_2)\right] \bullet (1 - p_3\gamma^2) \]

Where \( p_1 \), \( p_2 \) and \( p_3 \) are gotten from the parameter values of the tire, values gotten from testing of the tires.

4.1 Forces

The external forces acting on the tire of the car are defined below according to Milliken\(^4\) and Pacejka\(^{14} \), the longitudinal force, lateral force and normal force

4.1.1 Longitudinal Forces

Longitudinal Force \( (F_x) \) is the component of the force vector in the x-direction. This is the force responsible for the braking and driving force of a vehicle, the positive for the driving and the equal but negative force is needed for the braking. The Magic Formula becomes,

\[ F_x = D_x \sin[C_x \arctan\{B_x \kappa_x - E_x (B_x \kappa_x - \arctan B_x \kappa_x)\}] + S_{Vx} \]  

\[(62)\]
Where $\kappa_x$ longitudinal slip with horizontal shift in axis and $\kappa$ the longitudinal slip ratio is defined as

$$\kappa_x = \kappa + S_{Hx}$$  \hspace{1cm} (63)

$$\kappa = -\frac{V_{sx}}{|V_{cx}|}$$  \hspace{1cm} (64)

![Figure 4.3: Rolling and Slipping of a tire][14]

Where

$V_x$ is the velocity of the wheel in the x – direction

$V_{sx}$ is the slip velocity in the x – direction

$V_{cx}$ is the contact velocity in the x – direction

$V_{cx}^*$ is the longitudinal running speed

$S$ is the center of the rotation of motion or slip point
$C^*$ is the imaginary point of the center of the wheel

$C$ is the contact center of the tire

$\Omega$ is the angular speed

$\Omega_r$ is the angular speed of rolling tire

$r$ is the loaded tires radius

$r_e$ is the effective rolling radius, when the tire is in motion

It can be seen where longitudinal slip ratio is calculated from, slip is the relative motion between a tire and the road surface it is moving on. The above ratio is very essential in calculating the longitudinal force and then plotted with this force to observe the tires behavior. The stiffness factors (B), peak value of the curve (D), shape factor (C) and Curvature Factor (E) are variables that are very essential to this formula and are calculated from variables gotten during tire testing, the values are different compared to that of the lateral force equation.

Pure longitudinal force $F_{xw}$ and combined longitudinal force $F_x$ differ from each other; in combined longitudinal force, both the pure longitudinal force and lateral slip which is gotten from the pure lateral force calculation is needed and a graph showing both forces is shown below, the pure longitudinal force (green) can be seen to be greater than the combined longitudinal force (blue) because of the addition of slip angle values, they do not depend on camber angle. They are both equal when slip angle added is zero, the fourth graph seen with the highest amplitude, is the behavior when more normal forces are acting on the tire.
Figure 4.4: plot of Longitudinal force against Slip ratio

The simulation is run for a vehicle driving on a straight road that doesn’t negotiate any turn, so no cornering force is applied; the behavior of the longitudinal force on the tire is seen in time. The above plot is gotten for an instant time and it shows the relationship between force and the slip ratio, the plot below shows what that force behaves like in time and also the slip ratio behavior in time.
4.1.2 Lateral Forces

Lateral (cornering) Force \( (F_y) \) is the component of the force vector in the \( y \)-direction. This is the force responsible for the cornering force of a vehicle, when a car tries to negotiate a turn whether in a race or just turning a curve into a street, the cornering force is needed to make that turn. The Magic Formula becomes,

\[
F_y = D_y \sin[C_y \arctan\{B_y \alpha_y - E_y \left( B_y \alpha_y - \arctan B_y \alpha_y \right)\}] + S_{yx} 
\]

(65)

Where \( \alpha_y \) and \( \alpha^* \) the lateral slip angle are defined as

\[
\alpha_y = \alpha + S_{Hy} 
\]

(66)

\[
\alpha^* = -\frac{V_{cy}}{|V_{cx}|} 
\]

(67)

Where \( V_{cx} \) is the contact velocity in the \( x \) – direction and \( V_{cy} \) is the contact velocity in the \( y \) – direction, the above slip angle is very essential in calculating the lateral force and then plotted with this force to observe the tires behavior. Again, the stiffness factor (B), peak value of the
curve (D), shape factor (C) and Curvature Factor (E) is variables that are very essential to this formula and are calculated from variables gotten tire testing, the values are different compared to that of the lateral force equation.

Pure lateral force $F_{yo}$ and combine lateral force $F_y$ differ from each other; in combined lateral force, both the pure lateral force and longitudinal slip which is gotten from the pure longitudinal force calculation is needed and a graph showing both forces is shown below, they both depend on camber angle $\gamma$. The effect of positive and negative camber can be seen on the graph, positive camber angle increase the lateral force and negative camber angle decreases the lateral force, shown in cyan (positive $\gamma$) and green (negative $\gamma$). This is essential in design of race cars and other vehicles, depends on how sharp a turn the designer wants the vehicle to make. The pure lateral force (blue) can be seen to be greater than the combined longitudinal force (red) because of the addition of slip ratio values. They are both equal when slip ratio added is zero, the highest amplitude graph seen, is the behavior when more normal forces are acting on the tire.
The simulation is run for a vehicle driving on a road and negotiating a turn, the cornering force is applied; the behavior of the lateral force on the tire is seen in time. The above plot is gotten for an instant time and it shows the relationship between force and the angle, the plot below shows what that force behaves like in time and also the slip angle. When negotiating a curve, the outer wheel produce more normal forces which in instantly increases the lateral force, the outer force is seen below as the higher normal forces. Note- tire 1- FL (red), tire 2 FR (blue), tire 3 RL (green), tire 4 RR (black).

**Figure 4.6:** plot of lateral force against slip angle
Some of the other relevant plots such as the plot for slip angle, slip ratio, camber angle and velocity plotted with time are shown below. These graphs shows the behaviors of the tire
4.2 Moments

The external moments acting on the car are defined below,

a) Overturning moment \((M_x)\) is the component of the moment vector tending to rotate the vehicle about the \(x\) – axis, positive clockwise when looking in the positive direction of the \(x\) – axis.

b) Rolling resistance Moment \((M_y)\) is the component of the moment vector tending to rotate the vehicle about the \(y\) – axis, positive clockwise when looking in the positive direction of the \(y\) – axis.

c) Aligning torque \((M_z)\) is the component of the moment vector tending to rotate the vehicle about the \(z\) – axis, positive clockwise when looking in the positive direction of the \(z\) – axis.

For the purpose of this thesis as explained earlier in page 34, aligning force would only be discussed
4.2.1 Aligning Forces

This is the moment responsible for the cornering force of a vehicle, when a car tries to negotiate a turn whether in a race or just turning a curve into a street, the moment about the $z$-axis helps the vehicle make that turn. For the aligning torque to work, a pneumatic trail $t_o$ is the distance between the side force and the cornering force,

![Diagram of a vehicle cornering](image)

**Figure 4.10: Pneumatic Trail[^10]**

There are different equations for the pure pneumatic trail and combined pneumatic trail and they are listed below, the combined term is,

$$ t = D_i \cos[C_i \arctan\{B_i \alpha_{i,eq} - E_i (B_i \alpha_{i,eq} - \arctan(B_i \alpha_{i,eq}))\}] \cdot \cos^\prime \alpha $$

(68)

The pure term is,

$$ t_o = D_i \cos[C_i \arctan\{B_i \alpha_i - E_i (B_i \alpha_i - \arctan(B_i \alpha_i))\}] \cdot \cos^\prime \alpha $$

(69)

Where $\alpha_{i,eq}$ and $\alpha_i$ the slip angle for pneumatic trail are defined as

$$ \alpha_i = \alpha^\prime + S_{Ht} $$

(70)
\[
\alpha_{r,eq} = \sqrt{\alpha_t^2 + \frac{K_x}{K_{ya}}k^2} \cdot \text{sgn}(\alpha_t)
\]  \hspace{1cm} (71)

Where \(K_{ya}\) and \(K_{ak}\) are constant values gotten from the tire testing and the final aligning torque equation is then the product of the lateral force with the pneumatic trail and then added to a residual torque

\[
M_{cr} = D_r \cos(C_r \arctan(B_r \alpha_r))
\]  \hspace{1cm} (72)

\[
\alpha_r = \alpha^* + S_{HF}
\]  \hspace{1cm} (73)

The final equation is,

\[
M_c = -t \cdot F_y + M_{cr}
\]  \hspace{1cm} (74)

Pure aligning torque \(M_{cr}\) and combine lateral force \(M_c\) differ from each other; in combined aligning force, values from calculating the pure lateral force, pure longitudinal force and aligning torque are used. A graph showing both moments is shown below; they both depend on camber angle \(\gamma\). The effect of positive and negative camber like that of the lateral forces can be seen on the graph, positive camber angle increase the pure aligning torque and negative camber angle decreases the aligning torque, shown in cyan (positive \(\gamma\)) and green (negative \(\gamma\)). This is essential in design of race cars and other vehicles, depends on how sharp a turn the designer wants the vehicle to make. The pure aligning torque (blue) can be seen to be greater than the combined longitudinal force (yellow) because of the addition of slip ratio values. They are both equal when slip ratio added is zero, the highest amplitude graph seen, is the behavior when more normal forces are acting on the tire.
Figure 4.11: Plot of aligning torque against slip angle

The simulation is run for a vehicle driving on a road and negotiating a turn, the cornering force and with it an aligning moment is applied; the behavior of the lateral force on the tire is seen in time. The above plot is gotten for an instant time and it shows the relationship between torque and the angle, the plot below shows what that torque behaves like in time. A similar behavior should be seen like that of the lateral force in time, but different values and units.
Figure 4.12: Plot of aligning torque and pneumatic trail against time

4.3 Friction Circle

The lateral force and longitudinal force as functions of slip angle and traction/braking slip ratio is described in a friction circle. According to Milliken et al, at the origin the tire is free – rolling straight ahead with no longitudinal force, the slip angle and lateral force are zero. The vertical axis represents longitudinal force and various slip ratios are marked along the scale. Points on the vertical axis could be used to develop the curves of slip ratio vs. longitudinal force. The simulation runs a function that creates a friction circle, the plot below shows friction circle at initial run and at the end of the run for a cornering.
Figure 4.13: Friction circle at initial run

The blue line shows the direction of the velocity, the green line divides the circle into four quadrants to be able to show the different lateral force on the side and the longitudinal force along the velocity direction, the diameter of the circle shows the normal forces behavior. The graph below shows the graph at final run after cornering has been made. As the wheels move, an angle is created which is the slip angle, the angle between the velocity line (blue) and quadrant (green). Recall a slip angle as occurring when the steering wheel is turned from straight forming an angle between where the tire is pointed and where the car is actually going, the outer normal force is bigger compared to that of the inner tires.

Figure 4.14: Friction circle at final run
4.4 Cases

Different case runs are made to see how different values can affect the behavior of the tire, such include; comparing camber angle to non-camber angle and handling and steering.

4.4.1 Effect of positive, negative and zero camber angle

Simulation is run for cornering and comparing the forces and moment acting on the tire with time and understands the different behavior.
Wheel camber is defined as the angle between the wheel center plane and the normal to the road, the graph above shows the front view of a vehicle and its applied positive camber angle on the tires. The camber angle $\gamma$ as discussed in the lateral force section above influences the force and moment acting on the tire when cornering, test cases are shown to know how the effect of camber angle whether zero, positive or negative can affect cornering forces. The equation for camber angle is shown below

$$\sin \gamma = -n \cdot s$$  \hspace{1cm} (75)

Where

$\gamma$ is the camber angle

$n$ - known as the ground surface normal vector pointing up from the ground in the positive ground gradient direction.

$s$ is the vector normal to the wheel center plane

![Figure 4.17: Types of camber angle – (a.) zero, (b.) positive, (c.) negative]{17}
4.4.1.1 Zero camber angle

When zero camber angles are applied to the tires of the race car, the tires are straight and not bent inwards or outwards as shown in figure 4.18 above. The influences of camber angle is referenced in the appendix B. matlab code below, both lateral force and aligning torque equations are influenced by camber angle in the form of a term named spin_camber. The graph below shows the lateral force without the application of camber angle below

![Graph showing lateral force with zero camber angle](image)

**Figure 4.18:** Lateral force with zero camber angle

The graph below shows the tire angle at zero camber

![Graph showing tire angle at zero camber](image)

**Figure 4.19:** zero camber angle
4.4.1.2 Positive camber angle

The lateral forces are plotted also for positive camber angle, the two figures shown below describes the effect of positive camber angle, an increase in the lateral force can be seen compared to the zero camber, hence allowing for more cornering. The second graph shows the plot of tire angles with time and the camber angle plot can be seen.

![Lateral force with positive camber angle](image)

**Figure 4.20:** Lateral force with positive camber angle

![Slip Angle](image)

![Slip Ratio](image)

![Camber Angle](image)

![Velocity vs Time](image)

**Figure 4.21:** Positive camber angle
4.4.1.3 Negative camber angle

The lateral forces are plotted also for negative camber angle, the two figures shown below describe the effect of negative camber angle, a decrease in the lateral force can be seen compared to the zero camber and positive camber angle, hence allowing for less cornering. The second graph shows the plot of tire angles with time and the camber angle plot can be seen.

Figure 4.22: Lateral force with negative camber angle

Figure 4.23: Negative camber angle
4.4.2 Skid Pad Test

In this case, the steering ratio is adjusted to determine the behavior of the vehicle. A rack-and-pinion gearset is enclosed in a metal tube, with each end of the rack protruding from the tube. A rod, called a tie rod, connects to each end of the rack. The pinion gear is attached to the steering shaft. When you turn the steering wheel, the gear spins, moving the rack. The tie rod at each end of the rack connects to the steering arm on the spindle. The rack-and-pinion gearset converts the rotational motion of the steering wheel into the linear motion needed to turn the wheels. The steering ratio is the ratio of how far you turn the steering wheel to how far the wheels turn\textsuperscript{[4]}. For instance, if one complete revolution (360 degrees) of the steering wheel results in the wheels of the car turning 10 degrees, then the steering ratio is 360 divided by 10, or 36:1. The steering ratio is gotten by a steer – steer test, this is completed by using alignment tables with a steering scale. The load and ride height of the car should be known. A ride height is the amount of space between the base of an automobile tire and the underside of the chassis\textsuperscript{[2]}. The steering wheel is turned to the right at even intervals, noting the steer angles of both front wheels. The test continues by rotating the steering wheel back to the center, stopping at each angle and noting the front wheel for any hysteresis. Hysteresis is the influence of the previous history or treatment of a body on its subsequent response to a given force or changed condition\textsuperscript{[1][2]}. The skid pad test was performed for both 20% less steer ratio and 20% more steer ratio. This is done to determine the handling of the car, to be able to see if the vehicle is oversteer, understeer or neutral steer when the steering is adjusted while driving. Using section 2.1.4 which defines understeer coefficient, this case can determine the effect of steering on the vehicle. Six plots are seen below to explain this.
Figure 4.24: steering rack and pinion of a vehicle rack\cite{7}

Figure 4.25: Vehicle on a track at normal steer ratio

Figure 4.26: Radius made by the vehicle at normal steer ratio
The case where the steer ratio is decreased by 20% has values plotted below and can be seen it is stable as the case with less steer angle by viewing both graphs below, the vehicle uses lesser lateral force to negotiate a turn and the radius of the curve is larger. This is a method to fix understeer in vehicles.

**Figure 4.27**: Vehicle on a track at 20% less steer ratio

**Figure 4.28**: Radius made by the vehicle at 20% less steer ratio
The case where the steer ratio is increased by 20% has values plotted below and can be seen it is stable as the case with more steer angle by viewing both graphs below, the vehicle uses more lateral force to negotiate a turn and the radius of the curve is smaller. This is a method to fix under oversteer in vehicles.

**Figure 4.29:** Vehicle on a track at 20% more steer angle

**Figure 4.30:** Radius made by the vehicle at 20% more steer ratio
4.4.2.2 Constant Velocity

Another case is performed to determine the handling properties of the vehicle; a constant velocity skid pad test is performed in MatLab. Comparison is made between constant velocities of the two rear wheels to non–constant velocity of the two rear wheels, behavior of the vehicle can be read in the graphs shown below. The two rear wheels are linked together to have a constant rotational moment $\omega$, this then allows for the two rear wheels to move at the same rate. The graph below shows the cornering forces, radius, and velocity plots for both constant velocity and non-constant velocity cases.

![Graphs showing cornering forces, radius, and velocity plots for both constant velocity and non-constant velocity cases.]

**Figure 4.31:** Constant rotational moment on rear wheels
**Figure 4.32**: Non-constant rotational moment on rear wheels

From the two graphs above, differences can be read from the slip angle being affected all the way to the slip ratio, the tire velocity reduces in the constant rotational graph. There are other graphs shown below to help explain which case is better for the vehicle and why. The lateral force graphs below show a large difference that can affect cornering of the vehicle.
Figure 4.33: Constant rotational moment on rear wheels – lateral forces

Figure 4.34: non-constant rotational moment on rear wheels – lateral forces

In the constant rotation moment plot for the lateral forces, smaller lateral forces are acting on the rear wheels with lesser velocity in order to negotiate a more accurate turn. With so much lateral force acting on the rear wheels while trying to negotiate a turn, it will reduce the smoothness of the cornering of the vehicle.
A comparison of both the radius made while turning and aerodynamic drag and lift can be read in the graph below also

**Figure 4.35**: Constant rotational moment on rear wheels – radius

**Figure 4.36**: non - constant rotational moment on rear wheels – radius
With constant moment applied to the rear wheels, a constant in aerodynamic forces can be maintained and in the non-constant moment case, a drop in the aerodynamic forces can be seen. It can be seen from the graphs that for constant moment acting on the rear wheels, handling...
characteristics are better compared to the non–constant moment case, the constant moment radius graph shows an oversteer compared to that of the non–constant moment radius.

5. Conclusion and Recommendation

The goal for this thesis was to determine the behavior of pneumatic tires of a Lemans race car and be able to see that behavior with different cases, this done in order to design better, faster and safer tires for the racer.

The three different simulation cases were performed in order to have a better understanding of the race car tire, the theories have been met. The camber angle case shows how camber angle can affect cornering of a vehicle, most standard race car vehicles have negative camber angle applied to their tires, even this race car I am working on does. In order to understand why they have those angles applied a case was made, simulations were run to show the differences. It can be seen that the negative camber angles produce more lateral forces needed for cornering. Now, this doesn’t mean the other two are bad for vehicles, they have other applications and other vehicles might have the right use for zero camber and positive camber angle.

The other case was the comparison of steering ratio, typically looking on how much a driver can steer and how it affects handling of a car. The more steering provided by the driver, depending on the design of the steering rack and pinion, the chassis of the vehicle etc., more steering can affect the vehicles handling and make it oversteer which makes the car unstable. Again cases were made to have a physical understanding of how increasing or decreasing steering angle and ratio can affect the vehicle.

The last case is performed by designing the vehicle to have its rear wheels have constant angular moment and note is taken on how much difference when compared to that of a vehicle
with no constant angular moment. The lateral forces, aerodynamic forces and velocity are all affected by the difference.

The matlab code can essentially be used for other vehicle type, used to calculate the off-road ground vehicle, two wheel vehicle, a train, trailer, cart etc, several components might have to be changed and adjusted, that is because all vehicles are not the same and several things need to be considered when designing vehicles.

Although this chapter of the thesis is completed, there is still a work to be done, the stability of the vehicle for a linear and non-linear steady state and transient state, also making the code run a bit faster than its current time. Although, there are so many different functions attached to one main code (reason why it runs a bit slow), this can be fixed by differentiation and changing the method to second order which will make calculation faster and give the same result.

A more in-depth experience with automotive engineering will also be a good step to follow.
Appendix

A. Nomenclature

\( a \) = weight transfer length of wheelbase in the front section of the car in meters, \( m \)

\( b \) = weight transfer length of wheelbase in the rear section of the car in meter, \( m \)

\( \alpha \) = Slip angle in degrees

\( F_y \) = Lateral Force in Newton, \( N \)

\( \mu \) = coefficient of friction

\( F_z \) = Normal Force in Newton, \( N \)

\( C_a \) = Cornering Stiffness in N/deg

\( \alpha_{\text{max}} \) = Maximum slip angle reached in degrees

\( W_m \) = no min al load of vehicle in Newton, \( N \)

\( \rho \) = Air density kg/m\(^3\)

\( V \) = wind velocity m/s

\( A_F \) = frontal area of the vehicle m\(^2\)

\( C_{D,L,S} \) = Coefficient of drag related to the type of force

\( \rho \) = Drag Coefficient subscript

\( \alpha_f \) = front wheel slip angle in degrees

\( \alpha_r \) = rear wheel slip angle in degrees

\( m \) = mass of the body

\( \nu \) = acceleration

\( F_i \) = Force of the body

\( \delta \) = direction of force

\( j_o \) = moment of inertia kg \( \cdot \) m\(^2\)

\( \lambda \) = Lift Coefficient subscript

\( s \) = Side Force Coefficient subscript

\( \delta_f \) = steer angle in degrees

\( L \) = wheelbase in m

\( R \) = turning radius in m

\( K_{\text{us}} \) = understeer coefficient

\( \omega \) = angular velocity in rad/sec

\( r_{Bi} \) = rotation about point \( B \) of the particle acting on the \( i^{th} \) particle

\( M \) = moment acting on each axis in Nm

\( I \) = moment of inertia in kg \( \cdot \) m\(^2\)
\( c = \text{damping coefficient in N} \cdot \text{s/m}. \)

\( M_{\text{spring}} = \text{the moment acting on the spring} \)

\( M_{\text{dashpot}} = \text{the moment acting on the dashpot} \)

\( \vec{n} = \text{normal vector} \)

\( Y = \text{Output variable} F_x, F_y, \text{and} M_z \)

\( X = \text{Input variable} \tan \alpha \text{ or} \kappa \)

\( B = \text{stiffness factor} \)

\( D = \text{peak value of the curve} \)

\( C = \text{shape factor (determines the shape of the peak)} \)

\( E = \text{Curvature Factor} \)

\( S,_{\text{H}} \text{and} S,_{\text{V}} = \text{shifting values, they shift the curve horizontally and vertically.} \)

\( k_x = \text{longitudinal slip with horizontal shift in axis} \)

\( \kappa = \text{longitudinal slip ratio} \)

\( V_{sx} = \text{slip velocity in the} x - \text{direction} \)

\( V_{cx} = \text{contact velocity in the} x - \text{direction} \)

\( V_{sy} = \text{slip velocity in the} y - \text{direction} \)

\( V_{cy} = \text{contact velocity in the} y - \text{direction} \)

\( F_{xo} = \text{Pure longitudinal force} \)

\( F_{x} = \text{combined longitudinal force} \)

\( F_{yo} = \text{Pure lateral force} \)

\( F_{y} = \text{combined lateral force} \)

\( M_{yo} = \text{Pure Aligning Torque} \)

\( M_{z} = \text{Combined Aligning Torque} \)

\( \alpha_{y} = \text{lateral slip angle} \)

\( \gamma = \text{camber angle} \)

\( FL = \text{Front Left} \)

\( FR = \text{Front Right} \)

\( RL = \text{Right Left} \)
RR = Rear Right

\( x_{spring} = \text{the displacement vector} \) the distance and direction the spring
is deformed from its equilibrium length.

\( F_{spring} = \text{the resulting force vector} \) the magnitude and direction of the restoring
force the spring exerts

\( k = \text{the rate, spring constant} \) force constant of the spring,
a constant that depends on the spring’s material and construction

\( F_{dashpot} = \text{the resulting force vector} \) the magnitude and direction of
the restoring force the dashpot exerts

\( V_{sx} = \text{the slip velocity in the x direction} \)
\( V_{cx} = \text{the contact velocity in the x direction} \)
\( V_{cx} = \text{the longitudinal running speed} \)

\( S = \text{the center of the rotation of motion or slip point} \)
\( C' = \text{the imaginary point of the center of the wheel} \)
\( C = \text{the contact center of the tire} \)

\( \Omega = \text{the angular speed} \)
\( \Omega_r = \text{the angular speed of rolling tire} \)
\( r = \text{the loaded tires radius} \)
\( r_e = \text{the effective rolling radius} \)
B. MatLab Code

function [Fx,Fx0,Fy,Fy0,Mx,My,Mz,Mz0] = Magic_Formula(kappa,Fz,ntire,spin_camber,sangle)

% %Input new values for Pacejka Model 2002
% %-----------------------------------------------------------
% % insert the common global parameters into this function
% global_time
% % This data runs the coefficients needed to run the Lateral function
Magic_Formula_Coefficients();

format compact
format long e

dfz = ( Fz - Fz0(ntire) )/Fz0(ntire);
Fz0prime = lambdaFz0 * Fz0(ntire);

% ****************************************************
% Pure Longitudinal equation
% ****************************************************

SHx = (pHx1(ntire) + pHx2(ntire)*dfz)*lambdaHx;
kappax = kappa + SHx;
Cx = pCx1(ntire)*lambdaCx; % should be > 0
mux = (pDx1(ntire) + (pDx2(ntire)*dfz))*lammuxstar;
mux = max(mux,-1*mux);
Dx = mux*Fz*zeta1;
Ex = (pEx1(ntire) + (pEx2(ntire)*dfz) + (pEx3(ntire)*... + dfz*dfz))* (1 - (pEx4(ntire)*sign(kappax)))*lambdaEx; % should be <=1
Kxk = Fz * (pKx1(ntire)+(pKx2(ntire)*dfz))*exp(pKx3(ntire)*dfz)*...
lambdaKxk;
Bx = Kxk/((Cx*Dx) + etax);
SVx =  Fz*(pVx1(ntire) + pVx2(ntire)*dfz)*lammuxprime*zeta1;

Fx0 = Dx*sin(Cx*atan(Bx*kappax - Ex*(Bx*kappax - atan(Bx*... kappax)))) + SVx;

% ****************************************************
% Combined Longitudinal Force equation
% ****************************************************

SHxa = rHx1(ntire);
alphas = sangle + SHxa;
Exa = rEx1(ntire) + rEx2(ntire)*dfz;
Cxa = rCx1(ntire);
Bxa = rBx1(ntire) * cos(atan(rBx2(ntire)*kappa))*lambdaxa;
Gxa0 = cos(Cxa*atan(Bxa*SHxa - Exa*(Bxa*SHxa - atan(Bxa*SHxa))));
Gxa = cos(Cxa*atan(Bxa*alphas - Exa*(Bxa*alphas - atan(Bxa*alphas))))/Gxa0;

Fx = (Gxa*Fx0);

%**********************************************************************************************
% Pure Lateral equation
%**********************************************************************************************

SHy = (pHy1(ntire) + pHy2(ntire)*dfz)*lambdaHy + pHy3(ntire)*
      spin_camber*lambdaKyy*zeta0 + zeta4 - 1;
alphay = sangle + SHy;  
Cy =  pCy1(ntire)*lambdaCy;  
    % should be > 0 
muy = (pDy1(ntire) + pDy2(ntire)*dfz)*...
    (1 - pDy3(ntire)*spin_camber^2)*lammuystar;
Dy =  muy*Fz*zeta2;
Ey =  (pEy1(ntire) + pEy2(ntire)*dfz)*(1 - (pEy3(ntire) +... 
    pEy4(ntire)*spin_camber)*sign(alphay))*lambdaEy;
Ey = min(Ey,-1*Ey);
Kyao = pKy1(ntire)*Fz0prime*sin(2*atan(Fz/(pKy2(ntire)*... 
      Fz0prime)))*lambdaKya;
Kya =  Kyao*(1-pKy3(ntire)*spin_camber^2)*zeta3;
By =  Kya/((Cy*Dy));
SVy =  Fz*((pVy1(ntire) + pVy2(ntire)*dfz)*lambdaVyy + (pVy3(ntire)+... 
      pVy4(ntire)*dfz)*spin_camber*lambdaKyy)*lammuyprime*zeta2;

Fy0 = Dy*sin(Cy*atan(By*alphay -... 
      Ey*(By*alphay - atan(By*alphay)))) + SVy;

%**********************************************************************************************
% Combined Lateral Force equation
%**********************************************************************************************

DVyK = (muy*Fz)*(rVy1(ntire) + rVy2(ntire)*dfz + rVy3(ntire)*spin_camber)*...
      cos(atan(rVy4(ntire)*sangle))*zeta2;
SVyK = DVyK*sin(rVy5(ntire)*atan(rVy6(ntire)*kappa));
SHyK = rHy1(ntire) + rHy2(ntire)*dfz;
kappas = kappa + SHyK;
EyK = rEy1(ntire) + rEy2(ntire)*dfz;
EyK = min(EyK,-1*EyK);
CyK = rCy1(ntire);
ByK = rBy1(ntire)*cos(atan(rBy2(ntire)*(sangle-rBy3(ntire))))*lambdayK;
GyK0 = cos(CyK*atan(ByK*SHyK - EyK*(ByK*SHyK)- atan(ByK*SHyK)));
GyK = cos(CyK*atan(ByK*kappas - EyK*(ByK*kappas - atan(ByK*kappas))))/GyK0;
\[ F_y = (GyK*Fy0) + SVyK; \]

\% Pure Aligning Torque equation

\% Combined Aligning Torque equation

\[ \alpha_{t\_eq} = (\sqrt{\alpha_{t}^2 + ((KxK/Kyaprime)^2)*(kappa^2)})*\text{sign}(\alpha_{t}); \]
\[ \alpha_{r\_eq} = (\sqrt{\alpha_{r}^2 + ((KxK/Kyaprime)^2)*(kappa^2)})*\text{sign}(\alpha_{r}); \]
\[ s = R0(\text{ntire})*(sS1(\text{ntire}) + sS2(\text{ntire})*(Fy/Fy0prime) + sS3(\text{ntire}) + ... \]
\[ Mz0prime = -t0*Fy0; \]
\[ Mz0 = Mz0prime + Mz0; \]
% OverTurning moment equation

Mx = Fz*R0(ntire)*( qSx1(ntire) - qSx2(ntire)*spin_camber + qSx3(ntire)*Fy/Fz0(ntire) );

% Rolling Resistance equation

VroverVo = 1.0;
My = -Fz*R0(ntire)*( qSy1(ntire)*atan(VroverVo) + qSy2(ntire)*Fx/Fz0(ntire) );
REFERENCES


