EXTREMAL PROBLEMS FOR DEGREE SEQUENCES

by

CATHERINE C. ERBES

B.A., Carleton College, 2005

M.A., Indiana University, Bloomington, 2007

A thesis submitted to the

Faculty of the Graduate School of the

University of Colorado in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Applied Mathematics

2014

This thesis for the Doctor of Philosophy degree by

Catherine C. Erbes

has been approved for the

Department of Mathematical and Statistical Sciences

by

Michael Jacobson, Chair

Michael J. Ferrara, Advisor

Ellen Gethner

Stephen Hartke

Paul Wenger

April 29, 2014

Erbes, Catherine C. (Ph.D., Applied Mathematics)

Extremal Problems for Degree Sequences

Thesis directed by Associate Professor Michael J. Ferrara

ABSTRACT

A sequence of nonnegative integers π is *graphic* if it is the degree sequence of some graph G. In this case we say that G is a *realization* of π. Degree-sequence analogues of many classical problems in extremal graph theory appear throughout the literature. In this thesis we present results about two extremal problems for degree sequences, the potential number (the analogue of the classical Turán number) and the potential-Ramsey number.

A graphic sequence π is *potentially H-graphic* if there is a realization of π that contains H as a subgraph. The potential number of a graph H, denoted $\sigma(H, n)$, is the minimum even integer such that any graphic sequence of length n and sum at least $\sigma(H, n)$ is potentially H-graphic. The potential number has been determined asymptotically for general graphs H, and a family $\mathcal{P}(H)$ of extremal sequences that achieve this number is known.

Given nonincreasing graphic sequences $\pi_1 = (d_1, \ldots, d_n)$ and $\pi_2 = (s_1, \ldots, s_n)$, we say that π_1 *majorizes* π_2 if $d_i \geq s_i$ for all i, $1 \leq i \leq n$. In 1970, Erdős showed that for any K_{r+1}-free graph H, there exists an r-partite graph G such that $\pi(G)$ majorizes $\pi(H)$. In 2005, Pikhurko and Taraz generalized this notion and showed that for any graph F with chromatic number $r + 1$, the degree sequence of an F-free graph is, in an appropriate sense, nearly majorized by the degree sequence of an r-partite graph. Here, we give similar results for degree sequences that are not potentially H-graphic. In particular, we show that if π is a graphic sequence that is not potentially H-graphic, then π is close to being majorized by a sequence in $\mathcal{P}(H)$.

This shows that the structure of sequences that are not potentially H-graphic is close to that of the extremal sequences.

Secondly, we give a stability result for the potential problem, similar to the stability results of Erdős and Simonovits for the Turán problem. We say that a graph H is σ-stable if every graphic sequence with sum close to $\sigma(H, n)$ that is not potentially H-graphic can be transformed into a sequence in $\mathcal{P}(H)$ with $o(n)$ additions and subtractions. We show that, in contrast to the Turán problem, not all graphs are σ-stable. We also show that a large family of graphs are σ-stable.

Finally, we address a degree-sequence variant of the Ramsey number, recently introduced by Busch, et al. Given graphs G_1 and G_2, let $r_{pot}(G_1, G_2)$ be the minimum integer n such that for any graphic sequence π of length n, either π is potentially G_1-graphic or the complement of π is potentially G_2-graphic. We give several lower bounds on $r_{pot}(G_1, G_2)$, and also determine the values of $r_{pot}(C_s, C_t)$ and $r_{pot}(G_1, G_2)$ when G_1 is a fixed graph of order at most four and G_2 is arbitrary.

The form and content of this abstract are approved. I recommend its publication.

Approved: Michael J. Ferrara

To Rachel

ACKNOWLEDGMENTS

First, I would like to thank my advisor Mike Ferrara, for guiding me on this journey since day one. Thanks is also due to the members of my committee: Mike Jacobson (without whose advice I would never have even taken graph theory), Paul Wenger, Ellen Gethner, and Stephen Hartke, who have all helped me to become a better mathematician. I would also like to thank Ryan Martin, who along with Paul and Mike, helped me truly understand the necessity of a precisely written proof, no matter how ugly it might be.

I would also like to thank the NSF Grant #074324 "UCD GK-12 Transforming Experiences Project" and the Lynn Bateman Memorial Fellowship for funding portions of this research.

I would not have made it to this point without the support of many people. My fellow grad students, particularly Jenny Diemunsch and Tim Morris, my academic siblings, whose companionship in this process (also since day one) has made it much more fun, if not any easier; Axel Brandt, Brent Thomas, Mark Mueller, Sam Graffeo, and Devon Sigler, who were always happy to help with math, distraction, coffee, or drinks, whichever was most needed at the time; and Breeann Flesch and Eric Sullivan, who served as a reminder to me that it is possible to not only graduate, but also to find a job and succeed after grad school. My mom, dad, and brothers and sisters, who have always believed in me, and especially my sister Anissa, whose passion for flight and stories about learning to fly have been my greatest source of motivation. And finally, my wife Rachel, who always knew when I needed to play video games and when I needed to do math, and whose patience and support have meant the world to me.

TABLE OF CONTENTS

TABLES

Table

FIGURES

Figure

1. Introduction

1.1 Definitions and concepts

A *graph* G consists of a *vertex set* $V(G)$ and an *edge set* $E(G)$, where each edge is a set of vertices. Throughout the majority of this paper, we will restrict ourselves to standard graphs, in which each edge contains exactly two distinct vertices. Hypergraphs, in which an edge may contain more than two vertices (or even a single vertex), will be discussed briefly in the last chapter. In this thesis, we will only consider finite simple graphs, meaning that both the vertex and edge sets are finite, and there are no repeated edges or loops (edges consisting of one vertex multiple times).

To denote edges in G, we will write xy or $e = xy$ for an edge with endpoints x and y, and simply e if the endpoints are not relevant. If x and y are vertices of G and $xy \in E(G)$, we say that x is *adjacent* to y, and sometimes write $x \sim y$. When more than one graph is under consideration, we may specify that x is adjacent to y in G by writing $x \sim_G y$. If $xy \notin E(G)$ we sometimes write $x \not\sim y$ or $x \not\sim_G y$.

The *degree* of a vertex x, denoted $d(x)$ or $d_G(x)$, is the number of edges containing x, or, in the case of simple graphs, the number of vertices adjacent to x. The *neighborhood* of x is the set of vertices adjacent to x, and is written $N(x)$ or $N_G(x)$. So, we see that $d_G(x) = |N_G(x)|$. (The subscript G is used to specify a parameter within the graph or subgraph G. When there is only one graph being discussed or it is understood, we do not use the subscript.) The maximum degree and minimum degree of G are denoted $\Delta(G)$ and $\delta(G)$, respectively.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G, it is called *spanning* if $V(H) = V(G)$, and *induced* if for each $x, y \in V(H)$, $xy \in E(H)$ if and only if $xy \in E(G)$. If H is an induced subgraph of G, we write $H \leq G$. If $X \subseteq V(G)$, then the subgraph of G *induced by* X, denoted $G[X]$, is the graph $G - (V(G) \setminus X)$, which has vertex set X and edge set $\{xy \in E(G) \mid x, y \in X\}$.

There are several ways to create new graphs from other graphs. One such way is to take the *complement* of G, denoted $\overline{G}$, which has $V(\overline{G}) = V(G)$ and $e \in E(\overline{G})$ if and only if $e \notin E(G)$. The *disjoint union* of graphs G and H is denoted $G \cup H$, and has $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The *join* of G and H, denoted $G \vee H$ has vertex set $V(G) \cup V(H)$, and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G) \text{ and } y \in V(H)\}$.

A *path* is a simple graph in which the vertices can be ordered $v_1, \ldots, v_n$ such that $v_i \sim v_{i+1}$ for each $i \in \{1, \ldots, n-1\}$. If, in addition, $v_n \sim v_1$, then this graph is a *cycle*. We will often describe a path or cycle by its vertices, by writing $v_1 v_2 \cdots v_n$ for a path and $v_1 v_2 \cdots v_n v_1$ for a cycle. Let P_n denote the path on n vertices, and C_n the cycle on n vertices. We say that a graph is *connected* if there is a path between any pair of vertices. A *tree* is a connected graph that contains no cycles, and we say that a connected component of a graph is *nontrivial* if it has at least one edge.

A set of vertices that are pairwise adjacent is known as a *clique*. The complete graph on n vertices, denoted K_n, is a clique of order n. A set of vertices that are pairwise nonadjacent is an *independent set*, and the maximum order of an independent set in G is denoted $\alpha(G)$. A *matching* is a set of disjoint edges, that is a set of edges with no endpoints in common, and the maximum size of a matching in G is denoted $\alpha'(G)$.

A graph is *bipartite* if the vertex set can be partitioned into two independent sets, and *k-partite* if the vertex set can be partitioned into k independent sets. We will use $K_{r,s}$ to denote the *complete bipartite graph* with r vertices in one partite set and s vertices in the other.

A coloring of the vertices of a graph is called *proper* if adjacent vertices receive distinct colors. The *chromatic number* of G, denoted $\chi(G)$, is the minimum k such that G has a proper coloring using k colors.

Other terms will be defined as necessary, and for terms not defined in this dissertation, see [122].

1.2 An introduction to degree sequences

The *degree sequence* of a graph is a list of the vertex degrees of the graph. A sequence of nonnegative integers π is called *graphic* if it is the degree sequence of some graph G. In this case we say that G *realizes* π or is a *realization* of π, and we write $G = G(\pi)$, or $\pi = \pi(G)$. Unless otherwise noted, we will assume that all graphic sequences are written in nonincreasing order. We will often write $\pi = (d_1^{m_1}, \ldots, d_t^{m_t})$ to indicate that each degree d_i appears m_i times in π, although we will generally suppress the multiplicity of d_i if $m_i = 1$.

Given a graph, determining its degree sequence is a simple exercise. On the other hand, determining when a given sequence is graphic is more difficult. The first, and perhaps the simplest, characterization of graphic sequences is due independently to Havel and Hakimi.

Theorem 1.1 (Havel [59], Hakimi [56]) *Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers. The sequence π is graphic if and only if the sequence $\pi' = (d_2 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is graphic.*

Kleitman and Wang gave the following generalization of this theorem.

Theorem 1.2 (Kleitman and Wang [70]) *Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers, and let $i \in [n]$. If π_i is the sequence defined by*

$$\pi_i = \begin{cases} (d_1 - 1, \ldots, d_{d_i} - 1, d_{d_i+1}, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n) & \text{if } d_i < i \\ (d_1 - 1, \ldots, d_{i-1} - 1, d_{i+1} - 1, \ldots, d_{d_i+1} - 1, d_{d_i+2}, \ldots, d_n) & \text{if } d_i \geq i, \end{cases}$$

then π is graphic if and only if π_i is graphic.

Let π' be the sequence resulting from sorting π_i in nonincreasing order, and call π' the *residual sequence* obtained by *laying off* d_i.

The Havel-Hakimi and Kleitman-Wang Theorems give rise to efficient algorithms to test for graphicality. They also introduce so-called residual (sub-)sequence techniques for analyzing graphic sequences. These techniques involve using information that we know about the residual sequence, such as certain properties of its realizations, to gain knowledge about the original sequence. For example, if the sequence π' is potentially H-graphic (a property which will be defined in Section 2.2.2), then we know that π must also be potentially H-graphic.

In Chapter 3, we use a careful examination of the Kleitman-Wang algorithm to create a specific realization of a sequence that meets certain criteria. In Chapter 4, we will give some simple corollaries of the Kleitman-Wang algorithm that give us more information about the residual sequences obtained from applying it.

In addition to these residual-sequence characterizations of graphic sequences there are others, many of which take the form of systems of inequalities. The best-known of these is due to Erdős and Gallai.

Theorem 1.3 (Erdős and Gallai [37]) *A nonincreasing sequence* $\pi = (d_1, \ldots, d_n)$ *of nonnegative integers is graphic if and only if* $\sum_{i=1}^{n} d_i$ *is even and, for all* $p \in \{1, \ldots, n-1\}$,

$$\sum_{i=1}^{p} d_i \leq p(p-1) + \sum_{i=p+1}^{n} \min\{d_i, p\}. \tag{1.1}$$

In [60], Sierksma and Hoogeveen give six other characterizations of graphic sequences, and prove that they are all equivalent. Many others have given other characterizations or sufficient conditions for graphic sequences [66, 71, 115], as well as improvements on the Erdős-Gallai Criterion, in the form of a reduction of the number of inequalities that must be checked [116, 157].

A graphic sequence may have many different realizations. A common technique that is used to transform one realization into another is known as the *edge-exchange* or *2-switch*. If G is a realization of π with edges xy and uv such that xu and yv are

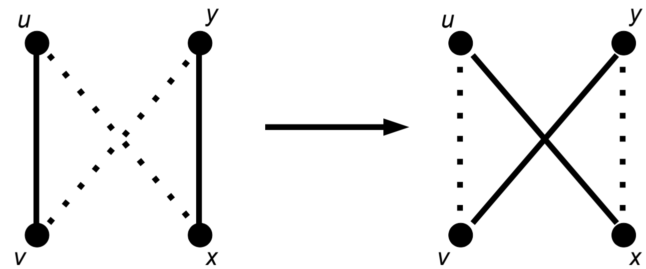


Figure 1.1: A 2-switch

not edges, we can delete the edges xy and uv and replace them with the edges xu and yv (see Figure 1.1). This maintains the degree of each vertex, so the new graph is also a realization of π.

Petersen [101] showed that given any pair of realizations of a graphic sequence, one can be obtained from the other by a sequence of 2-switches. This technique can be used to prove both the Havel-Hakimi and Kleitman-Wang Theorems, as well as many other results about degree sequences; a particularly nice example of this can be found in [45].

An *isomorphism* from a graph G to a graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. If there is an isomorphism from G to H, we say that G is *isomorphic to* H, or that they are in the same *isomorphism class.* While a 2-switch always changes the structure of a labeled graph (one where the names of the vertices are important), it may in fact preserve the isomorphism class of the (unlabeled) graph. Recently, Barrus [4] determined sufficient conditions for a 2-switch to change the isomorphism class of a graph.

2. Extremal Problems for Degree Sequences

2.1 An overview of extremal graph theory

Extremal graph theory can be thought of as the study of thresholds. In particular, we look for thresholds on graph invariants that guarantee that a graph has a certain property. That is, given an invariant $i(G)$, a class of graphs $\mathcal{F}$, and a graph property $\mathcal{P}$, the most general formulation of an extremal graph theory problem is: What is the minimum value m (or maximum value m') such that every graph $G \in \mathcal{F}$ with $i(G) > m$ ($i(G) < m'$) has property $\mathcal{P}$? Those graphs in $\mathcal{F}$ that satisfy $i(G) = m$ but do not have property $\mathcal{P}$ are known as *extremal graphs*.

2.1.1 The Turán problem

One of the first and most well-studied extremal problems is known as the Turán problem or simply the extremal problem. It is:

Problem 2.1 *Given a graph H, determine $ex(H, n)$, the maximum number of edges in a simple graph of order n that does not contain H as a subgraph.*

In this case, the graph invariant is the number of edges, the property is "contains H as a subgraph", and the class of graphs is all simple graphs. The first answer to a question of this type is due to Mantel, in 1907:

Theorem 2.2 (Mantel [94]) *The maximum number of edges in an n-vertex triangle-free simple graph is $\lfloor n^2/4 \rfloor$. That is, $ex(K_3, n) = \lfloor n^2/4 \rfloor$.*

The number $\text{ex}(H, n)$ is known as the *extremal number* or the *Turán number* for H, as the next result, due to Turán, was the first general solution to the problem. The graph $T_{n,r}$, known as the Turán graph, is the complete r-partite graph on n vertices, with partite sets as equal as possible. That is, if $n = kr + p$ where $0 \leq p < r$, then there are p partite sets of order $r + 1$ and $k - p$ partite sets of order r.

Theorem 2.3 (Turán's Theorem [117]) *For a positive integer r, $ex(K_{r+1}, n) = |E(T_{n,r})|$, and $T_{n,r}$ is the unique edge-maximal K_{r+1}-free graph of order n.*

Note that the Turán graph has $\binom{n}{2} - \frac{r(n+p-k)}{2} = (1-\frac{1}{r})\frac{n^2}{2} + O(n)$ edges.

While the exact value of the extremal function is known for very few graphs (for some examples, see [11, 18, 36, 103, 117]), in 1966 Erdős and Simonovits [40] extended previous work of Erdős and Stone [39] and determined $\mathrm{ex}(H, n)$ asymptotically for arbitrary (nonbipartite) H.

Theorem 2.4 (The Erdős-Stone-Simonovits Theorem [39, 40]) *If H is a graph with chromatic number $\chi(H) = r + 1 \geq 2$, then*

$$ex(H,n) = \left(1 - \frac{1}{r}\right)\binom{n}{2} + o(n^2).$$

This shows that asymptotically, the Turán graphs have the proper number of edges to be the extremal graphs for any H, not just for complete graphs. Subsequently, Simonovits [113] and Erdős independently proved the following, sometimes known as the First Stability Theorem.

Theorem 2.5 (Simonovits [113]) *Let H be a graph with $\chi(H) = r + 1$. For every $\epsilon > 0$, there exists a $\delta > 0$ and an n_ϵ such that if $n > n_\epsilon$ and G is an n-vertex H-free graph such that*

$$|E(G)| \geq ex(H,n) - \delta n^2,$$

then G can be obtained from $T_{n,r}$ by changing at most ϵn^2 edges.

With this theorem, Simonovits introduced the "stability method" for solving extremal problems. Recall that a general extremal problem seeks to find the minimum value m of a parameter (or graph invariant) $i(G)$ over a set of objects in a certain class $\mathcal{F}$ that will guarantee that every object for which $i(G) > m$ has property $\mathcal{P}$. The first step of the stability method is to find an asymptotic solution for the extremal problem, which usually also determines a set $\mathcal{C}$ of extremal objects. (If this set of extremal objects can be shown to be the complete set, then we can find an exact answer to the problem, so proving this is the ultimate goal.) The next step is to prove a stability result, which shows that for every object G in $\mathcal{F}$ for which $i(G)$ is close to but not greater than m, the structure of G is similar to that of the objects in $\mathcal{C}$. Using the stability result, the next step is to show that every extremal object is in fact in $\mathcal{C}$. This yields the exact answer to the extremal problem.

Simonovits used this method to determine exactly the extremal number for p disjoint copies of K_r, or pK_r [113]. Since then, stability methods have been used to attack a wide variety of extremal problems (see [7, 68, 97, 99, 102] for examples). Recently, stability methods have also been used to approach problems in Ramsey Theory [53, 54, 100], and have also proven particularly helpful in studying the hypergraph Turán problem (see [1], [96], or [98] for some examples).

2.1.2 Ramsey theory

Ramsey theory focuses on a different kind of extremal problem. In its original, most general form, Ramsey's Theorem [106] is about sets. Given a set S, we write $\binom{S}{r}$ to denote the set of r-element subsets of S. A set $T \subseteq S$ is called i-homogeneous if, under some coloring of the elements of $\binom{S}{r}$, all of the r-element subsets of T receive color i. If n is a positive integer, we let $[n] = \{1, \ldots, n\}$. With these definitions, we can now state Ramsey's Theorem.

Theorem 2.6 (Ramsey [106]) *Given positive integers r and $p_1, \ldots, p_k$, there exists an integer N such that every k-coloring of $\binom{[N]}{r}$ yields an i-homogeneous set of size p_i for some i.*

Graph theory provides a nice illustration of the case $r = k = 2$. In this case the sets of order 2 are edges, and we consider a 2-coloring of the edges of a complete graph. Thus, an i-homogeneous set of size p_i is a complete graph of order p_i in which all edges have the same color. In this language, Ramsey's Theorem says that given positive integers s and t, there is an n such that any red/blue coloring of the edges of K_n yields either a red K_s or a blue K_t. The number n is then called the *Ramsey number* and is denoted $r(s,t)$.

Ramsey numbers are notoriously difficult to compute. In fact, the exact value of $r(s,t)$ is known only for $s = 3$ and $t \in \{3, \ldots, 9\}$, or when $s = 4$ and $t = 4$, or $s = 4$ and $t = 5$ [105]. Asymptotically, we know the Ramsey number $r(3,t)$ within a constant factor:

$$\frac{ct^2}{\log t} \leq r(3,t) \leq \frac{c't^2}{\log t},$$

where currently the best known values of the constants are $c = 1/4$ and $c' = 1$ (see [105]).

A slight relaxation of this problem is to look for monochromatic copies of graphs other than complete graphs. Given graphs $G_1, \ldots, G_k$, the *k-color graph Ramsey number* $r(G_1, \ldots, G_k)$, is the minimum integer n such that any k-coloring of K_n yields a monochromatic G_i in color i for some i. The case $k = 2$ is the most well-studied, although there are some results for $k \geq 3$ when the graphs are small or relatively simple (see [105] for a thorough survey).

In Chapter 5 we will discuss a degree-sequence variant of graph Ramsey numbers.

2.2 Degree-sequence problems

2.2.1 Forcible problems versus potential problems

The goal of this dissertation is to examine degree-sequence analogues of classical extremal problems. Before we do that, however, we will discuss degree-sequence problems in general, beyond the simple characterization problems discussed in Chapter 1.

Recall that a graphic sequence can have many different realizations. Most degree-sequence problems deal with showing that the family of realizations of a sequence has certain properties. Degree-sequence questions can be categorized as either *forcible problems* or *potential problems.* A forcible problem asks for conditions on a graphic sequence that guarantee that *every* realization of the sequence has a certain property.

Potential problems, on the other hand, seek conditions which guarantee that there is *some* realization of the sequence with the desired property. That is, given a graphic sequence π and a graph property $\mathcal{P}$, we say that π is *forcibly $\mathcal{P}$-graphic* if every realization of π has property $\mathcal{P}$, and *potentially $\mathcal{P}$-graphic* if some realization of π has property $\mathcal{P}$.

The extremal problems that are the focus of this dissertation are potential problems, but we will give a few forcible problems here to illustrate the difference. Often, forcible degree sequence results do not look like degree sequence results at all. For example, consider the following theorem of Dirac:

Theorem 2.7 (Dirac [32]) *If G is a simple graph with n vertices such that $n \geq 3$ and $\delta(G) \geq n/2$, then G is Hamiltonian.*

We can rephrase this in terms of degree sequences as:

Theorem 2.8 (Theorem 2.7 rephrased) *Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence with $n \geq 3$. If $d_n \geq n/2$, then π is forcibly Hamiltonian.*

Another result on forcibly-Hamiltonian sequences is due to Chvátal:

Theorem 2.9 (Chvátal [26]) *Let G be a simple graph with vertex degrees $d_1 \leq \cdots \leq d_n$, where $n \geq 3$. If $i < n/2$ implies that $d_i > i$ or $d_{n-i} \geq n - i$, then G is Hamiltonian.*

While, for the property of Hamiltonicity, we only have sufficient conditions for forcibly $\mathcal{P}$-graphic sequences, characterizations of forcibly $\mathcal{P}$-graphic sequences have been found for many other graph properties. Chernyak, Chernyak, and Tyshkevich characterized forcibly chordal, strongly chordal, interval, and trivially perfect sequences in [22] and characterized the degree sequences of comparability graphs in [118]. S.B. Rao has characterized forcibly line-graphic [109], forcibly total-graphic, forcibly self-complementary, and forcibly planar graphic sequences [110]. These last three results are stated without proof in his survey of potential and forcible degree-sequence results [110], and he also developed a theory for proving characterizations of forcibly $\mathcal{P}$-graphic sequences when $\mathcal{P}$ is a hereditary property [111]. A proof of the characterization of forcibly planar-graphic sequences, by Zverovich, appears in [154]. Zverovich has also characterized forcibly 3-colorable [155] and forcibly 2-matroidal graphic sequences [156]. Choudum proved a characterization of forcibly outerplanar sequences [23], and gave sufficient conditions for a sequence to be forcibly connected [24].

The study of forcibly $\mathcal{P}$-graphic sequences has been changing in recent years. Hammer and Simeone gave a characterization of split graphs in terms of their degree sequences [57]. There is also a characterization of such graphs in terms of forbidden subgraphs (c.f. [5]). Inspired by this, Barrus, Kumbhat, and Hartke [5] investigated degree-sequence forcing sets, that is, a set of graphs $\mathcal{F}$ such that if any realization of a graphic sequence π is $\mathcal{F}$-free, then every realization of that sequence is $\mathcal{F}$-free. Bauer et al. gave necessary and sufficient conditions for a sequence to be forcibly t-tough [3], and sufficient conditions for a sequence to be forcibly k-factorable [2]. The conditions for t-toughness are a direct extension of the conditions in Theorem 2.9, both of which are shown to be best monotone, a property which we will discuss below. These kinds of results, which move beyond simple characterizations, show that the study of forcible degree sequence problems is far from complete.

To understand what is meant by a best monotone theorem, we first need a definition. Given two sequences $\pi = (x_1, \ldots, x_n)$ and $\pi' = (y_1, \ldots, y_n)$, both in either nonincreasing or nondecreasing order, we say that π *majorizes* π' if $x_i \geq y_i$ for each i, $1 \leq i \leq n$. Theorem 2.9 is best possible in the sense that if a sequence fails the condition at some position i, then it is majorized by a sequence that has a non-Hamiltonian realization. In particular, it is majorized by $\pi = (i^i, (n-i-1)^{n-2i}, (n-1)^i)$, which is uniquely realized by the graph $(\overline{K}_i \cup K_{n-2i}) \vee K_i$ (see Figure 2.1). Every graphic

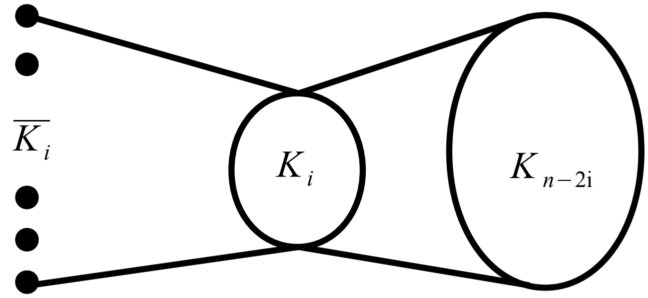


Figure 2.1: The graph $(\overline{K}_i \cup K_{n-2i}) \vee K_i$

sequence that majorizes this one, however, does satisfy the conditions of Theorem 2.9, and hence is forcibly Hamiltonian. This is what is meant by a best monotone theorem; in some sense it is the strongest possible degree-sequence condition that guarantees Hamiltonicity. As mentioned above, best monotone theorems for several other graph properties have been found, and this is an active area of research.

Now we turn our attention to potential degree-sequence problems. If we look simply at graph properties, there are several results on potentially $\mathcal{P}$-graphic sequences. There are characterizations of potentially $\mathcal{P}$-graphic sequences for each of the following properties: k-edge-connected [34], k-connected [120], k-factorable [72] and connected k-factorable (if the sequence is already potentially k-factorable) [108], and self-complementary [30]. In fact, "self-complementary" is the only property for which there is a characterization of both forcibly and potentially $\mathcal{P}$-graphic sequences.

There are also potential versions of classical graph theory problems. We present just a few examples before turning to the one that is the focus of this dissertation. Our first example is a potential version of Hadwiger's conjecture, a long-standing open problem in graph theory.

Conjecture 2.10 (Hadwiger's Conjecture [55]) *If $\chi(G) \geq k$, then G contains a K_k-minor.*

A slight relaxation of Hadwiger's Conjecture is Hajós' Conjecture, which says that a graph with chromatic number at least k must have a K_k-subdivision. Hajós' Conjecture has been shown to be false for almost all graphs (see [95] p. 45-56 for a nice proof). However, Dvořák and Mohar [33] recently proved that the potential version of Hajós' Conjecture is true. That is, they showed that if $\chi(\pi)$ is the maximum value of $\chi(G)$ over all realizations G of π, then for any graphic sequence π, there is a realization of π that contains a subdivided complete graph of order $\chi(\pi)$.

Another example of a potential version of a classical problem is a variant of the Graph Minor Theorem of Robertson and Seymour [112], which states that in every infinite set of graphs, there is a pair of graphs such that one is a minor of the other. This is not true, however, if "minor" is replaced by "induced subgraph." For the potential version, S.B. Rao [111] conjectured that in any infinite set of graphs, there is a pair of graphs, say G and H, such that $\pi(G)$ has a realization containing H as an induced subgraph. This was recently proved by Chudnovsky and Seymour [25].

There are also known results for potential versions of the Erdős-Sós conjecture on trees [137], the Sauer-Spencer graph packing problem [10], and a conjecture of Bollobás and Scott about spanning bipartite graphs [58].

2.2.2 Potentially H-graphic sequences

The property that we we are chiefly concerned with is that of having a subgraph isomorphic to a graph H. If some realization of π has this property, we say that π is *potentially H-graphic.* The study of this property will ultimately lead us to a potential version of the Turán problem, but first we give some useful results about potentially H-graphic sequences.

When a sequence is potentially H-graphic, it is often nice to know something about the realizations that contain H. The next two theorems, which we will use many times, tell us that we can find realizations of potentially H-graphic sequences that have nice properties. The first shows that we can find our subgraph H in a realization where H is on the vertices of highest degree.

Theorem 2.11 (Gould, Jacobson, Lehel [50]) *If $\pi = (d_1, \ldots, d_n)$ is potentially H-graphic, then there exists a realization G of π such that the vertices of H have the $|V(H)|$ largest degrees of π.*

If H is a complete split graph, $K_r \vee \overline{K}_t$, then we can draw a stronger conclusion.

Theorem 2.12 (Yin [127]) *A nonincreasing sequence $\pi = (d_1, \ldots, d_n)$ is potentially $K_r \vee \overline{K}_t$-graphic if and only if there is a realization of π containing a copy of $K_r \vee \overline{K}_t$ such that the vertices of the complete graph of order r have degrees $d_1, \ldots, d_r$ and the vertices of the independent set of order t have degrees $d_{r+1}, \ldots, d_{r+t}$.*

Let H be a graph with degree sequence $\pi(H)$. If $\pi(H) = (s_1, \ldots, s_k)$, then we say $\pi = (d_1, \ldots, d_n)$ is *degree sufficient for H* if $d_i \geq s_i$ for each i, $1 \leq i \leq k$. If both sequences are nonincreasing, then it is clear that being degree sufficient for H is a necessary condition for a sequence to be potentially H-graphic. However, it is not sufficient; for example, the sequence $(2, 2, 2, 2, 2)$ is degree sufficient for K_3, but the unique realization of this sequence is C_5, which does not contain K_3.

Because they will be needed in Chapter 5, we present here some characterizations of potentially H-graphic sequences for small H. These results serve to illustrate the point that degree-sufficiency for H is rarely enough to guarantee that a sequence has a realization containing H; there are usually exceptional families of sequences that prevent the obvious necessary condition from being sufficient.

Theorem 2.13 (Luo [92]) *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geq 3$. Then π is potentially K_3-graphic if and only if $d_3 \geq 2$ except for two cases: $\pi = (2^4)$ and $\pi = (2^5)$.*

Theorem 2.14 (Luo [92]) *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence, where the d_i are in nonincreasing order. Then π is potentially C_4-graphic if and only if the* *following conditions hold:*

1. $d_4 \geq 2$

2. $d_1 = n - 1$ *implies* $d_2 \geq 3$

3. *If* $n = 5, 6,$ *then* $\pi \neq (2^n)$.

Theorem 2.15 (Luo [92]) *A graphic sequence* $\pi = (d_1, d_2, \ldots, d_n)$ *is potentially* C_5*-graphic if and only if* π *satisfies the following conditions:*

1. $d_5 \geq 2$ *and* $\pi \neq (2^n)$ *for* $n = 6, 7$.

2. *For* $i = 1, 2,$ $d_1 = n - i$ *implies* $d_{4-i} \geq 3$.

3. *If* $\pi = (d_1, d_2, 2^k, 1^{n-k-2}),$ *then* $d_1 + d_2 \leq n + k - 2$.

The following characterization of potentially K_k-graphic sequences is due to Rao [107], although his paper was never published. Kézdy and Lehel [69] gave a proof using network flows, and Yin gave a constructive proof in [128]. For a graphic sequence π, we let $\sigma(\pi)$ denote the sum of the terms of π.

Theorem 2.16 (Rao [107]) *Let* $n \geq k$ *and* $\pi = (d_1, \ldots, d_n)$ *be a nonincreasing sequence of nonnegative integers.* π *is potentially* K_k*-graphic if and only if the following conditions hold:*

(i) $d_k \geq k - 1$,

(ii) $\sigma(\pi)$ *is even,*

(iii) For any s *and* t *with* $0 \leq s \leq k$ *and* $0 \leq t \leq n - k - 1$,

$$\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{k+i} \leq (s+t)(s+t-1) + \sum_{i=s+1}^{k} \min\{s+t, d_i - k + 1 + s\} + \sum_{i=k+t+1}^{n} \min\{s+t, d_i\}.$$

This theorem is difficult to use in practice, as it involves evaluating a large set of inequalities. The next theorem gives simple sufficient conditions for a sequence to be potentially K_k-graphic. The simplicity of these conditions make them much more practical to use, as we demonstrate several times throughout this dissertation.

Theorem 2.17 (Yin and Li [135]) *Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence and let k be a positive integer.*

(a) If $d_k \geq k-1$ and $d_i \geq 2(k-1)-i$ for $1 \leq i \leq k-2$, then π is potentially K_k-graphic.

(b) If $d_k \geq k-1$ and $d_{2k} \geq k-2$, then π is potentially K_k-graphic.

As can be seen from the previous theorems, characterizations of potentially H-graphic sequences are often highly technical. However, there are such characterizations for a large number of graphs and graph families. In particular, potentially C_k-graphic sequences were characterized for various k in [13, 92, 125, 126] and [145]. For $k < \ell$, graphic sequences with a realization containing cycles of each length between k and ℓ, inclusive, are known as potentially ${}_kC_l$-graphic. These have been characterized for $k = 3$ and $\ell = 4, 5, 6$, for $k = 4$ and $\ell = 5$, and for $k = 5$ and $\ell = 6$ [21, 132]. Characterizations of K_r-graphic and $K_r - H$-graphic sequences for small r have been studied by many different authors. Among these are characterizations for K_4 [93]; $K_4 - e$ [41, 75]; $K_5 - e$, K_5, and K_6 [148]; $K_5 - H$ where H is one of C_4 [63], $P_4, P_3 \cup K_2, K_3, K_{1,3}, 2K_2$ [62], $Z_1 = K_4 - P_3$ (the paw) [80]; P_5 and Y_4, where Y_4 is the tree with degree sequence $(3, 2, 1, 1, 1)$ [64]; $K_6 - C_6$ and $K_6 - 2C_3 = K_{3,3}$ [61], $K_6 - K_3$ [150], and $K_6 - C_5$ [124]. For larger complete graphs, we have characterizations for $K_{r+1} - e$ [146]; $K_{r+1} - P_3$ and $K_{r+1} - 2K_2$ [121] ($K_6 - 2K_2$ was done in [91]). A characterization of potentially $K_{1,4} + e$-graphic sequences is given in [80], and more generally for potentially $K_{1,t} + e$-graphic sequences in [16]. Finally, we have some characterizations for multipartite graphs, in particular for $K_{2,3}$ and $K_{2,4}$

[131]; $K_{1,1,s}$ for $s = 4, 5$ [151] and $s = 6$ [153]; and finally for $K_{1,1,s}$ when $s \geq 2$ and $n \geq 3s + 1$ [130]. Similar to the case for complete graphs, there is a characterization for a sequence to have a realization containing a complete split graph using a set of inequalities similar to Theorem 2.16 in [127], and simpler sufficient conditions are given in [129].

There is, however, another way to guarantee that a graphic sequence has a realization containing a specific subgraph. For that, we turn at last to the potential number.

2.3 The potential number

The degree sequence analogue of the Turán problem was introduced by Erdős, Jacobson, and Lehel in 1991 [38].

Problem 2.18 *Determine $\sigma(H, n)$, the minimum even integer such that every n-term graphic sequence π with $\sigma(\pi) \geq \sigma(H, n)$ is potentially H-graphic.*

We refer to $\sigma(H, n)$ as the *potential number* or *potential function* of H. When they proposed the problem, Erdős, Jacobson, and Lehel conjectured that $\sigma(K_k, n) = (k-2)(2n-k+1)+2$, based on the graphic sequence $((n-1)^{k-2}, (k-2)^{n-k+2})$ which is uniquely realized by $K_{k-2} \vee \overline{K}_{n-k+2}$. They proved this conjecture for $k = 3$. Gould, Jacobson, and Lehel [50] and Li and Song [86] proved the case $k = 4$ independently, and then Li and Song proved it for $k = 5$ [87] two years later. Finally, Li, Song, and Luo settled the conjecture for $k \geq 6$ and $n \geq \binom{k}{2} + 3$ [88].

Clearly, if a simple characterization of potentially H-graphic sequences is known for a given H (that is, one that gives simple conditions on the terms of the sequence and perhaps excludes exceptional families), then determining the potential number for H is not difficult. Thus, those graphs for which such characterizations of potentially H-graphic sequences are known also have known potential number. Often we do not have any characterization of potentially H-graphic sequences, or such characterizations are very complex, such as Theorem 2.16. Thus, it is sometimes easier to simply determine the potential number than it is to determine a simple characterization.

Aside from those listed above, the potential number is known for the following graphs and graph families. Note that in most cases n is assumed to be sufficiently large.

- pK_2 and C_4 [50]
- The fan graph, $F_{2m+i} = K_1 \vee P_{2m+i-1}$ [19]
- P_k and C^k, where C^k is a cycle with k chords incident to a vertex on the cycle [139]
- Single cycles: C_5, C_6 [73]; C_k [76]
- Sets of cycles: ${}_3C_l$ for $l = 4, 5, 6$ and $n \geq l$ [84]; ${}_3C_l$ for $3 \leq l \leq 8$ and $n \geq l$, and ${}_3C_9$ for $n \geq 12$ [85]; ${}_3C_l$ for all l and n sufficiently large [89]; ${}_kC_l$ for $l \geq 7$ and $3 \leq k \leq l$.
- Complete multipartite graphs: $K_{r,s}$ for $r = s = 3, 4$ [134], for $r \geq s \geq 3$ [141], for $r = 2$ [142]; $K_{1,1,2}$ [74]; $K_{1,1,3}$ [77]; $K_{1,1,t}$ for $n \geq t + 4$ [143, 14]; $K_{r_1,r_2,\ldots,r_l,2,s}$ for $s \geq 3$ and $n \geq 2s^2 + 8s + 3(r_1 + \cdots r_l) + 4$ [149]; $K_{r_1,r_2,\ldots,r_l,r,s}$ for $s \geq r \geq r_l \geq \cdots \geq r_1 \geq 0$, $r \geq 3$ [136]; K_s^t, which is the complete t-partite graph with s vertices in each part (in fact it is shown that $\sigma(K_s^t, n) = \sigma(K_j + K_{s,s}, n)$) [15]
- Friendship graphs, F_k, consisting of k triangles sharing a single vertex [44]
- Generalized friendship graphs, $F_{t,r,k}$, which are formed from k copies of K_t that overlap in a set of r vertices [133]
- Disjoint union of cliques [43]

- Small graphs: $K_5 - C_4$ [78]; $K_5 - P_4$ and $K_5 - P_5$, see [80]; $K_5 - e$ [144, 65]

- $K_{r+1} - e$ [135]

- $K_{r+1} - (kP_3 \cup tK_2)$ for $n \geq 4r + 10$, where r, k, and t satisfy $r + 1 \geq 3k + 2t$, $k + t \geq 2$, $k \geq 1$, and $t \geq 0$ [82]

- $K_{r+1} - pK_2$ for $r \geq 2$, $1 \leq p \leq \lfloor \frac{r+1}{2} \rfloor$ and $n \geq 3r + 3$ [138]

- $K_{r+1} - K_3$ [147]

- $K_{r+1} - P_3$ and $K_{r+1} - F$, where F is K_3-free and contains some tree on 4 vertices [81]

- $K_{r+1} - K_4$, $K_{r+1} - (K_4 - e)$, and $K_{r+1} - Z_1$ (recall $Z_1 = K_4 - P_3$), and $K_{r+1} - Z$, where Z is C_4-free and contains Z_1 [79]

- $K_{r+1} - U$, where U is C_4-free, Z_1-free, and contains $K_3 \cup P_4$ [83]

- Graphs with independence number 2 [47] (note that this result subsumes many of the previous results)

After many years of having only results for single graphs or small graph classes similar to those above, Ferrara, LeSaulnier, Moffatt, and Wenger [46] took a large step forward by determining the potential number asymptotically for all graphs H. Their result uses some ideas from [47], where the potential number was determined for graphs with independence number 2. Our work in Chapters 3 and 4 is based on this result, so we will describe it next in detail.

Let H be a graph on k vertices with at least one nontrivial connected component. For each $i \in \{\alpha(H) + 1, \ldots, k\}$, define

$$\nabla_i(H) = \min \{\Delta(F) : F \leq H, |V(F)| = i\},$$

where $F \leq H$ denotes that F is an induced subgraph of H. Let n be sufficiently large, and consider the sequence

$$\widetilde{\pi}_i(H,n) = ((n-1)^{k-i}, (k-i+\nabla_i(H)-1)^{n-k+i}).$$

This sequence is graphic provided that $n-k+i$ and $\nabla_i(H)-1$ are not both odd. If they are both odd, then reduce the last term of the sequence by 1. For each $i \in \{\alpha(H)+1, \ldots, k\}$, the sequence $\widetilde{\pi}_i(H,n)$ is realized by a graph G that consists of a clique on $k-i$ vertices joined to a (nearly) $(\nabla_i(H)-1)$-regular graph on $n-k+i$ vertices. This graph does not contain H, because any k-vertex subgraph of G must use i vertices from the $(\nabla_i(H)-1)$-regular graph; as such, it has an i-vertex induced subgraph with maximum degree $\nabla_i(H)-1$. However, H has no such subgraph, so H is not contained in G. Thus, $\widetilde{\pi}_i(H,n)$ is not potentially H-graphic, and consequently, $\sigma(H,n) \geq \max_i (\sigma(\widetilde{\pi}_i(H,n)))$.

Let $\widetilde{\sigma}_i(H) = 2(k-i)+\nabla_i(H)-1$. This is the leading coefficient of $\sigma(\widetilde{\pi}_i(H,n))$. Since we are concerned with the asymptotic behavior of $\sigma(H,n)$, we are chiefly concerned with the maximum value of $\widetilde{\sigma}_i(H)$. Note that the maximum value of $\widetilde{\sigma}_i(H)$ can be determined by finding the minimum value of $2i-\nabla_i(H)$. Thus, there may be more than one value of i for which $\widetilde{\sigma}_i(H)$ achieves its maximum value. Hence, we define $i^*(H)$ to be the smallest index $i \in \{\alpha(H)+1, \ldots, k\}$ that maximizes $\widetilde{\sigma}_i(H)$. We will often write just i^* instead of $i^*(H)$, when the context is clear.

The main result of [46] states that $\widetilde{\pi}_{i^*}(H,n)$ determines $\sigma(H,n)$ asymptotically for all H, which can be viewed as an Erdős-Stone-Simonovits-type theorem for the potential problem.

Theorem 2.19 (Ferrara, LeSaulnier, Moffatt and Wenger [46]) *If H is a graph* *and n is a positive integer, then*

$$\sigma(H,n) = \widetilde{\sigma}_{i^*}(H)n + o(n).$$

We define $\mathcal{P}(H)$ to be the set of sequences $\widetilde{\pi}_i(H,n)$ that achieve the maximum value of the leading coefficient. That is, $\mathcal{P}(H) = \{\widetilde{\pi}_i(H,n) : \widetilde{\sigma}_i(H) = \widetilde{\sigma}_{i^*}(H)\}$. Note that n does not play a role in the definition of $\mathcal{P}(H)$, because the coefficient of n in $\sigma(\widetilde{\pi}_i(H,n))$ is what puts the sequence into this set, not the actual sum of the sequence.

Example 1. Consider $H = K_k - Z_1$, where $Z_1 = K_4 - P_3$. Clearly $\alpha(H) = 3$, so we need to compute $\nabla_i(H)$ for $i \in \{4,\ldots,k\}$. The subgraph induced by the four vertices involved in the copy of Z_1 has maximum degree 2, so $\nabla_4(H) = 2$. For $i \geq 5$, every subgraph on i vertices has a dominating vertex, so $\nabla_i(H) = i-1$ for each $i \geq 5$. The minimum value of $2i - \nabla_i(H)$ is 6, and this is achieved by both $i = 4$ and $i = 5$. We have $\widetilde{\sigma}_4(H) = \widetilde{\sigma}_5(H) = 2k-7$, so $\sigma(H,n) \approx (2k-7)n$. In [79], Lai showed that

$$\sigma(H,n) = \begin{cases} (k-2)(2n-k+1) - 3(n-k+1) + 1 \text{ if } n \equiv k \mod 2 \\ (k-2)(2n-k+1) - 3(n-k+1) + 2 \text{ if } n \not\equiv k \mod 2. \end{cases}$$

So we see that the asymptotic result matches the exact result.

It is interesting to note that there are two sequences in $\mathcal{P}(H)$; they are $\widetilde{\pi}_4(H,n) = ((n-1)^{k-4},(k-3)^{n-k+4})$ and $\widetilde{\pi}_5(H,n) = ((n-1)^{k-5},(k-2)^{n-k+5})$ (where the last term of each may be decreased by 1, depending on parity). However, only the sequence $((n-1)^{k-4},(k-3)^{n-k+4})$ is given as an extremal sequence in [79], since $\sigma(\widetilde{\pi}_4(H,n)) = \sigma(\widetilde{\pi}_5(H,n)) - 2$. Thus, the method of determining extremal sequences by computing $\nabla_i(H)$ may actually yield more information than previous methods.

Chapters 3 and 4 explore $\mathcal{P}(H)$ and graphic sequences that are close to being in $\mathcal{P}(H)$ in more detail. In Chapter 3, we will explore the structure of sequences that are not potentially H-graphic, in terms of majorization of degree sequences. In Chapter 4, we prove a stability result for the potential number, and discuss what this result means for our study of the potential function.

3. The Shape of Graphic Sequences that are not Potentially H-Graphic

3.1 Introduction

In this chapter, we study the structure of degree sequences that are not potentially H-graphic. As the realizations of a graphic sequence may have a great deal of structural variety, it is perhaps more appropriate to say that we examine the "shape" of these sequences. One way to think about the shape of a graphic sequence is to draw a picture of it; that is, given a sequence $\pi = (d_1, \ldots, d_n)$, we draw rectangles of height $d_1, \ldots, d_n$ and line them up, similar to a Ferrers diagram, as in Figure 3.1. This gives us a graphical representation of the sequence that allows us to discern, at a glance, the relative differences in the values and multiplicities of terms in the sequence.

Given (not necessarily graphic) sequences $S_1 = (x_1, \ldots, x_n)$ and $S_2 = (y_1, \ldots, y_n)$, we say S_1 *majorizes* S_2 and write $S_1 \succeq S_2$ if $x_i \geq y_i$ for all i, $1 \leq i \leq n$. In terms of pictures, if S_1 majorizes S_2, then the drawing of S_2 fits "beneath" the drawing of S_1. Our study of the "shape" of graphic sequences is based on this idea of majorization, and what must be done to a sequence to ensure it will be majorized by another sequence. Since sequences that are not potentially H-graphic are the degree-sequence analogue of H-free graphs, our inspiration comes from several results on H-free graphs from the extremal literature.

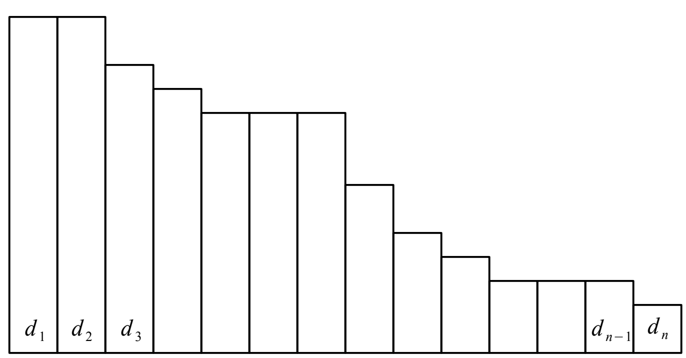


Figure 3.1: The shape of a graphic sequence

In [35], Erdős showed the following.

Theorem 3.1 *If G is a K_{r+1}-free graph of order n, then there exists an n-vertex r-partite graph F such that $\pi(F) \succeq \pi(G)$.*

We give the proof here for completeness.

Proof. We proceed by induction on r. When $r = 1$, the result is trivial, so suppose that $r > 1$ and the result is true for all smaller values of r. Let G be a K_{r+1}-free graph with $V(G) = \{v_1, \ldots, v_n\}$, such that v_1 has degree $\Delta(G)$ and $N(v_1) = \{v_2, \ldots, v_{\Delta(G)+1}\}$. Let H be the subgraph of G induced by $N(v_1)$. Clearly, H is K_r-free, so there is an $(r-1)$-partite graph F' with vertex set $N(v_1)$ such that $d_{F'}(v_i) \geq d_H(v_i)$ for all i with $2 \leq i \leq \Delta(G) + 1$. Create a new graph F on $V(G)$ by joining every vertex in $\{v_1, v_{\Delta(G)+2}, \ldots, v_n\}$ to the vertices of F'. This graph is r-partite, and since $\pi(F') \succeq \pi(H)$ and each vertex in $V(G) \setminus N(v_1)$ has degree at least $\Delta(G)$, $\pi(F) \succeq \pi(G)$. □

Since the sum of the degree sequence of a graph is twice the number of edges in the graph, this result says that for any K_{r+1}-free graph G, there is an r-partite graph with at least as many edges as G. Observing that the Turán graph $T_{n,r}$ is the unique edge-maximal r-partite graph of order n, Turán's Theorem (Theorem 2.3) follows as a corollary of Theorem 3.1.

Having established a result on the degree sequences of K_{r+1}-free graphs, we next discuss a similar result for H-free graphs for general H. To do this, we need to use another type of majorization, introduced by Pikhurko and Taraz in [104]. Given positive integers m and k, define $D_{k,m}(S_1)$ to be the sequence

$$(\underbrace{x_k - m, \ldots, x_k - m}_{k \text{ times}}, x_{k+1} - m, \ldots, x_n - m).$$

We say that S_2 (k,m)-*majorizes* S_1 if $S_2 \succeq D_{k,m}(S_1)$.

Theorem 3.2 (Pikhurko and Taraz [104]) *Let H be a graph with chromatic number $\chi(H) = r + 1 \geq 2$. For any $\epsilon > 0$ and $n \geq n_0(\epsilon, H)$, the degree sequence of an H-free graph G of order n is $(\epsilon n, \epsilon n)$-majorized by the degree sequence of some r-partite graph of order n.*

It was noted in [104] that both the operation of "leveling off" the first k terms of $\pi(G)$ and the operation of reducing all of the terms in $\pi(G)$ by m are necessary. For example, consider $F = K_{t,t}$. The graph $K_{t-1} \vee \overline{K}_{n-t+1}$ is F-free and has degree sequence $((n-1)^{t-1}, (t-1)^{n-t+1})$. However, $\chi(F) = 2$, so we have $r = 1$, and in this case an r-partite graph must be the empty graph with degree sequence (0^n). Clearly, we need both operations to reduce $((n-1)^{t-1}, (t-1)^{n-t+1})$ to a sequence of zeroes.

The operations used to create $D_{\epsilon n, \epsilon n}(\pi(G))$ in Theorem 3.2 reduce $\sigma(\pi(G))$ by at most $\epsilon^2 n^2$, so Theorem 3.2 says that reducing the degree sum of G by $o(n^2)$ results in a graph whose degree sequence is majorized by the degree sequence of an r-partite graph. This implies the Erdős-Stone-Simonovits Theorem (Theorem 2.4), just as Theorem 3.1 implies Turán's Theorem.

That the degree sequences of r-partite graphs appear as the bounding class in Theorems 3.1 and 3.2 is unsurprising given the central role played by the Turán graph in the extremal literature. As shown in Chapter 2, the sequences that form the bounding class for the potential problem are

$$\widetilde{\pi}_i(H, n) = ((n-1)^{k-i}, (k - i + \nabla_i(H) - 1)^{n-k+i}),$$

where H is a graph of order k. It is our goal to examine the structure of degree sequences that are not potentially H-graphic in a manner similar to Theorems 3.1 and 3.2, by comparing them to $\widetilde{\pi}_i(H, n)$.

3.2 Majorization of sequences that are not potentially H-graphic

Throughout the remainder of this chapter, unless otherwise noted we will assume that all sequences have minimum term at least 1. Given two n-term graphic sequences $\pi_1 = (d_1, \ldots, d_n)$ and π_2, and nonnegative integers a_1, a_2, and b with $a_1 \leq a_2$, we say that π_1 is $([a_1, a_2], b)$-*close* to π_2 if there is a (not necessarily graphic) sequence π_1' with $\pi_2 \succeq \pi_1'$ such that π_1' can be obtained from π_1 via the following two steps:

1. Create the sequence

$$S_1 = (d_1, \ldots, d_{a_1-1}, \underbrace{d_{a_2}, \ldots, d_{a_2}}_{a_2-a_1+1 \text{ times}}, d_{a_2+1}, \ldots, d_n).$$

2. Create π_1' from S_1 by subtracting a *total* of at most b from the terms of S_1.

We will refer to step (1) as *leveling off* terms a_1 to a_2 of π_1 and the procedure in step (2) as *editing* the sequence S_1.

In contrast to the idea of (k, m)-majorization, $([a_1, a_2], b)$-closeness leaves the first $a_1 - 1$ terms unchanged, and after the leveling off step allows for variable editing, provided that the total amount of editing in step (2) is at most b.

As an example, consider the sequences $\pi_1 = (19^5, 14^5, 10^5, 5^5)$ and $\pi_2 = (19^2, 8^{18})$. For the leveling off step, we can reduce terms 3 through 11 of π_1 to the value of the 11^{th} term to get the sequence $S_1 = (19^2, 10^{13}, 5^5)$. The sequence S_1 is still not majorized by π_2, so we edit by subtracting 2 from each term that is equal to 10; this is a total editing of 26, and results in $\pi_1' = (19^2, 8^{13}, 5^5)$, which is majorized by π_2. Thus, π_1 is $([3, 11], 26)$-close to π_2.

We show that if a sequence is not potentially H-graphic, then it is close (in the above sense) to being majorized by one of the sequences $\widetilde{\pi_i}(H, n)$. Our first result concerns sequences which fail to be potentially H-graphic simply by failing to be degree sufficient for H.

Theorem 3.3 *Let H be a graph with degree sequence $\pi(H) = (h_1, \ldots, h_k)$, and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence that is not degree sufficient for H. Further, let j be the largest integer for which $d_{k-j+1} < h_{k-j+1}$. If $j \geq \alpha(H) + 1$, then π is majorized by $\widetilde{\pi}_j(H, n)$. If $j < \alpha(H) + 1$, then π is $([k - \alpha(H), k - j + 1], 0)$-close to $\widetilde{\pi}_{\alpha(H)+1}(H, n)$.*

Proof. First note that for each $i \geq \alpha(H) + 1$, $h_{k-i+1} \leq k - i + \nabla_i(H)$. Otherwise, every i-vertex induced subgraph of H has maximum degree greater than $\nabla_i(H)$, contradicting the definition of $\nabla_i(H)$.

Similarly, if $i < \alpha(H) + 1$, then $h_{k-i+1} \leq k - \alpha(H)$. Otherwise, at most $i - 1 < \alpha(H)$ vertices have degree at most $k - \alpha(H)$, contradicting the fact that there are at least $\alpha(H)$ vertices in H with degree at most $k - \alpha(H)$.

Recall that j is the largest integer for which $d_{k-j+1} < h_{k-j+1}$. First suppose that $j \geq \alpha(H) + 1$. In this case, we show that π is majorized by $\widetilde{\pi}_j(H, n)$. Clearly, the first $k - j$ terms of π are majorized by the first $k - j$ terms of $\widetilde{\pi}_j(H, n)$. As $d_{k-j+1} < h_{k-j+1} \leq k - j + \nabla_j(H)$, the remaining terms of π are at most $k - j + \nabla_j(H) - 1$. Thus, π is majorized by $\widetilde{\pi}_j(H, n)$.

Now suppose that $j < \alpha(H)+1$. Here we show that π is $([k-\alpha(H), k-j+1], 0)$-close to $\widetilde{\pi}_{\alpha(H)+1}(H, n)$. We know that $d_{k-j+1} < h_{k-j+1} \leq k - \alpha(H)$. Since $k - \alpha(H) + \nabla_{\alpha(H)+1} - 2 \geq k - \alpha(H) - 1$, reducing terms $d_{k-\alpha(H)}$ through d_{k-j} of π to d_{k-j+1} results in a sequence that is majorized by $\widetilde{\pi}_{\alpha(H)+1}(H, n)$. □

The number of terms leveled off in Theorem 3.3 is best possible in light of the following example. Let $H = K_{k-r} \vee \overline{K}_r$, where r is at least 2, and for $1 \leq j < \alpha(H)+1$ let

$$\pi_j = \left(\left(\frac{n}{k+1}\right)^{k-j}, (k - r - 1)^{n-k+j}\right)$$

where n is sufficiently large. If the sum of π_j is even, then π_j is graphic; if the sum is odd, then reducing the last term by 1 yields a graphic sequence. Clearly π_j is not degree sufficient for H.

Note that the $(k-j+1)^{\text{st}}$ term of π_j is the first place that degree sufficiency for H fails. Since $k-r-1 \leq k-2$, the last $n-k+j$ terms of π_j are termwise dominated by the last $n-k+j$ terms of $\widetilde{\pi}_{\alpha(H)+1}(H,n)$. However, π_j has $k-j$ terms equal to $\frac{n}{k+1}$, of which only the first $k-\alpha(H)-1$ are dominated by $\widetilde{\pi}_{\alpha(H)+1}(H,n)$. Therefore, we need to reduce terms $d_{k-\alpha(H)}, \ldots, d_{k-j}$ of π_j to $d_{k-j+1} = k-r-1$, and each of these reductions is on the order of n. This yields a sequence that is majorized by $\widetilde{\pi}_{\alpha(H)+1}(H,n)$, but reducing any smaller number of terms would not suffice.

The case where π is degree sufficient for H seems to be much more technical, and requires both the leveling off and editing operations outlined above. Recall that $i^* = i^*(H)$ is the smallest i in $\{\alpha(H)+1, \ldots, k\}$ that minimizes $2i - \nabla_i(H)$.

Theorem 3.4 *Let H be a graph of order k with at least one nontrivial component and let π be an n-term graphic sequence that is degree sufficient for H. If π is not potentially H-graphic, then π is $([k-i^*+1, k], (6\alpha + 3)k^2 + \alpha^3 k)$-close to $\widetilde{\pi}_{i^*}(H,n)$.*

To show the sharpness of Theorem 3.4, consider $H = K_k$. Let

$$\pi_k = \left(n-1, (2k-5)^{2k-3}, 1^{n-2k+2}\right).$$

The Erdős-Gallai criteria show that π_k is graphic; clearly π_k is degree sufficient for K_k. Observe that π_k is potentially K_k-graphic if and only if the sequence $\pi'_k = \left((2k-6)^{2k-3}\right)$, obtained by performing the Havel-Hakimi algorithm, is potentially K_{k-1}-graphic. However, the complement of any realization of π'_k is a 2-regular graph, so the maximum size of a clique in any realization of π'_k is at most $k-2$. Hence π_k is not potentially K_k-graphic.

Since $2i - \nabla_i(K_k) = i + 1$ for each $i \in \{2, \ldots, k\}$, we have $i^* = 2$. Note that $\widetilde{\pi}_2(K_k, n) = ((n-1)^{k-2}, (k-2)^{n-k+3})$. As $k - i^* + 1 = k - 1$, leveling off terms $k-1$ and k of π'_k does not change the sequence. However, each entry from $k-1$ through $2k-2$ is larger than $k-2$, so we need to reduce each of these entries by $k-3$, for a total of $k(k-3)$ editing. Thus we perform a total of $O(\alpha k^2)$ editing.

This shows that Theorem 3.4 is in some sense best possible up to the coefficient of αk^2. However, since the amount of editing given in the theorem is $(6\alpha+3)k^2+\alpha^3 k$, the leading term of this expression is not k^2 when α is much larger than $k^{1/2}$. Thus, in this case we do not yet have any information about the sharpness of this result.

3.3 Lemmas

In addition to some of the results on degree sequences presented in Chapters 1 and 2, we will need the following results for the proof of Theorem 3.4. First, we have a lemma that is central to the proof of Theorem 3.4, and is likely also of independent interest. As the proof of this result is quite technical, we postpone it until Section 3.5.

Lemma 3.5 *Let r and k be positive integers with $r < k$, and let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence. Suppose that $d_{k-r} - d_k \geq r(k+2)$. If there are at least $r(k+r+1)$ terms among $d_{k+1}, \ldots, d_n$ with values in $\{k-r, \ldots, k-1\}$, then π has a realization containing the graph $K_{k-r,r}$ with vertices of degree $d_1, \ldots, d_{k-r}$ forming the partite set of order $k-r$.*

We also use the next lemma, which gives a bound on the length of a sequence with fixed maximum term that is not potentially H-graphic.

Lemma 3.6 *Let H be a graph with $\pi(H) = (h_1, \ldots, h_k)$ and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence with $d_1 \leq M$ such that there are terms $d_{i_1}, \ldots, d_{i_k}$ of π satisfying $d_{i_j} \geq h_j$ for $1 \leq j \leq k$. If π has at least $2M^2 + k$ positive terms, then there is a realization G of π with a copy of H that lies on vertices of degree $d_{i_1}, \ldots, d_{i_k}$.*

Proof. We may assume that $d_{i_j} = d_j$ for all j and also that $n \geq 2M^2 + k$ and $d_n \geq 1$. First note that if $M = 1$, then H must be a set of disjoint edges and isolated vertices, and π is potentially H-graphic as long as $n \geq k$. We therefore assume $M \geq 2$.

Let $V(H) = \{u_1, \ldots, u_k\}$, with the vertices in nonincreasing order by degree. In a realization G of π, let $S = \{v_1, \ldots, v_k\}$ be the vertices with the k highest degrees (in order) and let H_S be the graph with vertex set S and $v_i v_j \in E(H_S)$ if and only if $u_i u_j \in E(H)$. If all of the edges of H_S are in G, then H_S is a subgraph of G that is isomorphic to H.

Assume now that G is a realization of π that maximizes $|E(H_S) \cap E(G)|$, but this quantity is less than $|E(H_S)|$. Thus, there exist $v_i, v_j \in V(G)$ such that $v_i v_j \notin E(G)$ but $v_i v_j \in E(H_S)$. Since π is degree sufficient for H, it follows that v_i and v_j must each have a neighbor, say a_i and a_j, respectively, such that $v_i a_i, v_j a_j \notin E(H_S)$ but $v_i a_i, v_j a_j \in E(G)$. Note that possibly $a_i = a_j$.

Since the maximum degree in G is M, there are at most $M^2 + 1$ vertices at distance at most 2 from a_i, and at most $M^2 + 1$ vertices at distance at most 2 from a_j. Since a_i and a_j have distinct neighbors in S, there are at most $k - 2$ vertices in S that are distance at least 3 from both a_i and a_j. Therefore, there is a vertex w in $V(G) \backslash S$ that is distance at least 3 from both a_i and a_j. Let x be a neighbor of w; consequently x is not adjacent to a_i or a_j, and $xw \notin E(H_S)$. Exchanging the edges $v_i a_i, v_j a_j$, and wx for the non-edges $v_i v_j, a_i w$, and $a_j x$ yields a realization G' of π such that $|E(H_S) \cap E(G')| > |E(H_S) \cap E(G)|$, contradicting the maximality of G. □

3.4 Proof of Theorem 3.4

For the proof of Theorem 3.4, we actually prove a more technical result that follows below. First we define some terminology that is used in the proof.

Given a graphic sequence π that is degree sufficient for $K_r \vee \overline{K}_{k-r}$, we create a sequence π^w, called the *want sequence of* π *for* $K_r \vee \overline{K}_{k-r}$. Begin by finding a realization G of π on the vertices $\{v_1, \ldots, v_n\}$ with $d(v_i) = d_i$ that maximizes the sum of (a) the number of edges amongst $v_1, \ldots, v_r$ and (b) the number of edges joining $\{v_1, \ldots, v_r\}$ and $\{v_{r+1}, \ldots, v_k\}$. Let $G_r = G[v_{r+1}, \ldots, v_n]$, and let $\pi_0^w = (w_{r+1}, \ldots, w_n)$ be the degree sequence of G_r, indexed so that $w_i = d_{G_r}(v_i)$.

For each v_i with $i \leq r$, we want v_i to be adjacent to each of the vertices in the set $S_i = \{v_1, \ldots, v_k\} \setminus \{v_i\}$. Since π is degree sufficient for $K_r \vee \overline{K}_{k-r}$, we see that for each nonneighbor of v_i in S_i, there is a neighbor of v_i in $\{v_{k+1}, \ldots, v_n\}$, and each of these neighbors is distinct. Let W_i be a subset of $N_{G_r}(v_i) \cap \{v_{k+1}, \ldots, v_n\}$ that has size $k - 1 - d_{S_i}(v_i)$. Let W be the multiset $\cup_{i=1}^{k} W_i$.

To create the want sequence from π_0^w, we make the following modifications. Each time the vertex v_y appears in W, add 1 to entry w_y of π_0^w. For each j with $r + 1 \leq j \leq k$, subtract $r - d_{\{v_1,\ldots,v_r\}}(v_j)$ from w_j. The sequence that results from these modifications is the want sequence, π^w. Note that the largest value that can be subtracted from any entry is r, and the only entries that might be reduced are those with index at most k. Since π is degree sufficient for $K_r \vee \overline{K}_{k-r}$, no entry of π^w is negative and at most r terms of π^w are 0. The largest value that will be added to any entry of π_0^w is at most r, and only terms with index at least $k+1$ are increased, so the largest entry of π^w is at most the maximum of w_{r+1} and $w_{k+1} + r$.

If π^w is graphic, we can find a realization of π that contains $K_r \vee \overline{K}_{k-r}$. To do this, take the union of the complete split graph $K_r \vee \overline{K}_{k-r}$ on the vertices $\{v_1, \ldots, v_k\}$ (with the clique on the vertex set $\{v_1 \ldots, v_r\}$) and a realization of π^w on the vertices $\{v_{r+1}, \ldots, v_n\}$. Then join each vertex belonging to the clique of the complete split graph (that is, v_i such that $i \leq r$) to the vertices in $N(v_i) \cap \{v_{k+1}, \ldots, v_n\} \backslash W_i$. This graph has degree sequence π, so we have a realization of π that contains the desired complete split graph.

We will prove the following, more specific result than that stated in Theorem 3.4.

Theorem 3.7 *Let H be a fixed graph of order k with at least one non-trivial component. If π is a graphic sequence of length n that is degree sufficient for H but not potentially H-graphic, then π is*

$$([k-i^*+1,k], k^2+ki^*+2+(6k^2+(i^*-\nabla_{i^*}(H))^2k+\nabla_{i^*}(H))(i^*-\nabla_{i^*}(H)-2))\textit{-close}$$

to $\widetilde{\pi}_{i^*}(H,n)$.

Since

$$2i^*-\nabla_{i^*}(H) \leq 2(\alpha(H)+1)-\nabla_{\alpha(H)+1}(H) \leq 2\alpha(H)+1,$$

and $i^* \geq \alpha(H)+1$, we see that $i^*-\nabla_{i^*}(H) \leq \alpha(H)$. Thus,

$$\begin{aligned} k^2+ki^*+2+&(6k^2+(i^*-\nabla_{i^*}(H))^2k+\nabla_{i^*}(H))(i^*-\nabla_{i^*}(H)-2) \\ &\leq k^2+ki^*+2+(6k^2+\alpha^2k+\nabla_{i^*}(H))\alpha \\ &\leq 6\alpha k^2+\alpha^3k+3k^2+2. \end{aligned}$$

Hence Theorem 3.4 (which claims that $(6\alpha+3)k^2+\alpha^3k$ editing suffices) follows directly from Theorem 3.7.

Proof. Let $\pi = (d_1,\ldots,d_n)$ and label the vertices of H with $\{v_1,\ldots,v_k\}$ such that $d(v_i) \geq d(v_j)$ when $i < j$. To simplify notation, we will write α for $\alpha(H)$ and let $\ell^* = i^*-\nabla_{i^*}(H)$. Let $f(H) = 6k^2+(\ell^*)^2k+\nabla_{i^*}(H)$. With this notation, we prove that π is $([k-i^*+1,k], k^2+ki^*+2+f(H)(\ell^*-2))$-close to $\widetilde{\pi}_{i^*}(H,n)$.

If either $d_k \geq 2k-3$ or $d_{2k} \geq k-2$, then by Theorem 2.17, π is potentially K_k-graphic. As this would imply that π is potentially H-graphic, we assume henceforth that $d_k \leq 2k-4$ and, if $n \geq 2k$, that $d_{2k} \leq k-3$.

We will break the proof into several cases. In Cases 1-3, we will show that, after reducing the value of terms $d_{k-i^*+1}, \ldots, d_{k-1}$ to d_k, we only require at most $k^2 + ki^* + 2 + f(H)(\ell^* - 2)$ editing. In Case 4, we show that π is in fact potentially H-graphic, so no editing is required.

Case 1: $n < 2k$.

In this case, after leveling off terms d_{k-i^*+1} through d_{k-1}, we need to reduce at most $k+i^*$ terms of the sequence. Each of those terms is reduced by at most $k+\ell^*-3$, so the total amount of editing is at most $k^2+ki^*+(\ell^*-3)(k+i^*)$, which is less than $k^2 + ki^* + 2 + f(H)(\ell^* - 2)$.

Case 2: $2k \leq n < f(H)$.

Reducing the terms $d_{k-i^*+1}, \ldots, d_{k-1}$ of π to d_k creates a sequence where at most the first $k - i^*$ terms may be greater than $2k - 4$. Now in this sequence, each of the terms from position $k - i^* + 1$ to position $2k - 1$ is at most $2k - 4$, so they must be reduced by at most $k + \ell^* - 3$. Each term from position $2k$ to the end of the sequence is at most $k - 3$, so must be reduced by at most $\ell^* - 2$. Thus the amount of editing required for this sequence is at most

$$(k+i^*-1)(k+\ell^*-3)+(n-2k+1)(\ell^*-2) \leq (\ell^*-2)(f(H)-k+i^*)+(k-1)(k+i^*-1).$$

Case 3: $n \geq f(H)$ and $d_{f(H)} \leq k - \ell^* - 1$.

Here, we will edit only the terms up to $d_{f(H)-1}$, since all subsequent terms are at most $k - \ell^* - 1$. We again level off terms d_{k-i^*+1} through d_{k-1}, making them equal to d_k. Each of the terms from position $k - i^* + 1$ to position $2k - 1$ must be reduced by at most $k + \ell^* - 3$, and each term from position $2k$ to position $f(H) - 1$ must be reduced by at most $\ell^* - 2$. The total amount of editing required is at most

$$(k+i^*-1)(k+\ell^*-3)+(f(H)-2k)(\ell^*-2) \leq (\ell^*-2)(f(H)-k+i^*)+(k-1)(k+i^*-1).$$

Case 4: $n \geq f(H)$ and $d_{f(H)} > k - \ell^* - 1$.

Now we show that π is potentially H-graphic. If $\ell^* = 1$, then $d_{f(H)} \geq k - 1$, and since $f(H) \geq 2k$, this means $d_{2k} \geq k - 1$. Thus by part (b) of Theorem 2.17, π is potentially K_k-graphic. We assume henceforth that $\ell^* \geq 2$.

Let t be such that $d_t \geq k - 1$ but $d_{t+1} < k - 1$. We then have two cases.

Case 4a: $t < k - \ell^*$.

In this case, we wish to show that π has a realization that contains the complete split graph $K_t \vee \overline{K}_{k-t}$. First note that π is degree sufficient for such a graph because $d_t \geq k - 1$ and $d_k \geq k - \ell^* > t$. Let π_1^w be the want sequence of π for $K_t \vee \overline{K}_{k-t}$. Since every entry of π_0^w is at most $k - 2$ (because $d_{t+1} < k - 1$), and t is also at most $k - 1$, the largest entry of π_1^w is less than $2k$.

Zverovich and Zverovich [157] showed that a sequence with maximum term r and minimum term s is graphic as long as the length of the sequence is at least $\frac{(r+s+1)^2}{4s}$. Since π_1^w has length $n - t$ and up to t terms may be 0, π_1^w is graphic if $n - 2t \geq (k + 1)^2$. Since $t < k - \ell^*$, this is true if $n \geq k^2 + 4k + 1 - 2\ell^*$. The observation that $n \geq f(H) > 6k^2$ shows that this is true, and π_1^w is graphic.

Now observe that if π_1^w is graphic, then π is potentially $(K_t \vee \overline{K}_{k-t})$-graphic. If $t \geq k - \alpha(H)$, then this complete split graph contains a copy of H, and we are done. So we assume that $t < k - \alpha(H)$. Let F_i denote an i-vertex induced subgraph of H that achieves $\Delta(F_i) = \nabla_i(H)$. If π has a realization containing $K_t \vee \overline{K}_{k-t}$ with a copy of F_{k-t} on the vertices in the independent set, then π is potentially H-graphic.

By Theorem 2.12, we know that a realization G of π can be found that contains $K_t \vee \overline{K}_{k-t}$ such that the t vertices of degree $k-1$ are on the t highest-degree vertices of G, and the $k-t$ vertices of degree t are on the next $k-t$ highest-degree vertices of G. Delete the t vertices of highest degree in G, and let $\pi' = (d'_1, \ldots, d'_{n-t})$ be the degree sequence of the resulting subgraph of G. It follows that π' satisfies $d'_1 \leq k-2$ and $d'_{f(H)-t} \geq k - \ell^* - t \geq 1$. Since $\Delta(F_{k-t}) = \nabla_{k-t}(H)$ and $d'_{k-t} \geq d'_{f(H)-t} \geq k - \ell^* \geq \nabla_{k-t}(H)$, it follows that π' is degree sufficient for F_{k-t}. Applying Lemma 3.6 with $H = F_{k-t}$ and $M = k-2$, we see that π' is potentially F_{k-t}-graphic as long as at least $2(k-2)^2 + (k-t)$ terms of π' are positive. Since $f(H) - t \geq 6k^2$, this is true. Thus, there is a realization of π' that contains a copy of F_{k-t} on the vertices of highest degree; overlapping this with the vertices $v_{t+1}, \ldots, v_k$ of G, we get a realization of π containing $K_t \vee F_{k-t}$, which implies that π is potentially H-graphic.

Case 4b: $t \geq k - \ell^*$.

First, suppose $d_{k-\ell^*} - d_k \geq \ell^*(k+2)$. This implies that $d_{k-\ell^*} \geq \ell^*(k+2) + d_k$. Since $\ell^* \geq 2$, it follows that $d_{k-\ell^*} \geq 3k$. We claim that this implies that π is potentially H-graphic.

Since $f(H) > \ell^*(k+\ell^*+1)$ and $d_{f(H)} \geq k - \ell^*$, Lemma 3.5 yields a realization G of π containing the complete bipartite graph $K_{k-\ell^*,\ell^*}$, where the vertices $\{v_1, \ldots, v_{k-\ell^*}\}$ form the partite set of order $k - \ell^*$. Let S be the set of vertices in the copy of $K_{k-\ell^*,\ell^*}$, let $S' = \{v_1, \ldots, v_{k-\ell^*}\}$, and let $R = V(G) \setminus S$. If the vertices of S' induce a complete graph, then G contains $K_{k-\ell^*} \vee \overline{K}_{\ell^*}$; consequently G contains a copy of H since $k - \ell^* \geq k - \alpha$. Suppose there are vertices v_i and v_j in S' such that $v_i v_j \notin E(G)$. Since $d_S(v_i) \leq k-2$ and $d(v_i) \geq 3k-2$, we know that v_i has at least $2k$ neighbors in R. Similarly, v_j has at least $2k$ neighbors in R. If each neighbor of v_i in R is adjacent to each neighbor of v_j in R, then each of these vertices in R has degree at least $2k-1$.

Since v_i has at least $2k$ neighbors in R, it follows that G has at least $2k$ vertices with degree at least $2k-1$, contradicting the assumption that $d_k \leq 2k-4$. Thus there are vertices x and y in R such that $v_ix, v_jy \in E(G)$, and $xy \notin E(G)$. Hence we can replace the edges v_ix and v_jy with the non-edges xy and v_iv_j to obtain a new realization of π. Iteratively performing this process for each nonadjacent pair of vertices in S' yields a realization of π in which S' induces a complete graph. Consequently π is potentially H-graphic.

Finally, we must consider the case where $d_{k-\ell^*} < d_k + \ell^*(k+2) < 2k + \ell^*(k+2)$. Observe that π is degree sufficient for $K_{k-\ell^*} \vee \overline{K}_{\ell^*}$ since $t \geq k-\ell^*$ and $f(H) > k$. Let $\pi_2^w = (g_1, \ldots, g_{n'})$ be the want sequence of π for $K_{k-\ell^*} \vee \overline{K}_{\ell^*}$ where $n' = n-(k-\ell^*)$. Since constructing the want sequence increases each term by at most $k-\ell^*$, the facts that $d_{k-\ell^*+1} < 2k+\ell^*(k+2)$, $d_k < 2k-3$, and $d_{2k} < k-1$ imply that π_2^w has the following properties:

- $g_i < 3k + (k+1)\ell^*$ for $1 \leq i \leq \ell^*$,
- $g_i < 3k - \ell^*$ for $\ell^* + 1 \leq i \leq \ell^* + k$, and
- $g_i < 2k - \ell^*$ for $\ell^* + k + 1 \leq i \leq n'$.

Claim 3.1 *The sequence π_2^w is graphic.*

Proof of Claim 3.1. Note that $n' = n - (k - \ell^*) > 6k^2 + (\ell^*)^2 k$.

Tripathi and Vijay [116] showed that the Erdős-Gallai criteria (Theorem 1.3) need only be checked for certain values of p: it suffices to check all $p \leq s$, where s is the largest integer for which $d_s \geq s-1$, or to check only those values of p for which d_p is strictly greater than d_{p+1}. We will use the Erdős-Gallai criteria and this observation to show that π_2^w is graphic.

Since $g_i < 2k - \ell^*$ for large enough i, we only need to check the inequalities for indices up to $2k$. We can write the right side of the Erdős-Gallai inequality (Inequality

(1.1)) as

$$p(p-1) + \sum_{i=p+1}^{r} p + \sum_{i=r+1}^{n'} g_i,$$

where $r \geq p+1$ is the largest index such that $g_r \geq p$ but $g_{r+1} < p$. This then simplifies to

$$\begin{aligned} p(r-1) + \sum_{i=r+1}^{n'} d_i &\geq p(r-1) + n' - r \\ &= r(p-1) - p + n' \\ &\geq (p+1)(p-1) - p + n' \\ &= p^2 - p - 1 + n' \end{aligned}$$

So we need to show that $n' + p^2 - p - 1 \geq \sum_{i=1}^{p} g_i$ for each $p \leq 2k$.

First suppose $p \leq \ell^*$. Then

$$\sum_{i=1}^{p} g_i < p(3k + \ell^*(k+1)) \leq 3\ell^* k + (\ell^*)^2(k+1).$$

Since $n' \geq 6k^2 + (\ell^*)^2 k$, the desired inequality holds.

If $\ell^* + 1 \leq p \leq \ell^* + k$, then

$$\sum_{i=1}^{p} g_i \leq 3\ell^* k + (\ell^*)^2(k+1) + (p - \ell^*)(3k - \ell^*) \leq 2\ell^* k + (\ell^*)^2(k+1) + 3k^2.$$

For p in this range,

$$n' + p^2 - p - 1 \geq 6k^2 + (\ell^*)^2 k + (\ell^* + 1)^2 - 1 = 6k^2 + (\ell^*)^2(k+1) + 2\ell^*,$$

so the inequality holds.

Finally, if $\ell^* + k + 1 \le p \le 2k$, then

$$\sum_{i=1}^{p} g_i \le 2\ell^* k + (\ell^*)^2(k+1) + 3k^2 + (p - \ell^* - k)(2k - \ell^*) \le 5k^2 + (\ell^*)^2 k + 2(\ell^*)^2 - \ell^* k.$$

Now, $n' + p^2 - p - 1 > 6k^2 + (\ell^*)^2 k + 4k^2$, so the Erdős-Gallai inequality is satisfied.

Thus Claim 1 is proved.

Now that we have shown that π_2^w is graphic, we can use a realization of π_2^w to create a realization of π containing a copy of $K_{k-\ell^*} \vee \overline{K}_{\ell^*}$. Since $H \subseteq K_{k-\ell^*} \vee \overline{K}_{\ell^*}$, this implies that π is potentially H-graphic. □

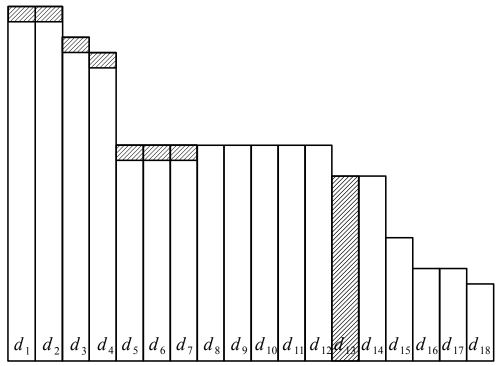
3.5 Proof of Lemma 3.5

Idea of the proof: The proof of Lemma 3.5 is based on a careful analysis of repeated applications of the Kleitman-Wang algorithm (Theorem 1.2). Observe that when laying off a term d_i from a graphic sequence, the d_i terms of highest degree, aside from d_i, are each reduced by 1. If there are many terms of the same value that will be reduced, the order in which these reductions occur does not matter. In particular, provided we reduce the correct number of terms, we may reduce any of the terms equal to d_{d_i} and will get the same residual sequence. This fact is the key to constructing a realization of π that contains $K_{k-r,r}$, as referenced in the statement of Lemma 3.5.

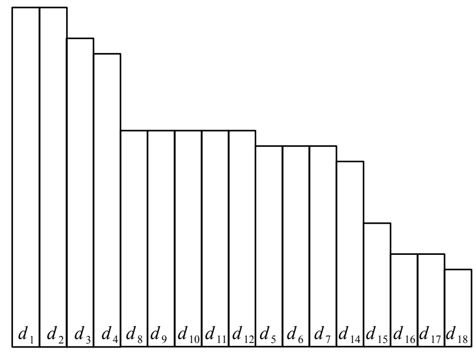
To see this more clearly, consider a sequence $\pi = (d_1, \ldots, d_{18})$, where $d_5 = d_6 = \cdots = d_{12}$, and $d_{13} = 7$. If we lay off d_{13} in the standard way, then we reduce terms d_1 through d_7 by one, and then reorder the sequence to make it nonincreasing. This gives us the residual sequence $\pi' = (d_1, d_2, d_3, d_4, d_8, d_9, d_{10}, d_{11}, d_{12}, d_5, d_6, d_7, d_{14}, \ldots, d_{18})$ (see Figures 3.2a and 3.2b). However, since $d_5 = \cdots = d_{12}$, we could reduce any three of these terms and the resulting sequence would be the same *except in the order of the* *terms*. So, we could reduce terms d_1 through d_4, d_6, d_9, and d_{10}, as in Figure 3.2c, and get the residual sequence $\pi' = (d_1, \ldots, d_4, d_5, d_7, d_8, d_{11}, d_{12}, d_6, d_9, d_{10}, d_{14}, \ldots, d_{18})$, but the values of the terms are the same as in the first case. Alternatively, we could reduce the first four terms of π and the last three terms of the constant subsequence (that is, d_{10}, d_{11}, and d_{12}), and then we would not have to reorder the sequence at all, as in Figure 3.2e. Figures 3.2b, 3.2d, and 3.2e show that while the order of the terms in the sequence changes depending on which terms we reduce, the *values* of those terms are the same, so as far as the sequences are concerned, it does not matter which terms we reduce.

However, if each term d_i is associated with a particular vertex v_i in a realization of π, it does matter which terms are reduced as we perform the Kleitman-Wang algorithm. Thus, we will perform the algorithm but carefully keep track of which terms we reduce and the order of the terms in the residual sequence. Sometimes we will reduce terms in the standard way prescribed by Theorem 1.2, but sometimes we will wish to change the order of the sequence as little as possible, and will reduce terms starting with the end of a constant subsequence as in Figure 3.2e. This will allow us to build a realization of π with the desired properties.

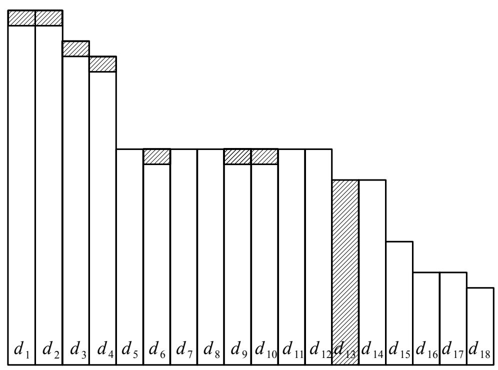
The Kleitman-Wang algorithm provides a means by which the desired realization can be constructed on the vertex set $V = \{v_1, \ldots, v_n\}$ so that the vertices v_j have degree d_j for $j = 1, \ldots, n$. The vertex v_j is associated with the jth term in π. When d_j is laid off, the resulting sequence, π', is a graphic sequence. We can use it to construct a graph on $V \setminus \{v_j\}$ with (the reordered) π' as its degree sequence. The vertex v_j is then added, adjacent to the first d_j members of $\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\}$. In this way, when we lay off a term d_j of π, we will say that the vertices associated with the terms that are reduced are assigned to the neighborhood of v_j. Repeating this process, we will create a realization of π containing $K_{k-r,r}$.



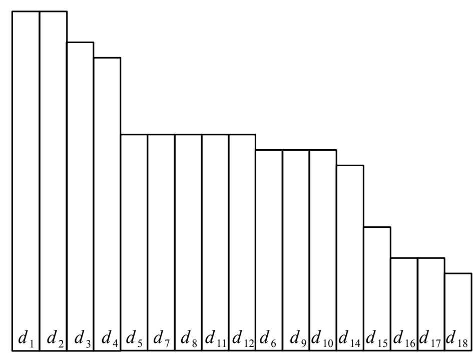
(a) Lay off d_{13} by reducing terms d_1 through d_7



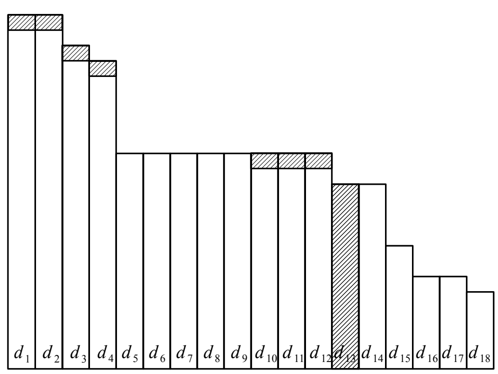
(b) The sequence that results from the reduction in 3.2a



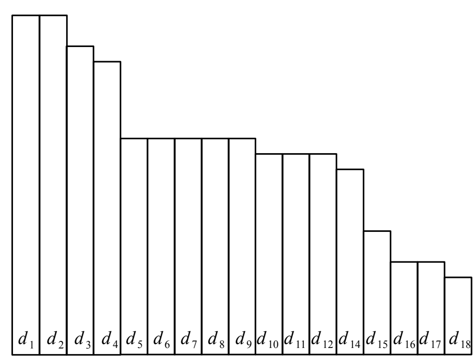
(c) Lay off d_{13} by reducing terms d_1 through d_4, d_6, d_9, and d_{10}



(d) The sequence that results from the reduction in 3.2c



(e) Lay off d_{13} by reducing terms d_1 through d_4, d_{10}, d_{11}, and d_{12}



(f) The sequence that results from the reduction in 3.2e

Figure 3.2: Different ways to reduce terms in the Kleitman-Wang algorithm

The problem with this procedure is that applying it more than once requires that each of the degree sequences must be reordered, which makes keeping track of the vertices that are assigned to a particular neighborhood difficult.

For clarity, we will often abuse terminology and say we lay off vertex v_j to mean we lay off the term of π whose value corresponds to the degree of v_j. This makes sense when we think about laying off a term d_j of π as assigning a set of vertices to the neighborhood of v_j. We will lay off at most $r(k+r+1)$ vertices with the aim of obtaining just r of them whose neighborhood contains $\{v_1, \ldots, v_{k-r}\}$. Our parameters are chosen just for this purpose. The entries in $\{d_{k+1}, \ldots, d_n\}$ that have value in $\{k-r, \ldots, k-1\}$ will be the candidates for entries to lay off. Because $d_{k-r} - d_k \geq r(k+2)$, we can guarantee that, for each of the degree sequences that result from the laying off procedure, the entries that correspond to $v_1, \ldots, v_{k-r}$ will always stay within the first $k-1$ entries.

Terms and definitions: Now we proceed to prove that the procedure outlined above does indeed produce the graph we want. We will lay off entries of π corresponding to vertices $v_{a_1}, v_{a_2}, \ldots, v_{a_p}, \ldots$, where v_{a_p} will be determined at step p. Let $V_0 = V$ and for $p = 1, 2, \ldots$, let $V_p = V_{p-1} - v_{a_p}$. The neighborhood we assign to v_{a_p}, which we will call N_p, is a subset of V_p. We will call the process of laying off $v_{a_1}, \ldots, v_{a_{r(k+2)}}$ the Laying-off Algorithm.

For $p = 0, 1, 2, \ldots$, we define $\hat{d}_p(v_i)$ to be the *remaining degree* of v_i after $v_{a_1}, \ldots, v_{a_p}$ are laid off. That is, for every vertex v_i, $\hat{d}_0(v_i) = d_i$ and for $p = 1, 2, \ldots, r(k+2)$, we have $\hat{d}_p(v_i) := d_i - |\{N_j : 1 \leq j \leq p \text{ and } v_i \in N_j\}|$. Iteratively,

$$\hat{d}_p(v_i) = \begin{cases} \hat{d}_{p-1}(v_i) & \text{if } v_i \notin N_p \\ \hat{d}_{p-1}(v_i) - 1 & \text{if } v_i \in N_p. \end{cases}$$

To determine which vertex v_{a_p} to lay off, for $p = 1, 2, \ldots$, we define $S_{p-1} \subset V_{p-1}$ to be the set of all vertices $w \in V_{p-1}$ for which $\hat{d}_{p-1}(w) \in \{k-r, \ldots, k-1\}$. Then choose

v_{a_p} to be a vertex in S_{p-1} for which $\hat{d}_{p-1}(v_{a_p})$ is minimum. Let $\ell_{p-1} = \hat{d}_{p-1}(v_{a_p})$; this is the number of vertices that will be assigned to N_p. Note that the neighborhood of v_{a_p} may not consist solely of the vertices in N_p. In particular, if $\hat{d}_{p-1}(v_{a_p}) < \hat{d}_0(v_{a_p})$, then v_{a_p} is in $N_{p'}$ for some $p' < p$. Thus, the neighborhood of v_{a_p} in our final graph contains $v_{a_{p'}}$ although $v_{a_{p'}}$ is not in N_p.

The natural ordering on $V = V_0$ is simply $(v_1, \ldots, v_n)$. This corresponds to the nonincreasing order of π. We say that v_i *naturally precedes* v_j if $i < j$, and will write $v_i \propto v_j$. We will define π_p to be the sequence given by each $\hat{d}_p(v_i)$, for all $v_i \in V_p$, that is nonincreasing and, when equality holds, to obey the natural ordering. That is, $\hat{d}_p(v_i)$ precedes $\hat{d}_p(v_j)$ in π_p if either (a) $\hat{d}_p(v_i) > \hat{d}_p(v_j)$, or (b) $\hat{d}_p(v_i) = \hat{d}_p(v_j)$ and $i < j$. This is simply the degree sequence obtained from π by p iterations of the Kleitman-Wang algorithm; thus, π_p is graphic.

Observe that in defining π_p, we have prescribed the order of the terms based on the vertices with which they are associated. This is because we need to keep track of not only the remaining degree of a vertex but also the position of that vertex in π_p. To make this precise, let τ_p be a function from $\{1, \ldots, |V_p|\} \to V_p$ in which $\tau_p(j)$ is the vertex in the jth position in the order defined by π_p, and let T_p be the sequence $\tau_p(1), \ldots, \tau_p(n-p)$. Thus, T_p is simply the sequence of vertices of V_p, ordered according to the position of their remaining degree in π_p. A subsequence $\tau_p(b_1), \ldots, \tau_p(b_m)$ of T_p is *consistent* if $\tau_p(b_i) \propto \tau_p(b_j)$ for all $b_i < b_j$. In essence, this means that all vertices in the subsequence are in order by index, from lowest to highest. We say that T_p itself is consistent if $\tau_p(1), \ldots, \tau_p(n-p)$ is consistent.

In the Kleitman-Wang algorithm, when the term d_i is laid off it is first removed from the sequence; then the first d_i terms of the resulting sequence are each reduced by one. To incorporate this into the Laying-off Algorithm, we define $\hat{\pi}_{p-1}$ to be π_{p-1} with the term associated with v_{a_p} removed. Then, $\hat{\tau}_{p-1}$ and $\hat{T}_{p-1}$ are the corresponding order function and sequence of vertices.

Finding the "neighborhoods" N_p: Now we can describe our modification of the Kleitman-Wang algorithm more precisely. At step p of the Laying-off Algorithm, we choose N_p in the following way:

1. If T_{p-1} is consistent, then simply let N_p be the first ℓ_{p-1} vertices in $\hat{T}_{p-1}$.

2. If T_{p-1} is not consistent but $\hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1})) > \hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1}+1))$, then we again let N_p consist of the first ℓ_{p-1} vertices in T_{p-1}.

3. If T_{p-1} is not consistent but $\hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1})) = \hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1}+1))$, then N_p consists of all vertices $w \in V_{p-1}$ for which $\hat{d}_{p-1}(w) > \hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1}))$, and the vertices x with the highest index for which $\hat{d}_{p-1}(x) = \hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1}))$.

In words, what we do is identify the vertices with largest $\hat{d}_p$ values and reduce their values by 1. If T_{p-1} is not consistent and we cannot reduce all of those with the same value, we reduce those with largest index (i.e., those that come later in the ordering, as in Figure 3.2e). When T_{p-1} is consistent, we still take the first ℓ_{p-1} vertices, even if all of those with the same value are not reduced. We will say that N_p is *good* if $\{v_1, \ldots, v_{k-r}\} \subseteq N_p$. The existence of at least r vertices among $\{v_{k+1}, \ldots, v_n\}$ such that laying off each gives a good N_p will yield the $K_{k-r,r}$ we seek.

An example: Let us do a small example to illustrate the way the Laying-off Algorithm works.

Begin with the graphic sequence $\pi = (9,9,9,9,8,8,7,7,7,7,4,4,4,4) = (d(v_1), \ldots, d(v_{14}))$. For the purposes of this example, we will only identify the neighborhoods of the vertices with degree 4.

Step 1 Since the original ordering of vertices is consistent, we assign the neighborhood of v_{14} to be $N_1 = \{v_1, v_2, v_3, v_4\}$. The new sequence is $\pi_1 = (8^6, 7^4, 4^3)$, and since $\hat{d}_1(v_4) \geq \hat{d}_1(v_5)$, there is no reordering of vertices and T_1 is consistent.

Step 2 Since T_1 is consistent, we can simply assign the set $N_2 = \{v_1, v_2, v_3, v_4\}$ to the neighborhood of v_{13}. Now $\pi_2 = (8, 8, 7^8, 4^2)$. However, the vertices are no longer in their original order; the sequence T_2 is: $v_5, v_6, v_1, v_2, v_3, v_4, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}$.

Step 3 Since T_2 is not consistent, we must consider $\hat{d}_2(\hat{\tau}_2(4))$. Since $\hat{d}_2(\hat{\tau}_2(4)) = \hat{d}_2(\hat{\tau}_2(5))$, we cannot simply assign the four highest-degree vertices to N_3. We begin with $N_3 = \{v_5, v_6\}$, the two highest-degree vertices. Then we need two more vertices, so we take the two vertices of degree $\hat{d}_2(\hat{\tau}_2(4)) = 7$ that have the highest index, that is v_9 and v_{10}. So $N_3 = \{v_5, v_6, v_9, v_{10}\}$. This leaves $\pi_3 = (7^8, 6^2, 4)$, and T_3 is consistent.

Step 4 Since T_3 is consistent, we let $N_4 = \{v_1, v_2, v_3, v_4\}$. Then $\pi_4 = (7^4, 6^6)$.

Observe that at each step π_i is exactly the sequence we would get after i iterations of the Kleitman-Wang algorithm if a term of value 4 is laid off each time.

Proof that the Laying-off Algorithm gives r good neighborhoods: Now we will show that this process does create r vertices among $\{v_{k+1}, \ldots, v_n\}$ that have good neighborhoods. We begin with several claims that develop useful properties of the Laying-off Algorithm, in particular the key observation that $\ell_p \geq \ell_{p-1}$ for all $p \leq rk$. Then, we show that at each iteration of the algorithm, the sequence T_p has a certain structure that allows us to easily count the number of iterations needed to find r good N_ps.

Claim 1. If $p \leq r(k+1)$ and $i < j$, then $\hat{d}_p(v_j) \leq \hat{d}_p(v_i) + 1$.

Proof of Claim 1. If $\hat{d}_p(v_j) \geq \hat{d}_p(v_i) + 2$ then, since $d_0(v_i) \geq d_0(v_j)$, there exists a p' such that $\hat{d}_{p'-1}(v_j) = \hat{d}_{p'-1}(v_i)$, $v_i \in N_{p'}$ and $v_j \notin N_{p'}$, and there also exists a p'' such that $\hat{d}_{p''-1}(v_j) = \hat{d}_{p''-1}(v_i) + 1$, $v_i \in N_{p''}$ and $v_j \notin N_{p''}$. But such a p'' cannot exist because if $\hat{d}_{p''-1}(v_j) > \hat{d}_{p''-1}(v_i)$, then $v_i \in N_{p''}$ implies v_j is also in $N_{p''}$. This contradiction proves Claim 1. □

Claim 2. If $p \leq r(k+1)$ and $j \leq k-r$, then $\hat{d}_{p-1}(v_j) \geq k$. In addition, if $S_{p-1} \cap N_p \neq \emptyset$, then N_p is good.

Proof of Claim 2. If v_j is not laid off, then d_j decreases by at most 1 at each step and so $\hat{d}_{p-1}(v_j) \geq d_j - (p-1)$. Because $d_{k-r} \geq d_k + r(k+2)$, we have the following:

$$\hat{d}_{p-1}(v_j) \geq d_j - (p-1) \geq d_{k-r} - (p-1) \geq d_k + r(k+2) - (p-1) \geq d_k + r.$$

The conditions on the sequence force $d_k \geq k-r$, giving $\hat{d}_{p-1}(v_j) \geq d_k + r \geq k$. As a result, if $S_{p-1} \cap N_p \neq \emptyset$, then N_p must contain every vertex with remaining degree greater than $\hat{d}_p(v_{a_p}) \leq k-1$. This includes all of $\{v_1, \ldots, v_{k-r}\}$ and so N_p must be good. □

Let g_p denote the number of good neighborhoods $N_{p'}$ with $p' \leq p$. We may assume that $g_p \leq r-1$ for all $p \leq r(k+1)$. Otherwise, we would have r good neighborhoods, hence our copy of $K_{k-r,r}$. In particular, by Claim 2 we can assume that there are at most $r-1$ values of p for which $S_{p-1} \cap N_p \neq \emptyset$.

Claim 3. If $p \leq r(k+1)$, then $|S_{p-1} \cap N_p| \leq r-1$ and $|S_{p-1}| > r(k+r+1) - g_{p-1}(r-1) - (p-1) \geq 2r$. In addition, every $v \in N_p$ has $\hat{d}_{p-1}(v)$ at least as large as the least value of $\hat{d}_{p-1}$ among members of S_{p-1}.

Proof of Claim 3. Consider the vertex v_{a_p}. It has degree at most $k-1$ when it is laid off. By Claim 2, there are at least $k-r$ vertices v_j with $\hat{d}_{p-1}(v_j) \geq k$ and so $|S_{p-1} \cap N_p| \leq (k-1) - (k-r) = r-1$. Because a vertex will only leave the set S_p if it has been laid off or assigned to the neighborhoods of enough other vertices that its remaining degree is too low,

$$|S_{p-1}| \geq r(k+r+1) - \left|\bigcup_{j=1}^{p-1}\{S_{j-1} \cap N_j\}\right| - (p-1) \geq r(k+r+1) - g_{p-1}(r-1) - (p-1).$$

Since $g_{p-1} \leq r-1$, we have $|S_{p-1}| \geq r(k+r+1) - (r-1)^2 - (p-1) \geq 2r$. If we include the vertices $\{v_1, \ldots, v_{k-r}\}$, there are a total of at least k vertices w for which $\hat{d}_{p-1}(w)$ is at least the minimum value of $\hat{d}_{p-1}$ among the members of S_{p-1}. This proves Claim 3. □

Claim 4. If $\ell_p < \ell_{p-1}$ for some $p \leq rk$, then at most r more iterations of the Laying-off Algorithm will create the desired copy of $K_{k-r,r}$.

Proof of Claim 4. By definition, $\ell_{p-1} = \hat{d}_{p-1}(v_{a_p})$ and $\ell_p = \hat{d}_p(v_{a_{p+1}})$. Since v_{a_p} was chosen to minimize $\hat{d}_{p-1}$ among S_{p-1}, we know that $\hat{d}_{p-1}(v_{a_{p+1}}) \geq \ell_{p-1} \geq k-r$.

Since $\hat{d}_p(v_{a_{p+1}}) \geq \hat{d}_{p-1}(v_{a_{p+1}}) - 1$, we get $\ell_p = \hat{d}_p(v_{a_{p+1}}) \geq \ell_{p-1} - 1$. Thus, $\ell_p = \ell_{p-1} - 1$. This means that $\hat{d}_{p-1}(v_{a_{p+1}}) = \ell_{p-1}$ and $v_{a_{p+1}} \in N_p$. Since $v_{a_{p+1}} \in S_{p-1}$ as well, Claim 2 gives that N_p is good.

Further, as ℓ_{p-1} is also the minimum remaining degree of any vertex in S_{p-1}, Claim 3 gives that every vertex w in N_p has $\hat{d}_{p-1}(w) \geq \ell_{p-1}$. Since $v_{a_{p+1}} \in N_p$, we conclude that $\hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1})) = \ell_{p-1}$. Since $\hat{d}_{p-1}(v_{k-r}) > \hat{d}_{p-1}(v_{a_{p+1}})$, there are at most $\ell_{p-1} - 1$ vertices w with $\hat{d}_{p-1}(w) \geq \hat{d}_{p-1}(v_{k-r})$.

So, if we can show that $\ell_{p'} \geq \ell_{p-1} - 1$ for all p' such that $p \leq p' \leq p + (r - g_p)$, then each $N_{p'}$ is good and we have the desired $K_{k-r,r}$ in at most r more steps. From Claim 3, there are at most $r-2$ vertices in $S_{p-1} \cap N_p$ that have remaining degree larger than $\hat{d}_{p-1}(v_{a_{p+1}}) = \ell_{p-1}$. Also from Claim 3, $|S_{p-1}| > r(k+r+1) - g_{p-1}(r-1) - (p-1)$.

Thus, there are at least

$$r(k+r+1) - g_{p-1}(r-1) - (p-1) - (r-2) > r(r - g_{p-1})$$

vertices of remaining degree equal to ℓ_{p-1} in S_{p-1}. Since Claim 3 gives that $|S_{p'-1} \cap N_{p'}| \leq r-1$, for all $p' \geq p$, each of the next $r - g_p$ iterations of the Laying-off Algorithm will remove at most r vertices from S_{p-1} which have remaining degree equal to ℓ_{p-1}.

Hence there is always a vertex in S_{p-1} with degree equal to ℓ_{p-1}. Thus, no vertex with degree $\ell_{p-1} - 1$ will be placed into $N_{p'}$, and Claim 4 is proved. □

We can thus assume that $\ell_p \geq \ell_{p-1}$ for all $p \leq rk$.

Now we are prepared to examine the structure of the sequence T_p. Claim 5 below is the main observation, that even when the Laying-off Algorithm results in an inconsistent sequence, the sequence that results is of a very specific form. Thus, the Laying-off Algorithm ensures that the number of iterations between consistent sequences is less than k.

To show this, we say the sequence T_p is of *proper form* if there is a partition of V_p into four ordered sets $\tau_p^{(1)}$, $\tau_p^{(2)}$, $\tau_p^{(3)}$ and $\tau_p^{(4)}$ (where the order is inherited from τ_p) such that $\hat{d}_p$ is constant on each of $\tau_p^{(2)}$ and $\tau_p^{(3)}$ and, when $i < j$ and $v_i, v_j \in V_p$, v_j precedes v_i if and only if $v_j \in \tau_p^{(2)}$ and $v_i \in \tau_p^{(3)}$. By Claim 1, we know that in this case $\hat{d}_p(v_j) \leq \hat{d}_p(v_i) + 1$. Note that this allows for $\tau_p^{(2)}$ and $\tau_p^{(3)}$ to be empty, in which case T_p is consistent.

We will abuse notation to let $\cup$ represent the "concatenation" of ordered sets; that is, $\tau_p^{(i)} \cup \tau_p^{(j)}$ is also an ordered set, where the elements of $\tau_p^{(i)}$ precede those of $\tau_p^{(j)}$, and within each set the original order is maintained. Thus, if T_p is of proper form, both $\tau_p^{(1)} \cup \tau_p^{(2)}$ and $\tau_p^{(3)} \cup \tau_p^{(4)}$ are consistent. For a sequence T_p that is of proper form, the *inconsistency* of T_p is $\left|\tau_p^{(2)} \cup \tau_p^{(3)}\right|$. A consistent sequence has inconsistency zero.

Claim 5. For all $p \in \{0, \dots, rk-1\}$, T_p is of proper form. If T_{p-1} is consistent or has inconsistency at least k, then N_p is good. If T_p is inconsistent, then $|\tau_p^{(1)} \cup \tau_p^{(3)}| \leq \ell_{p-1}$. If T_{p-1} has positive inconsistency, then either

- T_p is consistent (and N_p is good),
- $T_p = \hat{T}_{p-1}$ and N_p is good, or
- T_p has inconsistency strictly less than the inconsistency of T_{p-1}.

Proof of Claim 5. We will prove the claim by induction on p.

If $p = 0$, then $T_p = T_0$ is consistent. Moreover N_{p+1} is good because it is simply the first ℓ_p entries of $\hat{T}_p$, which must contain $v_1, \dots, v_{k-r}$. In fact, this is true for any consistent T_p and this will be our base case for the induction.

We assume the statement of the claim is true for $T_0, \dots, T_{p-1}$.

Case 1: T_{p-1} is consistent.

The set N_p is good because it is simply the first ℓ_{p-1} entries of $\hat{T}_{p-1}$, which must contain $v_1, \dots, v_{k-r}$. If T_p is consistent, then it is, by definition, of proper form.

If T_p is not consistent, then $\hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1})) = \hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1}+1))$. We can partition $\hat{T}_{p-1}$ into $\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(4)L} \cup \hat{\tau}_{p-1}^{(4)R}$, where $\hat{\tau}_{p-1}^{(4)L}$ contains all vertices with remaining degree exactly $\hat{d}_{p-1}(\hat{\tau}_{p-1}(\ell_{p-1}))$ and $\hat{\tau}_{p-1}^{(4)R}$ contains those with lower remaining degree. We can further partition $\hat{\tau}_{p-1}^{(4)L}$ into $\hat{\tau}_{p-1}^{(4)L_1}$ and $\hat{\tau}_{p-1}^{(4)L_2}$, where $\hat{\tau}_{p-1}^{(4)L_1}$ contains all vertices of $\hat{\tau}_{p-1}^{(4)L}$ that are included in N_p, and $\hat{\tau}_{p-1}^{(4)L_2}$ consists of those that are not.

Now T_p can be partitioned into

$$\tau_p^{(1)} = \hat{\tau}_{p-1}^{(1)}$$

$$\tau_p^{(2)} = \hat{\tau}_{p-1}^{(4)L_2}$$

$$\tau_p^{(3)} = \hat{\tau}_{p-1}^{(4)L_1}, \text{ and}$$

$$\tau_p^{(4)} = \hat{\tau}_{p-1}^{(4)R},$$

and it is of proper form. Clearly $|\tau_p^{(1)} \cup \tau_p^{(3)}| = \ell_{p-1}$.

Observe that if T_{p-1} is consistent and T_p is not, then $\{v_1, \ldots, v_{k-r}\}$ is contained in $\tau_p^{(1)} \cup \tau_p^{(2)} \cup \tau_p^{(3)}$.

Case 2: T_{p-1} is not consistent.

Recall that $\ell_{p-1} = |N_p|$, the number of vertices in V_p that are reduced by one when a vertex of T_{p-1} is laid off. The effect of the Laying-off Algorithm on T_p depends on the value of ℓ_{p-1}.

Note that $\ell_{p-1} \leq |\hat{\tau}_{p-1}^{(1)}|$ is not possible because Claim 4 allows us to assume that $\ell_{p-1} \geq \ell_{p-2}$. Since $\ell_{p-2} \geq |\tau_{p-1}^{(1)} \cup \tau_{p-1}^{(3)}| > |\hat{\tau}_{p-1}^{(1)}|$, this is a contradiction.

With this information, we can show that if the inconsistency of T_{p-1} is at least k, then N_p is good. The largest k entries of T_{p-1} are in $\tau_{p-1}^{(1)} \cup \tau_{p-1}^{(2)} \cup \tau_{p-1}^{(3)}$ and N_p contains all of $\hat{\tau}_{p-1}^{(1)}$. Because there are $r(k+r+1)$ vertices eligible to be laid off from $\{v_{k+1}, \ldots, v_n\}$, and we've laid off at most rk, the vertices $v_{k-r}, \ldots, v_k$ will not be laid off. If v_k is in $\hat{\tau}_{p-1}^{(3)}$, then its value is at most $d_k - 1$, and if v_k is in $\hat{\tau}_{p-1}^{(2)}$, then its value is at most d_k. But the degree of each of $v_1, \ldots, v_{k-r}$ is at least $d_{k-r} - kr \geq d_k + r(k+2) - kr > d_k$. So, each of $v_1, \ldots, v_{k-r}$ are in $\hat{\tau}_{p-1}^{(1)}$ and will be in N_p as long as the inconsistency is at least k.

Case 2a: $|\hat{\tau}_{p-1}^{(1)}| < \ell_{p-1} < |\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(2)}|$.

In this case, we can partition $\hat{\tau}_{p-1}^{(2)}$ into two pieces: $\hat{\tau}_{p-1}^{(2)L}$ and $\hat{\tau}_{p-1}^{(2)R}$. The members of $\hat{\tau}_{p-1}^{(2)R}$ are reduced when v_{a_p} is laid off, but those of $\hat{\tau}_{p-1}^{(2)L}$ are not.

After reordering, we obtain the following:

$$\begin{aligned}
\tau_p^{(1)} &= \hat{\tau}_{p-1}^{(1)}, \\
\tau_p^{(2)} &= \hat{\tau}_{p-1}^{(2)L}, \\
\tau_p^{(3)} &= \hat{\tau}_{p-1}^{(3)}, \\
\tau_p^{(4)} &= \hat{\tau}_{p-1}^{(2)R} \cup \hat{\tau}_{p-1}^{(4)}.
\end{aligned}$$

Moreover, $\ell_{p-1} \geq \ell_{p-2} \geq |\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(3)}| = |\hat{\tau}_p^{(1)} \cup \hat{\tau}_p^{(3)}|$. In addition, the inconsistency of T_p is $|\tau_p^{(2)} \cup \tau_p^{(3)}| = |\hat{\tau}_{p-1}^{(2)L} \cup \hat{\tau}_{p-1}^{(3)}|$, which is strictly less than the inconsistency of T_{p-1} because $\hat{\tau}_{p-1}^{(2)L}$ is a strict subset of $\hat{\tau}_{p-1}^{(2)}$.

To proceed through the next cases, we must partition $\hat{\tau}_{p-1}^{(4)}$ into two pieces: $\hat{\tau}_{p-1}^{(4)L}$ and $\hat{\tau}_{p-1}^{(4)R}$. The members of $\hat{\tau}_{p-1}^{(4)L}$ have the same remaining degree as those in $\hat{\tau}_{p-1}^{(3)}$, and those of $\hat{\tau}_{p-1}^{(4)R}$ have smaller remaining degree (either or both of these may be empty).

Case 2b: $|\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(2)}| \leq \ell_{p-1} \leq |\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(2)}| + |\hat{\tau}_{p-1}^{(4)L}|$.

In this case, the values of $\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(2)}$ as well as some of $\hat{\tau}_{p-1}^{(4)L}$ are reduced. Since the members of $\hat{\tau}_{p-1}^{(2)} \cup \hat{\tau}_{p-1}^{(3)}$ (and the unreduced values of $\hat{\tau}_{p-1}^{(4)L}$) now have the same value, reordering results in T_p being a consistent sequence.

Case 2c: $|\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(2)}| + |\hat{\tau}_{p-1}^{(4)L}| < \ell_{p-1} < |\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(2)} \cup \hat{\tau}_{p-1}^{(3)} \cup \hat{\tau}_{p-1}^{(4)L}|$.

In this case, we can partition $\hat{\tau}_{p-1}^{(3)}$ into two pieces: $\hat{\tau}_{p-1}^{(3)L}$ and $\hat{\tau}_{p-1}^{(3)R}$. The members of $\hat{\tau}_{p-1}^{(3)R}$ are reduced but those of $\hat{\tau}_{p-1}^{(3)L}$ are not.

After reordering, we obtain the following:

$$\begin{aligned}
\tau_p^{(1)} &= \hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(3)L}, \\
\tau_p^{(2)} &= \hat{\tau}_{p-1}^{(2)}, \\
\tau_p^{(3)} &= \hat{\tau}_{p-1}^{(3)R}, \\
\tau_p^{(4)} &= \hat{\tau}_{p-1}^{(4)}.
\end{aligned}$$

Moreover, $\ell_{p-1} \geq \ell_{p-2} \geq |\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(3)}| = |\hat{\tau}_p^{(1)} \cup \hat{\tau}_p^{(3)}|$. In addition, the inconsistency of T_p is $|\tau_p^{(2)} \cup \tau_p^{(3)}| = |\hat{\tau}_{p-1}^{(2)} \cup \hat{\tau}_{p-1}^{(3)R}|$, which is strictly less than the inconsistency of T_{p-1} because $\hat{\tau}_{p-1}^{(3)R}$ is a strict subset of $\hat{\tau}_{p-1}^{(3)}$.

Case 2d: $|\hat{\tau}_{p-1}^{(1)} \cup \hat{\tau}_{p-1}^{(2)} \cup \hat{\tau}_{p-1}^{(3)} \cup \hat{\tau}_{p-1}^{(4)L}| \leq \ell_{p-1}$.

In this case, no rearranging is necessary: the order of the vertices in T_p is the same as the order in $\hat{T}_{p-1}$.

Because the only vertices out of order are in $\hat{\tau}_{p-1}^{(2)} \cup \hat{\tau}_{p-1}^{(3)}$, N_p will contain all of the first ℓ_{p-1} vertices. Since $\ell_{p-1} \geq k - r$, the neighborhood N_p must contain $\{v_1, \ldots, v_{k-r}\}$ and thus be good.

This concludes the proof of Claim 5.

Given Claim 5, the proof of Lemma 3.5 follows easily. There can be at most $k-1$ neighborhoods that are not good between consecutive good neighborhoods. So after $(r-1)k+1$ iterations of the procedure, there will be r good neighborhoods, giving us the desired realization. $\square$

4. Stability with Respect to the Potential Number

In Chapter 3, we showed that graphic sequences that are not potentially H-graphic are close to being majorized by a sequence from $\mathcal{P}(H)$. In this chapter, we will show that for some graphs H, if the sequence also has the property that its sum is close to the potential number, then it is actually very close to being one of the sequences in $\mathcal{P}(H)$, not just to being majorized by one. However, this actually depends largely on the structure of H, so it is not true for all graphs.

Our goal is to prove a stability result akin to the result of Simonovits (Theorem 2.5) for the Turán problem, so we examine this theorem again in a slightly different light. We can rephrase Theorem 2.5 in terms of edit distance, which will help to motivate our definition of distance between graphic sequences. Given graphs G and G' on the same labeled vertex set, the *edit distance* between G and G', denoted $\text{dist}(G, G')$, is $|E(G) \triangle E(G')|$. With this terminology, we restate Theorem 2.5:

Theorem 4.1 (Simonovits [113]) *Let H be a graph with $\chi(H) = r + 1$. For every $\epsilon > 0$, there exists a $\delta > 0$ and an n_ϵ such that if $n > n_\epsilon$ and G is an n-vertex H-free graph such that*

$$|E(G)| \geq ex(H, n) - \delta n^2,$$

Then $dist(G, T_{n,r}) < \epsilon n^2$.

In order to define a stability concept for graphic sequences, we first need to define a measure of how far apart two graphic sequences are. The concept of $([a_1, a_2], b)$-closeness defined in Chapter 3 is not fine enough for our purposes, so instead we will use the standard ℓ^1 norm. For graphic sequences $\pi_1 = (x_1, \ldots, x_n)$ and $\pi_2 = (y_1, \ldots, y_m)$ where $m \leq n$, we let $\|\pi_1 - \pi_2\| = \sum_{j=1}^{n} |x_j - y_j|$, where we define $y_{m+1}, \ldots y_n$ to be 0 if $m \neq n$.

This is in keeping with our motivation. Indeed, if G is a graph such that $\text{dist}(G, T_{n,r}) < \epsilon n$, then we would like to have $\|\pi(G) - \pi(T_{n,r})\|$ be small as well. We know that, assuming appropriate divisibility conditions, $\pi(T_{n,r}) = ((\frac{r-1}{r}n)^n)$. Let

V_L be the set of vertices in G that have degree at least $\frac{r-1}{r}n$ and V_S be the set of vertices in G with degree less than $\frac{r-1}{r}n$. Then

$$\begin{aligned}\|\pi(G)-\pi(T_{n,r})\| &= \sum_{v\in V_L}\left(d(v)-\frac{r-1}{r}n\right)+\sum_{v\in V_S}\left(\frac{r-1}{r}n-d(v)\right)\\ &< 4\epsilon n.\end{aligned}$$

This justifies our use of the ℓ^1 norm to define a distance between graphic sequences.

We can now precisely define what it means for a graph to be stable with respect to the potential number.

Definition 4.2 *A graph H is* **stable with respect to the potential number***, or* **σ-stable***, if for any $\epsilon > 0$, there exists an $n_0 = n(\epsilon, H)$ and $\delta > 0$ such that for any graphic sequence π of length $n \geq n_0$ that is not potentially H-graphic and that satisfies*

$$\sigma(\pi) \geq \sigma(H, n) - \delta n,$$

there is some $\pi' \in \mathcal{P}(H)$ such that $\|\pi - \pi'\| < \epsilon n$.

We also define the following weaker version of σ-stability:

Definition 4.3 *A graph H is* **weakly σ-stable** *if for any $\epsilon > 0$, there exists an $n_0 = n(\epsilon, H)$ and $\delta > 0$ such that for any graphic sequence π of length $n \geq n_0$ that is degree sufficient for H but not potentially H-graphic and satisfies*

$$\sigma(\pi) \geq \sigma(H, n) - \delta n,$$

there is some $\pi' \in \mathcal{P}(H)$ such that $\|\pi - \pi'\| < \epsilon n$.

The difference between these conditions is that σ-stability does not require that a sequence be degree sufficient for H. There are graphs that are weakly σ-stable but

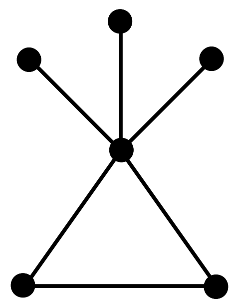


Figure 4.1: A graph with $\nabla_{\alpha+1} > 1$ and $2i^* - \nabla_{i^*} \leq 2\alpha$

not σ-stable; complete graphs are one example, which will be discussed in more detail in Section 4.2.

4.1 Graphs with degree-sequence stability

Recall that for a graph H of order k the sequences in $\mathcal{P}(H)$ are determined by the value $2i - \nabla_i(H)$ for $i \in \{\alpha(H)+1, \ldots, k\}$. Let $i^*(H)$ be the smallest index $i \in \{\alpha+1, \ldots, k\}$ such that $2i - \nabla_i$ is minimized. (When it is understood, we will suppress the argument H in our use of parameters like α, ∇_i, and i^*.) We have

$$2i^* - \nabla_{i^*} \leq 2(\alpha+1) - \nabla_{\alpha+1} \leq 2\alpha + 1.$$

If $\nabla_{\alpha+1}(H) > 1$, then $2(\alpha+1) - \nabla_{\alpha+1} \leq 2\alpha(H)$, so $2i^* - \nabla_{i^*} \leq 2\alpha(H)$. Thus, if $2i^* - \nabla_{i^*} = 2\alpha(H) + 1$, we must have $\nabla_{\alpha+1}(H) = 1$. In this case, we know that there is a set of $\alpha + 1$ vertices in H that induce a graph consisting of a matching and isolated vertices.

It is worth noting that $2i^* - \nabla_{i^*} \leq 2\alpha(H)$ does not necessarily mean that $\nabla_{\alpha+1}(H) > 1$. For example, if H is the graph in Figure 4.1, we have $\alpha(H) = 4$ with $\nabla_5(H) = 1$ and $\nabla_6(H) = 5$. Thus, $2i^* - \nabla_{i^*} = 2(6) - 5 = 7 < 2\alpha(H)$.

Our main theorem states that there is a large class of graphs that are σ-stable.

Theorem 4.4 *If H is a graph such that $2i^* - \nabla_{i^*}(H) \leq 2\alpha(H)$, then H is σ-stable.*

On the other hand, if $2i^* - \nabla_{i^*}(H) = 2\alpha(H)+1$ (and thus $\nabla_{\alpha+1}(H) = 1$), whether H is σ-stable depends more strongly on the structure of H, as can be seen in the next theorem. Let $S_{x,y}$ be the double star with central vertices of degree $x+1$ and $y+1$. That is, it is $K_{1,x} \cup K_{1,y}$ with an edge joining the vertices of degree greater than 1.

Theorem 4.5 *If H is a graph of order k such that*

(a) $2i^* - \nabla_{i^*}(H) = 2\alpha(H) + 1$,

(b) H has a set X of $\alpha(H) + 1$ vertices such that $H[X]$ has one edge, and

(c) $H \subseteq K_{k-\alpha(H)-2} \vee S_{b_1,b_2}$ for some b_1 and b_2 with $b_1 + b_2 = \alpha(H)$,

then H is σ-stable.

These results imply that complete split graphs, complete bipartite graphs, friendship graphs, and odd cycles, among many others, are σ-stable. We will say that H is *Type 1* if $2i^* - \nabla_{i^*}(H) \leq 2\alpha(H)$, and *Type 2* if $2i^* - \nabla_{i^*}(H) = 2\alpha(H) + 1$. Of those graphs just listed, the complete split graphs and complete bipartite graphs are Type 1, and the odd cycles and friendship graphs are Type 2.

4.2 Graphs that are not σ-stable

The hypotheses of Theorem 4.5 suggest that graphs that do not satisfy all of these conditions may not be σ-stable. This is in fact the case, at least if H satisfies condition (a) but not condition (c), as we see in the next theorem.

Theorem 4.6 *If H is a graph such that $2i^* - \nabla_{i^*}(H) = 2\alpha(H) + 1$, and $H \not\subseteq K_{k-\alpha(H)-2} \vee S_{b_1,b_2}$ for any b_1 and b_2 with $b_1 + b_2 = \alpha(H)$, then H is not σ-stable.*

Proof. Consider the sequence

$$\rho_\ell = \left((n-1)^\ell, \left(\frac{n-\ell}{2}\right)^2, (\ell+1)^{n-\ell-2} \right).$$

This is the degree sequence of the graph $K_\ell \vee S_{\frac{n-\ell}{2},\frac{n-\ell}{2}}$, and this graph is in fact the only realization of ρ_ℓ. Note that $\sigma(\rho_\ell) = 2(\ell+1)n - (\ell+2)(\ell+1)$.

If H is Type 2, then $\sigma(H,n) = 2(k-\alpha-1)n + o(n)$. Thus $\sigma(\rho_{k-\alpha-2}) \geq \sigma(H,n) - \delta n$ for any δ, provided n is large enough. The sequence $\rho_{k-\alpha-2}$ is not potentially H-graphic, for if $H \subseteq K_{k-\alpha-2} \vee S_{\frac{n-(k-\alpha-2)}{2},\frac{n-(k-\alpha-2)}{2}}$, then at least $\alpha+2$ vertices must come from the set of vertices that induce a double star; since at most α of these vertices can be independent in H, at most α of the vertices of degree $k-\alpha-1$ may be used. This implies that H is a subgraph of $K_{k-\alpha(H)-2} \vee S_{b_1,b_2}$ for some b_1 and b_2 with $b_1 + b_2 = \alpha$, contradicting our hypothesis.

It remains to show that $\|\rho_{k-\alpha-2} - \pi\| > \epsilon n$ for every $\pi \in \mathcal{P}(H)$ and some choice of ϵ. Since $2i^* - \nabla_{i^*} = 2\alpha+1$, we know that for each $\widetilde{\pi}_j(H,n) \in \mathcal{P}(H)$, we have $2j - \nabla_j(H) = 2\alpha+1$. We also know that $\widetilde{\pi}_{\alpha+1}(H) \in \mathcal{P}(H)$. Recall that $\widetilde{\pi}_j(H,n) = ((n-1)^{k-j}, (k-j+\nabla_j-1)^{n-k+j})$. When $j = \alpha+1$, we have $\widetilde{\pi}_{\alpha+1}(H,n) = ((n-1)^{k-\alpha-1}, (k-\alpha-1)^{n-k+\alpha+1})$, so

$$\|\rho_{k-\alpha-2} - \widetilde{\pi}_{\alpha+1}(H,n)\| = n-k+\alpha.$$

For any $j > \alpha+1$ with $\widetilde{\pi}_j(H,n) \in \mathcal{P}(H)$, we have

$$\begin{aligned}\|\rho_{k-\alpha-2} - \widetilde{\pi}_j(H,n)\| &= (j-\alpha-2)(n-1-(k-j+\nabla_j-1)) \\ &\quad + 2\left(\frac{n-k+\alpha+2}{2} - (k-j+\nabla_j-1)\right) \\ &\quad + (n-k+\alpha)(k-j+\nabla_j-1-(k-\alpha-1)) \\ &= 2n(j-\alpha-1) + (j-\alpha)(\alpha+j-\nabla_j-2k) + 4.\end{aligned}$$

Hence, $\|\rho_{k-\alpha-2} - \widetilde{\pi}_j(H,n)\| > \epsilon n$ for each $\widetilde{\pi}_j(H,n) \in \mathcal{P}(H)$ and any $\epsilon < 1$. □

Which graphs satisfy the conditions of Theorem 4.6? In Tables 4.1 and 4.2, we present all graphs of order at most 6 that satisfy the hypotheses of this theorem, and hence are not σ-stable. The graphs in Table 4.2 have the additional property that they are not *weakly* σ-stable, because the sequence $\rho_{k-\alpha-2}$ is degree sufficient for each of these graphs, demonstrating that the conditions for weak σ-stability cannot be met in these cases.

We can generalize many of the graphs in Tables 4.1 and 4.2 to find larger families of graphs that are not σ-stable.

Claim 4.1 *Let H be a graph of order k that is Type 2 and satisfies $H \not\subseteq K_{k-\alpha(H)-2} \vee S_{b_1,b_2}$ for any b_1 and b_2 with $b_1 + b_2 = \alpha(H)$. If $\alpha(H) \leq \frac{k+1}{2}$, then for all $p \geq 1$, the graph $H \vee K_p$ is not σ-stable.*

Proof. By Theorem 4.6, H itself is not σ-stable. Let $H_p = H \vee K_p$; we will use Theorem 4.6 to show that H_p is not σ-stable either. First note that $\alpha(H_p) = \alpha(H)$. To ensure that H_p is Type 2, we need to show that $2i^*(H_p) - \nabla_{i^*}(H_p) = 2\alpha(H) + 1$. We already know that $2i^*(H) - \nabla_{i^*}(H) = 2\alpha(H) + 1$, so we need to check that $2i - \nabla_i(H_p) \geq 2\alpha(H) + 1$ for each $i \in \{\alpha(H), \ldots, k + p\}$. For $i \leq k$, we have $\nabla_i(H_p) = \nabla_i(H)$, because for these values of i, any induced subgraph of H is also an induced subgraph of H_p. When $i > k$, we have $\nabla_i(H_p) = i - 1$, because there will be $i - k$ vertices of degree $i - 1$ in any set of i vertices. Thus, for $i > k$, we have $2i - \nabla_i(H_p) = i + 1$, which is smallest when $i = k + 1$. Since $\alpha(H) \leq \frac{k+1}{2}$, we know $(k + 1) + 1 \geq 2\alpha(H) + 1$, so H_p is Type 2.

We also need to show that $H_p \not\subseteq K_{k+p-\alpha(H)-2} \vee S_{b_1,b_2}$ for any b_1 and b_2 with $b_1 + b_2 = \alpha(H)$. This is clear, because if H_p were such a subgraph, then H would be a subgraph of $K_{k-\alpha-2} \vee S_{b_1,b_2}$. Thus, Theorem 4.6 implies that H_p is not σ-stable. $\square$

Table 4.1: Connected graphs of order at most 6 that are not σ-stable

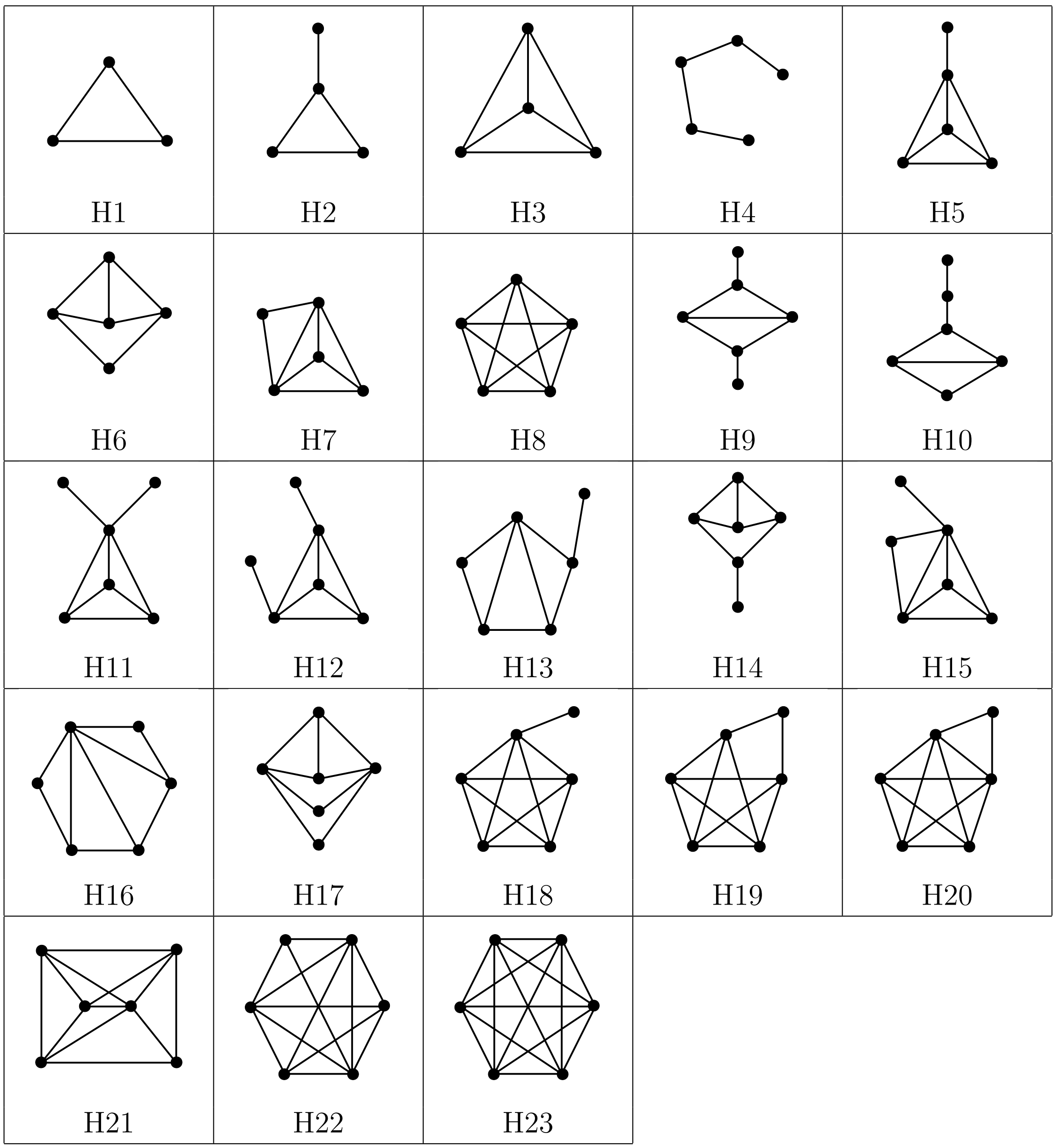
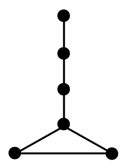
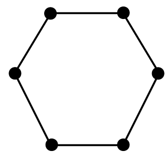
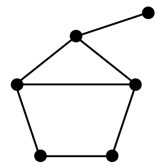
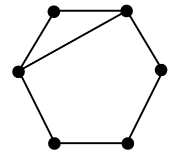
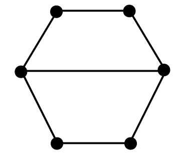
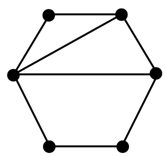
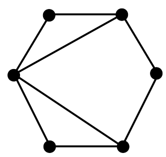
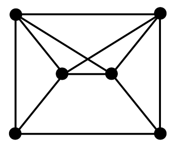


Table 4.2: Connected graphs of order at most 6 that are not weakly σ-stable

 F1	 F2	 F3	 F4	 F5
 F6	 F7	 F8		

In particular, this result shows that the complete graph K_k and the graphs $K_k - P_3 = K_{k-4} \vee Z_1$ and $K_{k-5} \vee P_5$ are not σ-stable. For $H = K_k$ or $H = K_k - P_3$, there is a set X of $\alpha + 1$ vertices such that $H[X]$ contains exactly one edge, showing that we cannot weaken the hypotheses of Theorem 4.5. For $H = K_{k-5} \vee P_5$, there is no such set. However, there is a set of $\alpha + 1$ vertices that induce a matching of size two and isolated vertices. We have not yet shown anything about the σ-stability of such graphs, so only know that they are not σ-stable if they fall under the hypotheses of Theorem 4.6. For further insight into these graphs, it is worth pointing out that graphs H4, H9, H10, H13, H14, and H16 from Table 4.1 and graphs F2 through F7 from Table 4.2 each have two edges in any matching induced by a vertex set of order $\alpha + 1$.

We can make other generalizations from the graphs in Tables 4.1 and 4.2. For example, there are several graphs consisting of a complete graph with one or more pendant vertices. For a graph G, let $\omega(G)$, called the *clique number* of G, be the order of the largest clique in G. Let $K_p(m_1, \ldots, m_p)$ denote the complete graph on vertex set $\{v_1, \ldots, v_p\}$ with m_j leaves incident to vertex v_j for each j, $1 \leq j \leq p$.

This graph has order $k = p + \sum_{i=1}^{p} m_i$ and, if any $m_i = 0$, the independence number is $\alpha = 1 + \sum_{i=1}^{p} m_i$ (if no $m_i = 0$, then the independence number is simply $\sum_{i=1}^{p} m_i$).

Claim 4.2 *The graph $K_p(0, 0, 1, 2, \ldots, p-2)$ is not σ-stable.*

Proof. First note that $k - \alpha - 2 = p - 3$ for any graph of this form. Let $G(\rho_{k-\alpha-2})$ be the unique realization of $\rho_{k-\alpha-2}$, which has clique number $k - \alpha = p - 1$. Since $K_p(m_1, \ldots, m_p)$ contains a clique of order p, we see that $K_p(m_1, \ldots, m_p)$ is not a subgraph of $G(\rho_{k-\alpha-2})$.

For $j \in \{1, \ldots, k-\alpha\}$, every set of $\alpha + j$ vertices in $K_p(0, 0, 1, \ldots p-2)$ induces a subgraph of maximum degree at least $2j - 1$. In fact, the subgraph induced by the set of all of the pendant vertices as well as vertices $v_1, v_2, \ldots v_j$ from the clique has maximum degree exactly $2j - 1$. Thus, $\nabla_{\alpha+j} = 2j - 1$ for each j, and in particular this means that the graph is Type 2. Theorem 4.6 then implies that it is not σ-stable.

□

The graph $K_p(0, 0, 1, \ldots, p-2)$ has an interesting property, namely that $2i - \nabla_i(K_p(0, 0, 1, \ldots, p-2)) = 2\alpha + 1$ for each $i \in \{\alpha + 1, \ldots k\}$, which means that the sequence $\widetilde{\pi}_i(K_p(0, 0, 1, \ldots, p-2), n)$ is in $\mathcal{P}(K_p(0, 0, 1, \ldots, p-2))$ for each i. That is, there are $k - \alpha$ sequences in $\mathcal{P}(K_p(0, 0, 1, \ldots, p-2))$, and none of them is within ϵn of $\rho_{k-\alpha-2}$. Any subgraph of $K_p(0, 0, 1, \ldots, p-2)$ that contains K_p has $2i^* - \nabla_{i^*} = 2\alpha + 1$, so is also not σ-stable. However, these subgraphs may not satisfy $\nabla_{\alpha+j} = 2j - 1$ for each j and do not have the same property.

It is interesting to note that $K_p(1, 1, \ldots, 1)$ is Type 2 but satisfies the conditions of Theorem 4.5, so is σ-stable, and $K_p(2, 2, \ldots, 2)$ is Type 1, which means it is also σ-stable. Thus, relatively small differences in the structure of a graph can change whether the graph is σ-stable.

The previous example points out another class of graphs that cannot be σ-stable. Since $\omega(G(\rho_\ell)) = \ell + 2$, if a graph that is Type 2 has clique number greater than $k - \alpha$, it cannot be a subgraph of $G(\rho_{k-\alpha-2})$. Since the clique number of a graph of order k cannot be larger than $k - \alpha + 1$, this means that any graph H that is Type 2 and has $\omega(H) = k - \alpha(H) + 1$ is not σ-stable.

We have shown that there are graphs that are not weakly σ-stable – are there graphs that *are* weakly σ-stable? In fact, complete graphs are weakly σ-stable even though they are not σ-stable.

Theorem 4.7 *The complete graph K_k is weakly σ-stable for all $k \geq 3$.*

Proof. To see this, consider Theorem 2.17. If a sequence $\pi = (d_1, \ldots, d_n)$ is degree-sufficient for K_k, then $d_k \geq k - 1$. Theorem 2.17 says that if in addition $d_{2k} \geq k - 2$, or if $d_i \geq 2(k-1) - i$ for each i with $1 \leq i \leq k - 2$, then π is potentially K_k-graphic. Thus if π is not potentially K_k-graphic, we must have $d_{2k} < k - 3$, and thus $d_j < k - 3$ for each $j \geq 2k$, as well as having some i with $i \leq k - 2$ such that $d_i < 2(k-1) - i$. Thus, the graphic sequence with the largest sum that is degree sufficient for K_k but not potentially K_k-graphic is $\pi = ((n-1)^{k-3}, (k-1)^{k+2}, (k-3)^{n-2k+1})$, which has sum $(2n - 2k)(k - 3) + (k - 1)(k + 2)$. As discussed in Chapter 2, the potential number for K_k is $\sigma(K_k, n) = (k-2)(2n-k+1)+2$. Thus, $\sigma(K_k, n) - \sigma(\pi) = 2n - 4k + 2$. This shows that among graphic sequences that are degree sufficient for K_k, but not potentially K_k-graphic, there are none that satisfy the condition that $\sigma(\pi) \geq \sigma(K_k, n) - \delta n$ for $\delta < 2$. Thus, taking $\delta = 1$ trivially satisfies the conditions for weak σ-stability. $\square$

Before closing this section, we point out some facts about the σ-stability of cycles. All cycles are Type 2. Odd cycles are σ-stable, because they satisfy the hypotheses of Theorem 4.5. The cycle C_6 is not σ-stable, because it is not a subgraph of $G(\rho_1) = G(\rho_{k-\alpha-2})$, and it is not weakly σ-stable because ρ_1 is degree sufficient for C_6. For a cycle of length $2p$ with $p > 3$, however, we have that $C_{2p} \subseteq K_{p-2} \vee S_{x,y}$ for some x, y with $x + y = p$. This does not imply that C_{2p} is σ-stable, though, because while it is Type 2, it has at least two edges in every graph of maximum degree one induced by $p + 1$ vertices, so does not satisfy the hypotheses of Theorem 4.5. Investigating the σ-stability of these and other Type 2 graphs with this property (containing a set of $\alpha + 1$ vertices that induces a matching with at least two edges) is the next step in our research.

Now we will turn our attention to the proofs of Theorems 4.4 and 4.5.

4.3 Technical lemmas

For a graph G, let $\mathcal{D}^{(t)}(G)$ denote the family of subgraphs of G obtained by deleting exactly t vertices from G. This is the family of induced subgraphs of G with order $|V(G)| - t$. We say that a graphic sequence π is potentially $\mathcal{D}^{(t)}(G)$-graphic if there is a realization of π containing any graph in $\mathcal{D}^{(t)}(G)$.

To prove Theorems 4.4 and 4.5, we will need the following useful consequences of Theorem 1.2, the Kleitman-Wang algorithm.

Corollary 4.8 *Let π_j be the sequence obtained from $\pi = (d_1, \ldots, d_n)$ by laying off the term d_j. Then:*

1. *There is a realization of π in which the vertex of degree d_j is adjacent to the d_j vertices of highest degree other than itself.*

2. *If π_j is potentially H-graphic, then π is potentially H-graphic.*

3. *If G is a graph with degree sequence π and v is a vertex in G, then if $\pi(G - v)$ is potentially H-graphic, then π is potentially H-graphic.*

4. *If $\pi = (n - 1, d_2, \ldots, d_n)$, then π is potentially H-graphic if and only if $\pi_1 = (d_2 - 1, \ldots, d_n - 1)$ is potentially $\mathcal{D}^{(1)}(H)$-graphic.*

Part 1 of Corollary 4.8 guarantees the existence of a realization of π in which the vertex of maximum degree, d_1, is adjacent to the next d_1 vertices of highest degree. Following [48], we call such a realization a *canonical realization* of π.

Lemma 4.9 *If H is a graph of order k and $t < k - \alpha(H)$, then there is an $F \in \mathcal{D}^{(t)}(H)$ such that $\sigma(H, n) - 2tn \geq \sigma(F, n)$.*

Proof. Since $t < k-\alpha(H)$, we know $\alpha(H) < k-t$. Thus, there is a graph $F \in \mathcal{D}^{(t)}(H)$ with $\alpha(F) = \alpha(H)$, for we could simply take a subgraph of H of order $k - t$ that contains a maximum independent set of H. For each $j \in \{\alpha(F) + 1, \ldots, k - t\}$, we have $\nabla_j(F) \leq \nabla_j(H)$, because every j-vertex subgraph F' of F is a j-vertex subgraph of H. This means that for each such j,

$$2j - \nabla_j(F) \geq 2j - \nabla_j(H) \geq 2i^* - \nabla_{i^*}(H),$$

since $2i^*(H) - \nabla_{i^*}(H)$ is the minimum value of $2j - \nabla_j(H)$ over the set $\{\alpha(H) + 1, \ldots, k\}$. In particular, we see that $2i^*(F) - \nabla_{i^*}(F) \geq 2i^*(H) - \nabla_{i^*}(H)$. Recall from Theorem 2.19 that $\sigma(H, n) = (2(k - i^*(H)) + \nabla_{i^*}(H) - 1)n + o(n)$. We compute

$$\begin{aligned}
\sigma(F, n) &= (2(k - t) - 2i^*(F) + \nabla_{i^*}(F) - 1)n + o(n) \\
&= (2k - (2i^*(F) - \nabla_{i^*}(F)) - 1)n - 2tn + o(n) \\
&\leq (2k - (2i^*(H) - \nabla_{i^*}(H)) - 1)n - 2tn + o(n) \\
&= \sigma(H, n) - 2tn.
\end{aligned}$$

□

Our proof of Theorem 4.4 follows very closely the proof of Theorem 2.19 in [46]. As such, we will need the following result from that paper.

Theorem 4.10 (The Bounded Max Degree Theorem [46]) *Let H be a graph of order k and $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence with n sufficiently large satisfying the following:*

1. *π is degree sufficient for H, and*

2. *$d_n \geq k - \alpha(H)$.*

There exists a function $f = f(\alpha(H), k)$ such that if $d_1 < n - f(\alpha(H), k)$, then π is potentially H-graphic.

For our purposes, it is useful to know that the function f is given by

$$\begin{aligned} f(\alpha(H), k) &= \binom{k}{k - \alpha(H)} \left[2\binom{k - \alpha(H)}{2} + \alpha(H) - 1 \right] + 4k^2 + k + 1 \\ &= O\left(\binom{k}{\lceil k/2 \rceil} (k^2 - 1) \right). \end{aligned}$$

4.4 Proofs of Theorems 4.4 and 4.5

Theorem 4.4 is an immediate corollary of the following lemma. Theorem 4.5 will follow with a little more work; in particular when $2i^* - \nabla_{i^*}(H) = 2\alpha(H) + 1$, then Lemma 4.11 may result in a realization of π containing $K_{k-\alpha-1} \vee \overline{K}_{\alpha(H)+1}$. In this case we must do further analysis to show that H is σ-stable.

Lemma 4.11 *Let H be a graph of order k and let $\epsilon > 0$ be given. There exists an $n_0 = n(\epsilon, H)$ and $\delta < \frac{1}{k}$ such that any graphic sequence π of length $n \geq n_0$ with $\sigma(\pi) \geq \sigma(H, n) - \delta n$, either*

1. *π is potentially H-graphic;*

2. *$\|\pi - \pi'\| < \epsilon n$ for some $\pi' \in \mathcal{P}(H)$; or*

3. *π is potentially $(K_{k-\alpha(H)-1} \vee \overline{K}_{\alpha(H)+1})$-graphic.*

Our goal is to create a realization of π containing a supergraph of H, or show that $\|\pi - \pi'\| < \epsilon n$ for some $\pi' \in \mathcal{P}(H)$. This will be done in three stages. Stage 1 describes an algorithm that iteratively builds a realization of π with a desired structure, while keeping track of changes made so that we can discover more about the terms of the sequence as we proceed. Stage 2 uses this realization and the information gained from the algorithm to show that in many cases, this realization actually does contain H (if H is Type 1), or it contains $(K_{k-\alpha(H)-1} \vee \overline{K}_{\alpha(H)+1})$ (if H is Type 2). Finally, Stage 3 shows that in those cases where we do not have one of these desirable realizations, we can show that π is close (in our metric) to one of the sequences in $\mathcal{P}(H)$.

Proof of Lemma 4.11. Let π be a graphic sequence of length n, where we assume n is sufficiently large. Suppose also that for some $\delta < \frac{1}{k}$, $\sigma(\pi) \geq (2(k - i^*) + \nabla_{i^*} - 1 - \delta)n$.

Stage 1: In this stage, we describe the algorithm that reduces π to a residual sequence π_ℓ, assuming at each step that the sequence obtained does not satisfy the conditions of the Bounded Max Degree Theorem (Theorem 4.10) for a certain complete split graph.

Define

$$b_H = \begin{cases} 0 & \text{if } H \text{ is Type 1} \\ 1 & \text{if } H \text{ is Type 2.} \end{cases}$$

We initialize the algorithm by applying the Kleitman-Wang algorithm (Theorem 1.2) to π to obtain a sequence π_0 with minimum degree at least $k - i^* + \frac{\nabla_{i^*}-1-\delta}{2}$. Since $\delta < 1$, we do this by iteratively laying off terms of value at most $k - i^* + \frac{\nabla_{i^*}-2}{2}$, beginning with the smallest such term. After laying off the first term, we get a sequence π_1 with $n-1$ terms such that

$$\begin{aligned}\sigma(\pi_1) \geq &(2(k-i^*)+\nabla_{i^*}-1-\delta)n-(2(k-i^*)+\nabla_{i^*}-1)+1 \\ = &(2(k-i^*)+\nabla_{i^*}-1-\delta)(n-1)+(1-\delta).\end{aligned}$$

As we continue to lay off terms of low value, we obtain a sequence π_j of length $n-j$ with

$$\sigma(\pi_j) \geq (2(k-i^*)+\nabla_{i^*}-1-\delta)(n-j)+j(1-\delta).$$

If $j(1-\delta) > 2\delta(n-j)$, then

$$\sigma(\pi_t') \geq (2(k-i^*)+\nabla_{i^*}-1)(n-j)+\delta(n-j) \geq \sigma(H, n-j),$$

implying that π_j is potentially H-graphic. By Part 2 of Corollary 4.8, this means that π is also potentially H-graphic. Thus, laying off at most $\frac{2\delta}{1+\delta}n$ terms must result in a sequence with the desired minimum degree. Call this sequence $\pi^{(0)}$. Note the following properties of $\pi^{(0)}$: it is not potentially H-graphic, it has length $n_0 \geq n(1-\frac{2\delta}{1+\delta})$, and its smallest term is at least $k-i^*+\frac{\nabla_{i^*}-1-\delta}{2} \geq k-\alpha-b_H$.

After this initialization, we perform the following steps to create sequences $\pi^{(1)}, \ldots, \pi^{(\ell)}$ for some $\ell < k-\alpha(H)-b_H$. The purpose of each step is to reduce the maximum term and increase the minimum term of each successive sequence. For each $t \geq 0$, let $\pi^{(t)} = (d_1^{(t)}, \ldots, d_{n_t}^{(t)})$ be the sequence that results from the t'th iteration of the algorithm, and let R_t be a canonical realization of $\pi^{(t)}$ on the vertex set $\{v_1^{(t)}, \ldots, v_{n_t}^{(t)}\}$ such that $d(v_j^{(t)}) = d_j^{(t)}$.

Starting with $t=0$, the algorithm proceeds as follows:

(i) Remove the non-neighbors of the vertex $v_1^{(t)}$ from R_t to obtain a graph $\widehat{R}_t$. Let $\widehat{\pi}^{(t)} = \pi(\widehat{R}_t)$. Note that in $\widehat{R}_t$, the vertex $v_1^{(t)}$ is a dominating vertex; since R_t was a canonical realization of $\pi^{(t)}$, this means that if vertex $v_j^{(t)}$ was removed from R_t, then vertex $v_p^{(t)}$ is also removed if $p > j$.

(ii) Lay off the largest term of $\widehat{\pi}^{(t)}$, and call the resulting sequence $\breve{\widehat{\pi}}^{(t)}$. This is the degree sequence of the neighborhood of $v_1^{(t)}$ in $\widehat{R}_t$.

(iii) As in the initialization step, apply Theorem 1.2 to $\breve{\widehat{\pi}}^{(t)}$, laying off terms of smallest value until we have a sequence with minimum term at least

$$k - \frac{2i^* - \nabla_{i^*}}{2} - \frac{1 + (2+t)\delta}{2} - (t+1).$$

Let $\pi^{(t+1)}$ be the sequence that results from this step.

(iv) Terminate if $d_1^{(t+1)} < n_{t+1} - \binom{k}{\lceil k/2 \rceil}(k^2 - 1)$ or $t + 1 = k - \alpha(H) - b_H$. Otherwise, return to Step (i).

When the algorithm terminates, let $\pi^{(\ell)} = \pi^{(t+1)}$. The remainder of the proof deals with determining properties of $\pi^{(\ell)}$ and its realizations.

Let us first examine the effects of the algorithm. Step (i) yields a graph with a dominating vertex, whose degree sequence has largest term $\widehat{n}_t - 1$, where $\widehat{n}_t$ is the length of $\widehat{\pi}^{(t)}$. Since the algorithm stops when $d_1^{(t+1)}$ is too small, the number of vertices removed is at most $\binom{k}{\lceil k/2 \rceil}(k^2 - 1)$, and Theorem 2.17 (sufficient conditions for a graphic sequence to be potentially K_k-graphic) gives us a bound on the size of those terms. Laying off the largest term of $\widehat{\pi}^{(t)}$ in Step (ii) removes a dominating vertex from the graph. Finally, in Step (iii), we lay off terms to obtain a graphic sequence with minimum degree at least $k - (t + 1) - \alpha - b_H$.

Before moving on to Stage 2, we state and prove several claims which, along with the discussion in the previous paragraph, will help us determine if we can achieve the desired realization of π or show that it is close to an extremal sequence.

Claim 4.3 *If $\pi^{(\ell)}$ is potentially $\mathcal{D}^{(\ell)}(H)$-graphic, then π is potentially H-graphic. Additionally, there is a realization G of π such that G contains $K_\ell \vee G^{(\ell)}$, where $G^{(\ell)}$*

is a realization of $\pi^{(\ell)}$.

Proof of Claim 4.3. First we will show that for each t with $0 \leq t \leq \ell$, if $\pi^{(t)}$ is potentially $\mathcal{D}^{(t)}(H)$-graphic, then π is potentially H-graphic. The proof is by induction on t. If $\pi^{(0)}$ is potentially $\mathcal{D}^{(0)}(H)$-graphic, then by repeated applications of Part 2 of Corollary 4.8, π is potentially H-graphic. Assume that the statement is true for some $t < \ell$, and suppose $\pi^{(t+1)}$ is potentially $\mathcal{D}^{(t+1)}(H)$-graphic. Again, repeated applications of Part 2 of Corollary 4.8 show that $\breve{\pi}^{(t)}$ is potentially $\mathcal{D}^{(t+1)}(H)$-graphic. Then, as $\breve{\pi}^{(t)}$ is created from $\widehat{\pi}^{(t)}$ by the removal of a dominating vertex, we see that $\widehat{\pi}^{(t)}$ is potentially $\mathcal{D}^{(t)}(H)$-graphic. This is an application of Part 4 of Corollary 4.8. Finally, Part 3 of Corollary 4.8 implies that $\pi^{(t)}$ is potentially $\mathcal{D}^{(t)}(H)$-graphic, and the induction hypothesis shows that π is potentially H-graphic.

Now we will show that for each $t \leq \ell$, π has a realization containing $K_t \vee G^{(t)}$, where $G^{(t)}$ is a realization of $\pi^{(t)}$. We proceed by induction on t. Let $G^{(0)}$ be a realization of $\pi^{(0)}$. By repeated applications of Part 2 of Corollary 4.8, π is potentially $G^{(0)}$-graphic, establishing the base case of our induction. Suppose the statement is true for some t with $0 \leq t < \ell$, and let $G^{(t+1)}$ be a realization of $\pi^{(t+1)}$. Again, we use Part 2 of Corollary 4.8 several times to show that $\breve{\pi}^{(t)}$ is potentially $G^{(t+1)}$-graphic. Part 4 of Corollary 4.8 then shows that $\widehat{\pi}^{(t)}$ is potentially $G^{(t+1)}$-graphic, and we know that moreover, since we have added a dominating vertex to the graph, there is a realization of $\widehat{\pi}^{(t)}$ that contains $K_1 \vee G^{(t+1)}$. To get from $\pi^{(t)}$ to $\widehat{\pi}^{(t)}$, we removed the nonneighbors of a vertex $v_1^{(t)}$ from a canonical realization R_t of $\pi^{(t)}$. Doing this does not change the graph induced by $N_{R_t}(v_1^{(t)})$. Thus we may assume that $G^{(t)}$ is a realization of $\pi^{(t)}$ in which $G^{(t+1)} \subseteq N_{G^{(t)}}(v_1^{(t)})$; that is, we can assume that $v_1^{(t)}$ is the vertex acting as K_1 in the copy of $K_1 \vee G^{(t+1)}$. Now the induction hypothesis gives us a realization of π that contains $K_t \vee G^{(t)}$, and we see that we can view $G^{(t)}$ as $K_1 \vee G^{(t+1)}$ with additional vertices that are not adjacent to $v_1^{(t)}$. This gives us a realization of π containing $K_{t+1} \vee G^{(t+1)}$, as desired. Thus, there is a realization of π

that contains $K_\ell \vee G^{(\ell)}$. □

The construction of the realization of π containing $K_\ell \vee G^{(\ell)}$ shows that after the t'th step of the algorithm, there is a realization of π in which t vertices are adjacent to all but a fraction of the vertices in the graph. The exact value of this fraction will be determined after the next claim, as we need some of the facts developed in that claim in order to calculate it.

Claim 4.4 $\sigma(\pi^{(t)}) \geq (2(k-i^*) + \nabla_{i^*}(H) - 1 - (1+t)\delta - 2t)n_t$, *and at most* $\frac{(t+3)\delta}{1+\delta}n_t$ *iterations of the Kleitman-Wang algorithm are needed at each implementation of Step (iii).*

Proof of Claim 4.4. First we will prove the lower bound on $\sigma(\pi^{(t)})$. Again the proof is by induction on t. The claim holds for $\pi^{(0)}$ by hypothesis. Suppose $t \geq 0$ and $\sigma(\pi^{(t)}) \geq (2(k-i^*) + \nabla_{i^*}(H) - 1 - (1+t)\delta - 2t)n_t$. We will show that the inequality holds for $\pi^{(t+1)}$. Let $M = 2(k-2)\binom{k}{\lceil k/2 \rceil}(k^2-1)$. Since $d_1^{(t)} > n_t - \binom{k}{\lceil k/2 \rceil}(k^2-1)$, creating $\widehat{\pi}^{(t)}$ from $\pi^{(t)}$ entails removing at most $\binom{k}{\lceil k/2 \rceil}(k^2-1)$ vertices, each of which has degree at most $k-2$ (by Theorem 2.17). Thus, $\sigma(\widehat{\pi}^{(t)}) \geq \sigma(\pi^{(t)}) - M$. We have assumed that n is sufficiently large, so in particular, since n_t increases with n, we may assume $M \leq \delta n_t$, which means that

$$\begin{aligned}\sigma(\widehat{\pi}^{(t)}) \geq& (2(k-i^*) + \nabla_{i^*}(H) - 1 - (t+1)\delta - 2t)n_t - \delta n_t \\ =& (2(k-i^*) + \nabla_{i^*}(H) - 1 - (1+(t+1))\delta - 2t)n_t \\ \geq& (2(k-i^*) + \nabla_{i^*}(H) - 1 - (1+(t+1))\delta - 2t)\widehat{n}_t,\end{aligned}$$

where $\widehat{n}_t$ is the length of $\widehat{\pi}^{(t)}$. Creating $\breve{\pi}^{(t)}$ from $\widehat{\pi}^{(t)}$ requires laying off a term of value $\widehat{n}_t - 1$, so $\sigma(\breve{\pi}^{(t)}) = \sigma(\widehat{\pi}^{(t)}) - 2(\widehat{n}_t - 1)$. We now have

$$\begin{aligned}\sigma(\breve{\pi}^{(t)}) \geq & (2(k - i^*) + \nabla_{i^*}(H) - 1 - (1 + (t+1))\delta - 2t)\widehat{n}_t - 2(\widehat{n}_t - 1) \\ \geq & (2(k - i^*) + \nabla_{i^*}(H) - 1 - (1 + (t+1))\delta - 2(t+1))\widehat{n}_t \\ \geq & (2(k - i^*) + \nabla_{i^*}(H) - 1 - (1 + (t+1))\delta - 2(t+1))\breve{n}_t,\end{aligned}$$

where $\breve{n}_t$ is the length of $\breve{\pi}^{(t)}$. Finally, the last step of our algorithm involves laying off terms from $\breve{\pi}^{(t)}$ that have value less than $k - \frac{2i^* - \nabla_{i^*}}{2} - \frac{1+(t+2)\delta}{2} - (t+1)$. The first iteration of this process yields a new sequence $\breve{\pi}_1^{(t)}$ with

$$\begin{aligned}\sigma(\breve{\pi}_1^{(t)}) &\geq \sigma(\breve{\pi}^{(t)}) - [2(k - i^*) + \nabla_{i^*}(H) - 1 - (1 + (t+1))\delta - 2(t+1)] \\ &\geq [2(k - i^*) + \nabla_{i^*}(H) - 1 - (1 + (t+1))\delta - 2(t+1)](\breve{n}_t - 1).\end{aligned}$$

Repeating this j times yields the sequence $\breve{\pi}_j^{(t)}$ with

$$\begin{aligned}\sigma(\breve{\pi}_j^{(t)}) &\geq [2(k - i^*) + \nabla_{i^*}(H) - 1 - (1 + (t+1))\delta - 2(t+1)](\breve{n}_t - j) \\ &= [2(k - i^*) + \nabla_{i^*}(H) - 1 - (1 + (t+1))\delta - 2(t+1)]\breve{n}_{t,j},\end{aligned}$$

where $\breve{n}_{t,j}$ is the length of $\breve{\pi}_j^{(t)}$. Since $\pi^{(t+1)} = \breve{\pi}_j^{(t)}$ for some j, we have established the lower bound on $\sigma(\pi^{(t+1)})$.

If we analyze this final step more carefully, we get the second half of the claim. We have seen that $\sigma(\breve{\pi}^{(t)}) \geq (2(k - i^*) + \nabla_{i^*}(H) - 1 - (1 + (t+1))\delta - 2(t+1))\breve{n}_t$. When we lay off a term of $\breve{\pi}^{(t)}$ whose value is too small, we subtract $2(k - i^*) + \nabla_{i^*} - 2 - 2(t+1)$ from $\sigma(\breve{\pi}^{(t)})$, so that, if we call the resulting sequence $\breve{\pi}_1^{(t)}$, we see that $\sigma(\breve{\pi}_1^{(t)})$ is at least

$$[(2(k-i^*)+\nabla_{i^*}(H)-1-(t+2))\delta-2(t+1)]\breve{n}_t$$
$$-[2(k-i^*)+\nabla_{i^*}-1-2(t+1)]+1$$
$$=[(2(k-i^*)+\nabla_{i^*}(H)-1-(t+2))\delta-2(t+1)](\breve{n}_t-1)$$
$$+(1-(t+2)\delta).$$

After repeating this j times, we get a sequence $\breve{\pi}_j^{(t)}$ with sum at least

$$[(2(k-i^*)+\nabla_{i^*}(H)-1-(t+2))\delta-2(t+1)](\breve{n}_t-j)+(1-(t+2)\delta)j.$$

Now, if $\sigma(\breve{\pi}_j^{(t)}) \geq \sigma(H, \breve{n}_t - j) - 2(t+1)(\breve{n}_t - j)$, then by Lemma 4.9, $\sigma(\breve{\pi}_j^{(t)}) \geq \sigma(F, \breve{n}_t - j)$ for some $F \in \mathcal{D}^{(t+1)}(H)$, so $\breve{\pi}_j^{(t)}$ is potentially $\mathcal{D}^{(t+1)}(H)$-graphic. This will happen if $(1-(t+2)\delta)j \geq (t+3)\delta(\breve{n}_t - j)$, which means that if $j \geq \frac{(t+3)\delta}{1+\delta} n_t \geq \frac{(t+3)\delta}{1+\delta}\breve{n}_t$, then $\breve{\pi}_j^{(t)}$ is potentially $\mathcal{D}^{(t+1)}(H)$-graphic. Since $\pi^{(t+1)} = \breve{\pi}_j^{(t)}$ for some j, this in turn means $\pi^{(t+1)}$ is potentially $\mathcal{D}^{(t+1)}(H)$-graphic. Finally, Claim 4.3 shows that in this case, π is potentially H-graphic. Thus, at most $\frac{(t+3)\delta}{1+\delta} n_t$ iterations of the Kleitman-Wang algorithm result in the desired sequence $\pi^{(t+1)}$. $\square$

We would like to know how many terms have been removed from π in the creation of $\pi^{(\ell)}$. That is, what is the value of $n - n_\ell$? At each iteration of the algorithm, the vertex $v_1^{(t)}$ misses at most $\binom{k}{\lceil k/2 \rceil}(k^2-1)$ vertices, so in Step (i) at most this many vertices are removed. In Step (ii), only one vertex is removed. Then, Claim 4.4 shows that at most $\frac{(3+k)\delta}{1+\delta} n$ terms are laid off in Step (iii). Since we do the algorithm at most $k - \alpha - 1$ times, and do the initialization step once, this means that

$$n - n_\ell \leq (k-\alpha-1)\left(\binom{k}{\lceil k/2 \rceil}(k^2-1) + 1 + \frac{(3+k)\delta}{1+\delta} n\right) + \frac{2\delta}{1+\delta} n. \qquad (4.1)$$

Claim 4.3 shows that there is a realization of π that contains $K_\ell \vee G^{(\ell)}$, where $G^{(\ell)}$ is a realization of $\pi^{(\ell)}$. The ℓ vertices in the clique are adjacent to every vertex in $G^{(\ell)}$, so they are only nonadjacent to vertices that have been removed or laid off through the course of the algorithm. Thus, they are not adjacent to at most $n - n_\ell$ vertices; that is, the expression in Equation (4.1) gives an upper bound on how many vertices are not adjacent to the clique in this realization.

Stage 2: In Stage 2, we analyze the realizations of $\pi^{(\ell)}$ with the goal of showing that some realization contains a supergraph of H. Recall that in Stage 1, our algorithm stops either when we have iterated "enough" times, or when the sequence $\pi^{(t)}$ fails the maximum degree condition of Theorem 4.10.

If the algorithm stops because we have iterated $k - \alpha(H) - b_H$ times, then the graph $K_\ell \vee G^{(\ell)}$ contains $K_{k-\alpha(H)-b_H} \vee \overline{K}_{\alpha(H)+b_H}$, which is a supergraph of H when H is Type 1, so in this case π is potentially H-graphic. If H is Type 2, then we are left with a realization of π containing $K_{k-\alpha(H)-1} \vee \overline{K}_{\alpha(H)+1}$. This suffices to prove Parts 1 and 3 of the lemma.

Therefore we assume that the algorithm stops when $t + 1 = \ell < k - \alpha(H) - b_H$ and $d_1^{(\ell)} < n_\ell - \binom{k}{\lceil k/2 \rceil}(k^2 - 1)$. Now $\pi^{(\ell)}$ has maximum term $d_1^{(\ell)}$, which is less than $n_\ell - \binom{k}{\lceil k/2 \rceil}(k^2 - 1)$ and minimum term $d_{n_\ell}^{(\ell)}$, which is at least $k - \alpha(H) - b_H - \ell$. If $\pi^{(\ell)}$ is also degree sufficient for $S_\ell = K_{k-\alpha-b_H-\ell} \vee \overline{K}_{\alpha+b_H}$, then the Bounded Max Degree Theorem (Theorem 4.10) implies that $\pi^{(\ell)}$ is potentially S_ℓ-graphic, because the value $\binom{k}{\lceil k/2 \rceil}(k^2 - 1)$ was chosen so that $\binom{k}{\lceil k/2 \rceil}(k^2 - 1) > f(\alpha + b_H, k - \ell)$, where f is the function given by Theorem 4.10. Since $K_\ell \vee S_\ell = K_{k-\alpha-b_H} \vee \overline{K}_{\alpha+b_H}$, when H is Type 1 this shows that π is potentially H-graphic. If H is Type 2, then there is a realization of π containing $K_{k-\alpha-1} \vee \overline{K}_{\alpha+1}$. Again, this suffices to prove Parts 1 and 3 of the lemma.

We can therefore assume that $\pi^{(\ell)}$ is not degree sufficient for S_ℓ. Let $p = \max\{j : d_j^{(\ell)} \geq k - \ell - 1\}$. Since $\pi^{(\ell)}$ is not degree sufficient for S_ℓ but the minimum term of

$\pi^{(\ell)}$ is at least $k - \alpha(H) - b_H - \ell$, we know that $p < k - \alpha(H) - b_H - \ell$. Thus, $\pi^{(\ell)}$ is degree sufficient for $K_p \vee \overline{K}_{k-\ell-p}$.

For $j \in \{\alpha + 1, \ldots, k\}$, let F_j be a j-vertex induced subgraph of H such that $\Delta(F_j) = \nabla_j(H)$. We will show that if $\pi^{(\ell)}$ is degree sufficient for $K_p \vee F_{k-\ell-p}$, then π is potentially H-graphic. Unfortunately, we cannot simply apply the Bounded Max Degree Theorem for this graph, because we do not know what $\alpha(F_{k-\ell-p})$ is. Instead we will use the BMDT to show that we have a realization containing $K_p \vee \overline{K}_{k-\ell-p}$, and then create a copy of $F_{k-\ell-p}$ on the vertices of the independent set.

To begin, note that $\pi^{(\ell)}$ satisfies the conditions of the Bounded Max Degree Theorem (Theorem 4.10) for $K_p \vee \overline{K}_{k-\ell-p}$, since the minimum term of $\pi^{(\ell)}$ is at least $k - \alpha(H) - b_H - \ell > p = |V(K_p \vee \overline{K}_{k-\ell-p})| - \alpha(K_p \vee \overline{K}_{k-\ell-p})$ and $d_1^{(\ell)} < n_\ell - \binom{k}{\lceil k/2 \rceil}(k^2 - 1) < n_\ell - f(k - \ell - p, k - \ell)$.

Let G_ℓ be a realization of $\pi^{(\ell)}$ on the vertex set $\{v_1, \ldots, v_{n_\ell}\}$ that contains $K_p \vee \overline{K}_{k-\ell-p}$ on the vertices $\{v_1, \ldots, v_{k-\ell}\}$ such that the first p vertices induce a clique. Such a realization exists by Lemma 2.12. Delete the vertices of the clique from G_ℓ to get the graph G'_ℓ, and let $\mu = \pi(G'_\ell)$, with the order of the vertices maintained. We want to show that there is a realization of μ that contains a copy of $F_{k-\ell-p}$ on the vertex set $\{v_{p+1}, \ldots, v_{k-\ell}\}$. Note that since the minimum term of $\pi^{(\ell)}$ is greater than p, the minimum term of μ is at least 1.

To construct the desired realization, place a copy of $F_{k-\ell-p}$ on the vertices $v_{p+1}, \ldots, v_{k-\ell}$. Since $\pi^{(\ell)}$ is degree sufficient for $K_p \vee F_{k-\ell-p}$, we know μ is degree sufficient for $F_{k-\ell-p}$. Thus, the degrees of the vertices in $\{v_{p+1}, \ldots, v_{k-\ell}\}$ prescribed by μ are at least as large as the degrees of these vertices in $F_{k-\ell-p}$. If, after doing this, any vertex in $\{v_{p+1}, \ldots, v_{k-\ell}\}$ does not yet have the degree prescribed by μ, join that vertex to distinct vertices among the remaining $n_\ell - (k - \ell)$ vertices. The sequence μ' obtained by subtracting the degrees of the vertices in the graph constructed so far from the corresponding terms of μ has at least $n_\ell - (k - \ell) - (k - \ell - p)(k - \ell - 3)$

positive terms and maximum term at most $k-\ell-2$. By the Erdős-Gallai criteria (Theorem 1.3), the sequence μ' is graphic. Combining a realization of μ' with the constructed graph gives us the desired realization of μ, showing that $\pi^{(\ell)}$ is potentially $(K_p \vee F_{k-\ell-p})$-graphic. Since $K_p \vee F_{k-\ell-p}$ is a supergraph of a graph in $\mathcal{D}^{(\ell)}(H)$, Claim 4.3 implies that π is potentially H-graphic.

Stage 3: We have shown that in many cases, we can construct a realization of π that contains either a supergraph of H or $K_{k-\alpha-1} \vee \overline{K}_{\alpha+1}$. This gives us the conclusions in Parts 1 and 3 of the lemma. Now we will show that in the remaining case, that is when $\pi^{(\ell)}$ is not degree sufficient for $K_p \vee F_{k-\ell-p}$, we have $\|\pi - \pi'\| < \epsilon n$ for some $\pi' \in \mathcal{P}(H)$, establishing Part 2.

To do this, we first create a sequence η from $\pi^{(\ell)}$ by adding ℓ vertices to a realization of $\pi^{(\ell)}$ and taking the degree sequence of the resulting graph. Then we will compute

- $\|\pi - \eta\|$,
- $\|\eta - \widetilde{\pi}_{k-\ell-p}(H, n_\ell + \ell)\|$, and
- $\|\widetilde{\pi}_{k-\ell-p}(H, n_\ell + \ell) - \widetilde{\pi}_{k-\ell-p}(H, n)\|$.

After showing that $\widetilde{\pi}_{k-\ell-p}(H, n) \in \mathcal{P}(H)$, we will use the triangle inequality to show that $\|\pi - \widetilde{\pi}_{k-\ell-p}(H, n)\| < \epsilon n$.

We begin by noting that if $\pi^{(\ell)}$ is not degree sufficient for $K_p \vee F_{k-\ell-p}$, then it has the following properties:

- $d_j^{(\ell)} \leq n_\ell - 1$ for $j \leq p$.
- $d_j^{(\ell)} \leq k - \ell - 2$ for $p + 1 \leq j \leq k - \ell - 1$.
- $d_j^{(\ell)} \leq \nabla_{k-\ell-p} + p - 1$ for $j \geq k - \ell$.

We thus have the following upper bound on $\sigma(\pi_\ell)$:

$$
\begin{aligned}
\sigma(\pi^{(\ell)}) &\le p(n_\ell - 1) + (k-\ell-1-p)(k-\ell-2) + (n_\ell - (k-\ell-1))(\nabla_{k-\ell-p} + p - 1) \\
&= (2p + \nabla_{k-\ell-p} - 1)n_\ell + (k-\ell-1)(k-\ell-1-\nabla_{k-\ell-p} - 2p) \\
&= (2p + \nabla_{k-\ell-p} - 1)n_\ell + o(n).
\end{aligned}
$$

We also know, by Claim 4.4, that

$$
\begin{aligned}
\sigma(\pi^{(\ell)}) &\ge (2(k-i^*) + \nabla_{i^*} - 1 - (\ell+1)\delta - 2\ell)n_\ell \\
&\ge (2(k-\ell) - (2(k-\ell-p) - \nabla_{k-\ell-p}) - 1 - (\ell+1)\delta)n_\ell \\
&= (2p + \nabla_{k-\ell-p} - 1 - \delta(\ell+1))n_\ell.
\end{aligned}
$$

Let η be the degree sequence of $K_\ell \vee G^{(\ell)}$, where $G^{(\ell)}$ is a realization of $\pi^{(\ell)}$. Then

$$
\begin{aligned}
\sigma(\eta) &= \sigma(\pi^{(\ell)}) + 2\ell n_\ell + \ell^2 - \ell \\
&\ge (2p + \nabla_{k-\ell-p} - 1 - \delta(\ell+1))n_\ell + 2\ell n_\ell + \ell^2 - \ell.
\end{aligned}
$$

Now consider the sequence $\widetilde{\pi}_{k-\ell-p}(H, n_\ell+\ell)$. We need to show that $\widetilde{\pi}_{k-\ell-p}(H, n_\ell+\ell) \in \mathcal{P}(H)$. The sum of this sequence is

$$
\begin{aligned}
\sigma(\widetilde{\pi}_{k-\ell-p}(H, n_\ell+\ell)) &= (2p + \nabla_{k-\ell-p} - 1)n_\ell - p^2 - p\nabla_{k-\ell-p} + 2\ell n_\ell + \ell^2 - \ell \\
&\ge \sigma(\eta).
\end{aligned}
$$

After simplifying, we see that the coefficient of $n_\ell + \ell$ in $\sigma(\widetilde{\pi}_{k-\ell-p}(H, n_\ell+\ell))$ is $2(\ell+p) + \nabla_{k-\ell-p} - 1$. If $\widetilde{\pi}_{k-\ell-p}(H, n_\ell+\ell)$ is not in $\mathcal{P}(H)$, then

$$
(2(\ell+p) + \nabla_{k-\ell-p} - 1)(n_\ell+\ell) < (2(k-i^*) - \nabla_{i^*} - 1)(n_\ell+\ell). \tag{4.2}
$$

However, we also know that

$$\sigma(\widetilde{\pi}_{k-\ell-p}(H, n_\ell+\ell)) \geq \sigma(\eta) \geq (2(k-i^*)-\nabla_{i^*}-1-(\ell+1)\delta)n_\ell+2\ell n_\ell+\ell^2-\ell. \quad (4.3)$$

Inequalities (4.2) and (4.3) together imply that

$$(2(k-i^*)-\nabla_{i^*}-1-(\ell+1)\delta)n_\ell+2\ell n_\ell+\ell^2-\ell \leq (2(k-i^*)-\nabla_{i^*}-1)(n_\ell+\ell),$$

which is not true for sufficiently large n_ℓ. Thus, $\widetilde{\pi}_{k-\ell-p}(H, n_\ell+\ell) \in \mathcal{P}(H)$.

In order to calculate $\|\eta - \widetilde{\pi}_{k-\ell-p}(H, n_\ell+\ell)\|$, note that the only terms of $\eta = (a_1, \ldots, a_{n_\ell+\ell})$ that are greater than those of $\widetilde{\pi}_{k-\ell-p}(H, n_\ell+\ell)$ are those a_j for which $p+\ell+1 \leq j \leq k-1$. Thus, we can bound the distance between these sequences by taking the absolute difference of their sums and adding two times the difference of the terms in that range. Since the largest terms of η in that range are at most $k-2$, and the corresponding terms of $\widetilde{\pi}_{k-\ell-p}$ equal $\nabla_{k-\ell-p}+p+\ell-1$, we add at most $2(k-\ell-p-1-\nabla_{k-\ell-p})(k-\ell-p-1)$ to the difference of sums. This yields

$$\begin{aligned}\|\eta - \widetilde{\pi}_{k-\ell-p}(H, n_\ell+\ell)\| &\leq \sigma(\widetilde{\pi}_{k-\ell-p}(H, n_\ell+\ell)) - \sigma(\eta) + 2k^2 \\ &\leq \delta(\ell+1)n_\ell + 2k^2.\end{aligned}$$

Now we calculate $\|\pi - \eta\|$; this is where we use all of the information we gained in Stage 1 about the structure of π. Recall from Equation (4.1) that at most

$$\ell\left(\binom{k}{\lceil k/2 \rceil}(k^2-1)+1+\frac{(3+k)\delta}{1+\delta}n\right)+\frac{2\delta}{1+\delta}n$$

terms are removed from π to create $\pi^{(\ell)}$. Of these, the terms that were laid off in Step (ii) because they were the first term of some $\pi^{(t)}$ have high degree, and those that were removed during Steps (i) and (iii) of the algorithm have very low degree.

Theorem 2.17 implies that those terms removed in Steps (i) and (iii) are in fact at most $k-3$, and the hypotheses of the algorithm imply that the terms removed in Step (ii) are at least $n_\ell - \binom{k}{\lceil k/2 \rceil}(k^2-1)$. The terms of η that give the degrees of vertices in the clique in $K_\ell \vee G^{(\ell)}$ correspond to the terms of π that were removed in Step (ii). So the difference between these terms in the two sequences is at most $\binom{k}{\lceil k/2 \rceil}(k^2-1)$. The terms of π that were removed in Steps (i) and (iii) correspond to zeros in η. Thus, the difference between these terms is at most $k-3$. We see that since $n_\ell \leq n_t \leq n$ and $\ell \leq k - \alpha(H) - 1$, we have

$$\begin{aligned}\|\pi - \eta\| \leq &\left(\frac{(2+k-\alpha(H))\delta}{1+\delta}n + \binom{k}{\lceil k/2 \rceil}(k^2-1)\right)(k-3)(k-\alpha(H)-1) \\ &\qquad + \binom{k}{\lceil k/2 \rceil}(k^2-1)(k-\alpha(H)-1) + \frac{2\delta}{1+\delta}n \\ \leq &\frac{(k^3+2)\delta}{1+\delta}n + \binom{k}{\lceil k/2 \rceil}k^4.\end{aligned}$$

Finally, we know that $\|\widetilde{\pi}_{k-\ell-p}(H, n_\ell+\ell) - \widetilde{\pi}_{k-\ell-p}(H,n)\| = (n-(n_\ell+\ell))(2(k - i^*) + \nabla_{i^*} - 1)$, and by Equation (4.1), this is at most

$$\begin{aligned}&\left((k-1)\binom{k}{\lceil k/2 \rceil}(k^2-1) + k\frac{(3+k)\delta}{1+\delta}n\right)(2(k-i^*)+\nabla_{i^*}-1) \\ &\qquad \leq \binom{k}{\lceil k/2 \rceil}k^4 + 2k^2\frac{(3+k)\delta}{1+\delta}n.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|\pi - \widetilde{\pi}_{k-\ell-p}(H,n)\| &\leq \|\pi - \eta\| + \|\eta - \widetilde{\pi}_{k-\ell-p}(H, n_\ell+\ell)\| \\ &\qquad + \|\widetilde{\pi}_{k-\ell-p}(H, n_\ell+\ell) - \widetilde{\pi}_{k-\ell-p}(H,n)\| \\ &\leq \delta(\ell+1)n + k^2 + \frac{(k^3+2)\delta}{1+\delta}n + 2\binom{k}{\lceil k/2 \rceil}k^4 + 2k^2\frac{(3+k)\delta}{1+\delta}n \\ &\leq \left(\delta(k-\alpha) + \frac{(k^3+2)\delta}{1+\delta} + 2k^2\frac{(3+k)\delta}{1+\delta}\right)n + 2\binom{k}{\lceil k/2 \rceil}k^4 + k^2.\end{aligned}$$

Thus, as δ approaches 0 this expression is less than ϵn. $\square$

From the proof of Lemma 4.11, we see that when H is Type 1, then we only reach the conclusions in Parts 1 and 2 of the lemma, thus implying that H is σ-stable. Thus, Theorem 4.4 is an immediate corollary. When H is Type 2, however, we may end up in the case where Lemma 4.11 can only guarantee a realization of π containing $K_{k-\alpha(H)-1} \vee \overline{K}_{\alpha(H)+1}$. Now, to prove Theorem 4.5, we analyze what happens in this case.

Proof of Theorem 4.5. Let H be a graph that satisfies the hypotheses of Theorem 4.5. Let $\epsilon > 0$ be given, and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence of length n that is not potentially H-graphic and such that $\sigma(\pi) \geq (2(k - i^*) + \nabla_{i^*} - 1 - \delta)n$, where $\delta < \frac{1}{k}$. By Lemma 4.11, either $\|\pi - \pi'\| < \epsilon n$ for some $\pi' \in \mathcal{P}(H)$, or π is potentially $(K_{k-\alpha(H)-1} \vee \overline{K}_{\alpha(H)+1})$-graphic.

Since the former case implies that H is σ-stable, we may assume the latter case holds. By Theorem 2.12, we may assume that there is a realization G of π on the vertex set $\{v_1, \ldots, v_n\}$ such that $d(v_i) = d_i$ for $1 \leq i \leq n$, the vertices $v_1, \ldots, v_{k-\alpha(H)-1}$ induce the clique and the vertices $v_{k-\alpha(H)}, \ldots, v_k$ form the independent set in the complete split graph. Let $S = \{v_{k-\alpha(H)}, \ldots, v_k\}$. If S is not an independent set in G, then $\{v_1, \ldots, v_k\}$ induces a supergraph of H, a contradiction. We may therefore assume that S is an independent set in G.

If every vertex in S has degree $k-\alpha-1$, then since π is nonincreasing, $d_j \leq k-\alpha-1$ for each $j \geq k - \alpha$. Since $\widetilde{\pi}_{\alpha+1}(H, n) = ((n-1)^{k-\alpha-1}, (k-\alpha-1)^{n-(k-\alpha-1)})$, we see that each term of π is less than or equal to the corresponding term of $\widetilde{\pi}_{\alpha+1}(H, n)$. Thus, $\|\pi - \widetilde{\pi}_{\alpha+1}(H, n)\| \leq \sigma(\widetilde{\pi}_{\alpha+1}(H, n)) - \sigma(\pi) \leq \delta n$.

If at least two vertices in S, say u and v, have degree at least $k - \alpha$ in G, then we can do an edge exchange to create a supergraph of H, implying that π is potentially H-graphic. Without loss of generality, suppose $d(u) \geq d(v) \geq k - \alpha$. Since S is an independent set in G, there must be vertices a_1 and a_2 in $V(G) - \{v_1, \ldots, v_k\}$ such that $u \sim a_1$ and $v \sim a_2$. It is possible that $a_1 = a_2$. By Theorem 2.17, we may assume that each vertex in $V(G) - \{v_1, \ldots, v_k\}$ has degree at most $2k-4$. Thus, there are at most $(2k)^2+1$ vertices at distance at most 2 from a_1, and at most $(2k)^2+1$ vertices at distance at most 2 from a_2. Since n is sufficiently large, there is a vertex w that is at distance at least 3 from both a_1 and a_2. Since $\sigma(\pi) \geq \sigma(H,n) - \delta n$, that vertex must have positive degree in G. Let x be a neighbor of w; note that x is not adjacent to a_1 or a_2, but it is possible that $x \in \{v_1, \ldots, v_k\} \setminus \{u, v\}$. We can exchange the edges ua_1, va_2, and wx for the nonedges uv, wa_1, and xa_2 to get a realization of π that contains $(K_{k-\alpha-1} \vee \overline{K}_{\alpha+1}) + e$, where e is the edge uv in the independent set. Since H has a set of $\alpha+1$ vertices that induces one edge, this shows that π is potentially H-graphic.

We are left with the case where exactly one vertex in S has degree greater than $k-\alpha-1$. We may assume that this vertex is $v_{k-\alpha}$, so $d_{k-\alpha} \geq k-\alpha$ but $d_j \leq k-\alpha-1$ for all $j \geq k-\alpha+1$. For $j \in \{1, \ldots, k-\alpha-1\}$, let W_j be the set of neighbors of $v_{k-\alpha}$ that are not adjacent to v_j. Recall that $H \subseteq K_{k-\alpha-2} \vee S_{b_1,b_2}$ for some b_1 and b_2. Suppose $b_1 \geq b_2$. If $d_{k-\alpha} \geq (k-\alpha-1)(b_2+1)$, then for some $p \in \{1, \ldots, k-\alpha-1\}$ we have $|W_p| \geq b_2$. Let $x \in W_p$. Then we can exchange the edges $xv_{k-\alpha}$ and $v_p v_{k-\alpha+1}$ for the nonedges xv_p and $v_{k-\alpha}v_{k-\alpha+1}$. We can do a similar edge exchange for b_2 vertices in W_p and the vertices $v_{k-\alpha+2}, \ldots, v_{k-\alpha+b_2}$ until $v_{k-\alpha}$ is adjacent to each of $v_{k-\alpha+1}, \ldots, v_{k-\alpha+b_2}$, while v_p is no longer adjacent to these vertices. Thus, we create a realization of π where v_p and $v_{k-\alpha}$ are the centers of a double star S_{b_1,b_2} that is joined to a complete graph on the vertices $\{v_1, \ldots, v_{k-\alpha-1}\} \setminus \{v_p\}$, which means that π is potentially H-graphic.

So $d_{k-\alpha} \leq (k-\alpha-1)(b_2+1)$. In this case we can show that $\|\pi - \widetilde{\pi}_{\alpha+1}(H,n)\| < \epsilon n$. Since $\widetilde{\pi}_{\alpha+1}(H,n) = ((n-1)^{k-\alpha-1}, (k-\alpha-1)^{n-(k-\alpha-1)})$, we see that $\widetilde{\pi}_{\alpha+1}(H,n)$

majorizes π except at the $(k-\alpha)^{th}$ term. Thus,

$$\begin{aligned} \|\pi - \widetilde{\pi}_{\alpha+1}(H,n)\| &< (\sigma(\widetilde{\pi}_{\alpha+1}(H,n)) - \sigma(\pi)) + (d_{k-\alpha} - (k-\alpha-1)) \\ &< \delta n + (k-\alpha-1)b_2. \end{aligned}$$

As δ becomes smaller, we see that this is less than ϵn, so H is σ-stable. □

4.5 Implications of stability

Once we know that a graph H is σ-stable, the next step in the stability method is to use this fact to show that the sequences in $\mathcal{P}(H)$ are the only extremal sequences. If the full set of extremal sequences for a graph is known, then we also know the precise value of the potential number for that graph. This is one of the main goals in proving a stability result for an extremal problem. However, in our case the property of not being σ-stable is also very interesting, because it draws attention to the existence of other extremal sequences for such graphs.

We showed in Section 4.2 that if H is a graph of order k and independence number α that is not a subgraph of $K_{k-\alpha-2} \vee S_{x,y}$ for some x, y, with $x+y=\alpha$, then H is not σ-stable. The reason such a graph is not σ-stable is that the sequence $\rho_\ell = ((n-1)^\ell, \left(\frac{n-\ell}{2}\right)^2, (\ell+1)^{n-\ell-2})$, with $\ell = k-\alpha-2$, is not potentially H-graphic, although its sum is arbitrarily close to the potential number for large n. This implies that the sequences in $\mathcal{P}(H)$ are not the only extremal sequences for these graphs.

As an example, consider the complete graph on k vertices, K_k. We have shown that K_k is not σ-stable, but it is weakly σ-stable. This implies that the sequence $\widetilde{\pi}_2(H,n) = ((n-1)^{k-2}, (k-2)^{n-k+2})$, which is the only extremal sequence noted in the literature, is not the only extremal sequence for K_k. In fact, the sequence $\rho_{k-3} = ((n-1)^{k-3}, \left(\frac{n-k+3}{2}\right)^2, (k-2)^{n-k-5})$ has the same sum as $\widetilde{\pi}_2(H,n)$, and is also not potentially K_k-graphic, as shown in Section 4.2. Thus, this sequence also belongs in the set of extremal sequences for K_k.

The fact that K_k is weakly σ-stable is reflected in the observation that ρ_{k-3} is not degree sufficient for K_k, and in fact if there are other extremal sequences for K_k, they must also fail to be degree sufficient. If a graph H is weakly σ-stable but not σ-stable, any extremal sequence that is not in $\mathcal{P}(H)$ must not be degree sufficient for H. If a graph is not weakly σ-stable, then any extremal sequences in addition to those of $\mathcal{P}(H)$ may or may not be degree sufficient for the graph. A natural question to ask is whether the sequence $\rho_{k-\alpha-2}$ is the only additional extremal sequence for graphs that are not σ-stable, or if others can be found.

Currently we know very little about graphs that are Type 2 but have a set of $\alpha+1$ vertices that induce a matching of size at least 2. We have shown that if such a graph is not a subgraph of $K_{k-\alpha-2} \vee S_{x,y}$, then it is not σ-stable, but we have no positive results on the σ-stability of such graphs. Determining more about such graphs is the next step in our study of σ-stability.

5. Potential-Ramsey Numbers

5.1 Introduction

As discussed in Chapter 2, the k-color graph Ramsey number $r(G_1, \dots, G_k)$, where $G_1, \dots, G_k$ are graphs, is the minimum integer n such that any k-edge-coloring of K_n yields a monochromatic G_i in color i for some i. The potential version of this problem was introduced by Busch, Ferrara, Hartke and Jacobson in [9], and is defined first for two colors (we will discuss a multi-color version in Chapter 6). Given graphs G_1 and G_2 and a graphic sequence $\pi = (d_1, \dots, d_n)$, we write $\pi \to (G_1, G_2)$ if either π is potentially G_1-graphic or $\overline{\pi}$ is potentially G_2-graphic, where $\overline{\pi} = (n-1-d_1, n-1-d_2, \dots, n-1-d_n)$ is the complementary degree sequence of π. The *potential-Ramsey number* of G_1 and G_2, denoted $r_{pot}(G_1, G_2)$, is the minimum integer n such that if π is a graphic sequence of length at least n, then $\pi \to (G_1, G_2)$.

We can easily show that $r_{pot}(G_1, G_2) \le r(G_1, G_2)$, because a realization of a graphic sequence of length n can be thought of as giving a 2-edge-coloring of K_n. That is, a realization of π determines the red edges in K_n, and the complementary realization of $\overline{\pi}$ determines the blue edges. Thus, if every red/blue coloring of the edges of K_n produces a red G_1 or blue G_2, then every graphic sequence of length n is either potentially G_1-graphic or its complement is potentially G_2-graphic. This bound is sharp in some cases; in particular we have the following result from [9].

Lemma 5.1 (Busch et al. [9]) *Let $r = r(G_1, G_2)$ and let G be a graph of order $r-1$ such that $G_1 \nsubseteq G$ and $G_2 \nsubseteq \overline{G}$. If $\pi(G)$ is unigraphic, then $r(G_1, G_2) = r_{pot}(G_1, G_2)$.*

Busch et al. used this and a result of Gerencsér and Gyárfás [49] to show that $r_{pot}(P_s, P_t) = r(P_s, P_t) = s + \lfloor \frac{t}{2} \rfloor - 1$.

Determining the Ramsey number $r(K_s, K_t)$ is one of the foremost open problems in combinatorics. A well-known lower bound on $r(K_t, K_t)$ is $r(K_t, K_t) \ge \frac{\sqrt{2}}{e} t 2^{t/2}$, proven by Spencer in [114]. However, the potential-Ramsey number is comparatively simple; in fact, Busch et al. determined $r_{pot}(K_s, K_t)$ for all s, t, thus showing that

$r_{pot}(K_t, K_t)$ is linear in t (and that the simple bound in Lemma 5.1 can be very far from sharp).

Theorem 5.2 (Busch et al. [9]) *For $s \geq t \geq 3$, $r_{pot}(K_s, K_t) = 2s + t - 4$, except when $s = t = 3$, in which case $r_{pot}(K_3, K_3) = 6$.*

They also determined the potential-Ramsey numbers $r_{pot}(C_s, K_t)$ and $r_{pot}(P_s, K_t)$.

Theorem 5.3 (Busch et al. [9]) *If $s \geq 3$ and $\lfloor \frac{2s}{3} \rfloor \geq t \geq 2$, then $r_{pot}(C_s, K_t) = s + t - 2$. If $s \geq 4$ and $t > \lfloor \frac{2s}{3} \rfloor \geq 2$, then $r_{pot}(C_s, K_t) = 2t - 2 + \lceil \frac{s}{3} \rceil$.*

Theorem 5.4 (Busch et al. [9]) *For $s \geq 6$ and $t \geq 3$,*

$$r_{pot}(P_s, K_t) = \begin{cases} s + t - 2 & \text{if} \quad t \leq \lfloor \frac{2s}{3} \rfloor \\ 2t - 2 + \lfloor \frac{s}{3} \rfloor & \text{if} \quad t > \lfloor \frac{2s}{3} \rfloor. \end{cases}$$

In this chapter, we present results on $r_{pot}(H, G)$, where H is one of $K_2, 2K_2, K_3, P_3$, or P_4, and G is an arbitrary graph of order t, as well as $r_{pot}(H, G)$ for all H and G of order at most 4. We will also state and prove the value of $r_{pot}(C_s, C_t)$.

5.2 Preliminaries

We will assume from now on that G is a graph of order t. If H is a nontrivial graph, then it is easy to see that $r_{pot}(H, G) \geq t$, for if $\pi = (0^{t-1})$, $\pi \not\to (H, G)$. It is also easy to see that $r_{pot}(H, tK_1) = r_{pot}(tK_1, H) = t$. We can thus assume that all graphs have at least one nontrivial component. We begin with a simple observation about the potential Ramsey number of graphs with isolated vertices.

Proposition 5.5 *Suppose $G = G_1 \cup K_1$ and $r_{pot}(H, G_1) = r$. If $r \leq |V(G)|$, then $r_{pot}(H, G) = |V(G)|$, and if $r > |V(G)|$, then $r_{pot}(H, G) = r$.*

Proof. Let $n = |V(G)|$. First suppose $r \leq n$, and let π be a graphic sequence of length n. If π is not potentially H-graphic, then we know that $\overline{\pi}$ has a realization R

containing G_1 as a subgraph. Since $|V(G_1)| < n$, there is a vertex in R that is not used by the copy of G_1. Adding this vertex gives us a copy of G in R, so $\pi \to (H, G)$. Since $r_{pot}(H, G) \geq |V(G)|$, the conclusion holds.

If $r > n$, let π be a graphic sequence of length r. If π is not potentially H-graphic, then again $\overline{\pi}$ has a realization R containing G_1, and since $r > n$, there is a vertex in R that is not used by the copy of G_1, so there is in fact a copy of G in R. Thus, $\pi \to (H, G)$. Since $r_{pot}(H, G_1) = r$, we know that $r_{pot}(H, G) \geq r$, so $r_{pot}(H, G) = r$. □

We may henceforth assume that G and H do not have any isolated vertices.

5.2.1 General lower bounds

We now determine several lower bounds on $r_{pot}(H, G)$.

Lemma 5.6 *If H is a graph of order t that is not a subgraph of $K_{1,t-1}$ and G has no isolated vertices, then $r_{pot}(H, G) \geq t + 1$.*

Proof. Consider $\pi = (t - 1, 1^{t-1})$. This sequence is uniquely realized by the star $K_{1,t-1}$, so π is not potentially H-graphic. The complement of π is uniquely realized by $K_{t-1} + K_1$, which can contain no graph on t vertices unless that graph has at least one isolate. So $r_{pot}(H, G) \geq t + 1$. □

The maximum size of a matching in the complement of G, $\alpha'(\overline{G})$, turns out to play an important role in determining $r_{pot}(H, G)$. To simplify notation, we will often write $\overline{\alpha}'$ in place of $\alpha'(\overline{G})$ when G is understood.

Lemma 5.7 *If G is a graph of order $t \geq 3$ and H is a graph with a connected component of order at least 3, then $r_{pot}(H, G) \geq \max\{2(t - \alpha'(\overline{G}) - 1) + 1, t\}$.*

Proof. We have already seen that $r_{pot}(H, G) \geq t$. Thus we may assume $2(t - \overline{\alpha}' - 1) + 1 \geq t$. Let $\pi = (1^{2(t-\overline{\alpha}'-1)})$. This degree sequence is uniquely realized by a matching of size $t - \overline{\alpha}' - 1$, so it does not contain a copy of any graph H with a component of order at least 3. Any graph on t vertices in the complement of this realization must use at least $\overline{\alpha}' + 1$ pairs of vertices that are endpoints of one of the edges, which requires a matching of size at least $\overline{\alpha}' + 1$ in $\overline{G}$. Thus, $\pi \not\to (H, G)$ and $r_{pot}(H, G) \geq 2(t - \overline{\alpha}' - 1) + 1$. $\square$

Recall that if the degree sequence of G is $(x_1, \ldots, x_t)$ and $\pi = (d_1, \ldots, d_n)$, then π is degree sufficient for G if $d_i \geq x_i$ for each i, $1 \leq i \leq k$. A simple necessary condition for π to be potentially G-graphic is that π be degree sufficient for G. Thus, sequences that fail to be degree sufficient for H and G often play a role in determining $r_{pot}(H, G)$. In particular, the minimum degree of G becomes important when $\alpha'(\overline{G})$ is large and the lower bound provided by Lemma 5.7 is simply t. When this happens, the following lemma may be used.

Lemma 5.8 *If G is a graph of order $t \geq 5$ such that $\delta(G) \geq \lceil \frac{t+1}{2} \rceil$ and H is a graph with three or more vertices of degree at least 2, then $r_{pot}(H, G) \geq t + 2$.*

Proof. Suppose first that t is odd, so $\delta(G) \geq \frac{t+1}{2}$. Consider the sequence $\pi_1 = ((t + 1 - \delta)^2, 1^{t-1})$, which is graphic by the Erdős-Gallai criteria (Theorem 1.3). Clearly, π_1 is not potentially H-graphic because it is not degree sufficient for H. On the other hand, $\overline{\pi}_1$ is not degree sufficient for G, so $\pi_1 \not\to (H, G)$.

If t is even, consider the sequence $\pi_2 = (t + 2 - \delta, t + 1 - \delta, 1^{t-1})$, which has $\overline{\pi}_2 = ((t - 1)^{t-1}, \delta - 1, \delta - 2)$. Since $\delta \geq \frac{t+2}{2}$, the Erdős-Gallai criteria again show that π_2 is graphic. However, π_2 is not degree sufficient for H, so is not potentially H-graphic, while $\overline{\pi}_2$ is not degree sufficient for any graph of order t and minimum degree δ. The result follows. $\square$

5.2.2 Towards an upper bound

We will often want to show that a graph H is a subgraph of the complement of another graph, G. To do this, we will look for a mapping f with the property that if $f(u) \sim_G f(v)$, then $u \nsim_H v$. If there is such a mapping, and it is injective, then $H \subseteq \overline{G}$.

As we will discuss, when H is P_3, P_4, or K_3, sequences that are not potentially H-graphic must end with a string of 1's or 0's. Thus, these sequences are realized by graphs containing connected components with at least three vertices, and a set of disjoint edges and vertices. The following results will therefore help us to determine upper bounds on the potential Ramsey number.

Lemma 5.9 *If G is a graph of order $t \geq 3$ and R is a graph of order at least* $\max\{2(t - \alpha'(\overline{G}) - 1) + 1, t\}$ *such that $\Delta(R) \leq 1$, then G is a subgraph of $\overline{R}$.*

Proof. First suppose $\overline{\alpha}' < t/2$, so that $\max\{2(t - \overline{\alpha}' - 1) + 1, t\} = 2t - 2\overline{\alpha}' - 1$. If $|R| = 2(t - \overline{\alpha}' - 1) + 1$, then there is at least one isolated vertex in R, and at most $t - \overline{\alpha}' - 1$ (disjoint) edges. By taking each isolated vertex and at most one endpoint of each edge, we can find $t - \overline{\alpha}'$ independent vertices in R. To find the remaining $\overline{\alpha}'$ vertices needed for G, we must pick both endpoints of at most $\overline{\alpha}'$ edges in R. We can then identify the edges of R with a matching in $\overline{G}$, and this gives a mapping of G into $\overline{R}$, showing that $G \subseteq \overline{R}$.

If $|R| > 2(t - \overline{\alpha}' - 1) + 1$, there are again at least $t - \overline{\alpha}'$ independent vertices in R; choosing the largest independent set in R, at most $\overline{\alpha}'$ more vertices are needed to create a copy of G. This means at most $\overline{\alpha}'$ edges of R must be used, and this can also be identified with a matching in $\overline{G}$ as before.

If $\overline{\alpha}' = t/2$, then $\max\{2(t - \overline{\alpha}' - 1) + 1, t\} = t$. In this case, if $|R| = t$, then there are at most $t/2$ disjoint edges in R, and the edges of a matching in $\overline{G}$ can be identified with these edges and any isolated vertices in R to see that $G \subseteq \overline{R}$. If $|R| > t$, it is easy to see that $G \subseteq \overline{R}$. $\square$

We now have the following easy corollary.

Corollary 5.10 *Let G be graph of order $t \geq 3$, and R a graph of order at least $2(t - \alpha'(\overline{G}) - 1) + 1$ such that $R = R_1 \cup R_2$, where R_1 is the maximal subgraph of R with $\Delta(R_1) \leq 1$. Suppose there is a subgraph G_2 of G such that $G_2 \subseteq \overline{R}_2$, and let G_1 be the subgraph of G induced by $V(G) \setminus V(G_2)$. If $|R_1| \geq \max\{2(|G_1| - \alpha'(\overline{G_1}) - 1) + 1, |G_1|\}$, then $G \subseteq \overline{R}$.*

Proof. Lemma 5.9 shows that if $|R_1| \geq \max\{2(|G_1| - \alpha'(\overline{G_1}) - 1) + 1, |G_1|\}$, then $G_1 \subseteq \overline{R}_1$. Since there are no edges between R_1 and R_2 in R, this implies that $G \subseteq \overline{R}$. $\square$

Lemma 5.11 *Let G and R be as given in Corollary 5.10, and suppose $|R_2| = q + a$ where $a = \alpha(R_2)$, and $q' = |R| - (2(t - \overline{\alpha}' - 1) + 1)$. If*

(i) $\overline{\alpha}' > \lfloor \frac{t-a}{2} \rfloor$ and $|R| - t \geq q$, or

(ii) $\overline{\alpha}' \leq \lfloor \frac{t-a}{2} \rfloor$ and $q' - q + a \geq 0$,

then $G \subseteq \overline{R}$.

Proof. Choose a set V_1 of $t - a$ vertices of G so that $\alpha'(\overline{G[V_1]})$ is maximized, and let $G_1 = G[V_1]$. We claim that $\alpha'(\overline{G_1}) = \min\{\lfloor \frac{t-a}{2} \rfloor, \overline{\alpha}'\}$. Clearly, this is an upper bound on the size of $\alpha'(\overline{G_1})$. Let M be a maximum matching in $\overline{G}$. To maximize $\alpha'(\overline{G_1})$, we begin by selecting for V_1 only those vertices that are saturated by M. If $t - a \geq 2\overline{\alpha}'$, then V_1 contains all vertices saturated by M, so $\alpha'(\overline{G_1}) \geq \overline{\alpha}'$. Otherwise, we can choose up to $t - a$ vertices that are saturated by M so that we have at least $\lfloor \frac{t-a}{2} \rfloor$ edges of M included. In this case, $\alpha'(\overline{G_1}) = \lfloor \frac{t-a}{2} \rfloor$.

Suppose that condition (i) holds. Then $|R_1| = |R| - (q+a) \geq q+t-(q+a) = t-a$. Since $\overline{\alpha}' > \lfloor \frac{t-a}{2} \rfloor$, we have $\alpha'(\overline{G_1}) = \lfloor \frac{t-a}{2} \rfloor$ which implies that $\max\{2((t-a)-\alpha'(\overline{G_1})-1)+1, t-a\} = t-a$, so Corollary 5.10 shows that $G \subseteq \overline{R}$.

If condition (ii) holds, then $\alpha'(\overline{G_1}) = \alpha'(\overline{G})$, and

$$|R_1| = |R| - (q+a) = q' + 2t - 2\alpha'(\overline{G}) - 1 - q - a$$
$$= q' - q + a + (2(t-a) - 2\alpha'(\overline{G}) - 1) \geq 2(t-a) - 2\alpha'(\overline{G_1}) - 1.$$

Thus, Corollary 5.10 again shows that $G \subseteq \overline{R}$. □

Finally we present a lemma which can be used when a sequence is not degree sufficient for a graph with at least three vertices of degree at least 2.

Lemma 5.12 *Let G be a graph of order t, and let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence such that $d_3 \leq 1$ and $n \geq 2(t - \alpha'(\overline{G}) - 1) + 1$. If $\alpha'(\overline{G}) \leq \frac{t-3}{2}$, then $\overline{\pi}$ is potentially G-graphic.*

Proof. Since $d_3 \leq 1$, $\pi = (d_1, d_2, 1^{(n-2-\ell)}, 0^{\ell})$. Every realization of this sequence is a graph $R = R_1 \cup R_2$, where $\Delta(R_1) \leq 1$ and R_2 is either empty (that is, $R_1 = R$), a star, two disjoint stars, or a double star. If there is any realization of π where $\Delta(R_1) = 0$, then there are at least t independent vertices in R and thus $G \subseteq \overline{R}$ ($\overline{\pi}$ is potentially G-graphic).

If $\Delta(R_1) = 1$ in every realization of π, we can apply Lemma 5.11. Clearly $q' \geq 0$ and, since R_2 is a star, two stars, or a double star, $a \geq 2$ and $q \leq 2$. This implies that $-q + a \geq 0$, so part (ii) of the lemma holds. Since $\overline{\alpha}' \leq \frac{t-3}{2}$ implies $2(t - \overline{\alpha}' - 1) + 1 \geq t + 2$, part (i) of the lemma also holds. Thus, $G \subseteq \overline{R}$. □

5.2.3 Potentially P_4-graphic sequences

Before we can address potential-Ramsey numbers for small graphs, we will prove the following characterization of potentially P_4-graphic sequences. This proof serves as an illustration of common degree-sequence techniques such as the use of 2-switches and residual sequences, as discussed in Chapter 1.

Proposition 5.13 *If $\pi = (d_1, \ldots d_n)$ is a nonincreasing graphic sequence of length $n \geq 4$, then π is potentially P_4-graphic if and only if $d_2 \geq 2$ and $d_4 \geq 1$.*

A key step in the proof is the next lemma.

Lemma 5.14 *If $\pi = (d_1, d_2, \ldots, d_m)$ is a nonincreasing graphic sequence such that $\pi_1 = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_m)$ is potentially P_n-graphic, then π is potentially P_{n+1}-graphic.*

Proof. Let H be a realization of π_1 containing P, a path of order n. Label the vertices of P with $w_1, \ldots, w_n$, such that $w_i \sim w_{i+1}$ for $1 \leq i \leq n-1$. Add a vertex v of degree d_1 to H by making v adjacent to the vertices of degree $d_2 - 1, \ldots, d_{d_1+1} - 1$, and call this new graph H'. Note that H' is a realization of π. If $v \sim w_1$ or $v \sim w_n$, then there is clearly a P_{n+1} in H'. Also note that if v has consecutive neighbors on P, then by detouring through v we get a P_{n+1}, so we can assume this does not happen.

Suppose v has two neighbors, x_1 and x_2, that are not on P. Then v, x_1, and x_2 are each not adjacent to w_1 or w_2, or we get a longer path. Exchange the edges vx_1 and w_1w_2 with the nonedges vw_2 and x_1w_1 to get a realization of π in which $x_2vw_2w_3 \cdots w_n$ is a path of order $n+1$.

Thus we may assume that v has all but at most one of its neighbors on P. Let w_i and w_j be two neighbors of v on P. If $w_i \nsim w_j$, then we can exchange the edges $w_{i-1}w_i$ and vw_j with the non-edges vw_{i-1} and w_iw_j to get $v \sim w_{i-1}$, which creates a path of order $n+1$. Thus, all neighbors of v on P must be adjacent. This means that if w_i is adjacent to v, then w_i has at least $d_1 - 2$ neighbors on P that are neighbors of v, two neighbors on P that are not neighbors of v, and one neighbor not on P (v itself) for a degree of at least $d_1 + 1$. But every vertex in H has degree at most d_1, so this is not possible. Thus, π is P_{n+1}-graphic. □

Now we can prove the desired characterization. Note that the sequence $\pi = (d_1, \ldots, d_n)$ is potentially P_3-graphic if and only if $d_1 \geq 2$.

Proof of Proposition 5.13. If π is P_4-graphic, then clearly the conditions must hold.

So assume $\pi = (d_1, d_2, \ldots, d_n)$ with $d_2 \geq 2$ and $d_4 \geq 1$. If $d_2 \geq 3$, then π_1 (as defined in Lemma 5.14) is potentially P_3-graphic, so Lemma 5.14 shows that π is potentially P_4-graphic.

If $d_2 = 2$, let v_1 and v_2 be vertices of degree d_1 and d_2, respectively, in a realization H of π. Let x and y be the neighbors of v_2. Suppose that $x = v_1$. If $v_1 \sim y$, then we have a 3-cycle v_1v_2y in H. Since $d_4 \geq 1$, there is a fourth vertex w in H and either $w \sim v_1$, $w \sim y$, or there is another edge, say wz, in H. In the first two cases, wv_1v_2y or wyv_1v_2 is a P_4. In the second case we can exchange the edges wz and v_1v_2 with nonedges wv_1 and zv_2 to get the path zv_2yv_1.

If $v_1 \nsim y$, then since $d_1 \geq 2$ there must be another vertex w that is adjacent to v_1, and we again get the path wv_1v_2y.

If $v_1 \nsim v_2$, but v_1 and v_2 have a common neighbor, say x, then v_1xv_2y is a P_4. If they do not have any common neighbors, then there are at least two vertices w, z adjacent to v_1. If $w \sim x$ (or y) then v_1wxv_2 is a P_4. If not, then we can exchange the edges v_1w and v_2x with the nonedges v_1v_2 and wx to get the path zv_1v_2y. Thus, π is potentially P_4-graphic. □

5.3 Potential-Ramsey numbers for small graphs versus arbitrary graphs

We are now prepared to determine $r_{pot}(H, G)$ for small H.

Proposition 5.15 *If G is any graph of order $t \geq 2$, then $r_{pot}(K_2, G) = t$.*

Proof. We know that $r_{pot}(K_2, G) \geq t$. To see that this is enough, let π be any graphic sequence of length t. If no realization of π contains K_2, then $\pi = (0^t)$, and the unique realization of $\overline{\pi}$ is K_t, which contains any graph on t vertices. So $r_{pot}(K_2, G) = t$. □

Given this result, we will no longer consider K_2 in the determination of the following potential-Ramsey numbers. That is, we will in general assume that all our graphs have order at least 3.

Proposition 5.16 *If G is a graph of order $t \geq 3$, then $r_{pot}(2K_2, G) = t + 1$ unless $G = K_t$, where $r_{pot}(2K_2, K_t) = t + 2$.*

Proof. If π is a graphic sequence of length n that is not potentially $2K_2$-graphic, then π is one of the following: $(1, 1, 0^{n-2})$, $(2, 2, 2, 0^{n-3})$, or $(2, 1, 1, 0^{n-3})$, which are the degree sequences of $K_2 \cup (n-2)K_1$, $K_3 \cup (n-3)K_1$, and $P_3 \cup (n-3)K_1$, respectively. The complements of each of these graphs are $K_n - e$, $K_n - K_3 = K_{n-3} \vee \overline{K}_3$, and $K_n - P_3$, respectively. If $G = K_t$, then G is a subgraph of $K_n - K_3$ or $K_n - P_3$ if and only if $n \geq t + 2$, and it is a subgraph of $K_n - e$ if and only if $n \geq t + 1$. Thus if $n \geq t + 2$, then $\pi \to (2K_2, K_t)$.

If G is not a complete graph of order t, then $\alpha'(\overline{G}) \geq 1$. Thus, G is a subgraph of $K_n - K_3$ or $K_n - P_3$ as long as $n \geq t + 1$, and G is a subgraph of $K_n - e$ if $n \geq t$. This means that if $n \geq t + 1$, then $\pi \to (2K_2, G)$. □

It is easy to see that a graphic sequence $\pi = (d_1, \ldots, d_n)$ is potentially P_3-graphic if and only if $d_1 \geq 2$. We will use this in the next result.

Proposition 5.17 *If G is a graph of order $t \geq 3$, then*

$$r_{pot}(P_3, G) = \begin{cases} 2(t - \alpha'(\overline{G}) - 1) + 1 & \text{if } \alpha'(\overline{G}) < t/2 \\ t & \text{if } \alpha'(\overline{G}) = t/2. \end{cases}$$

Proof. By Lemma 5.7, we know that $r_{pot}(P_3, G) \geq \max\{2(t - \overline{\alpha}' - 1) + 1, t\}$, so we only need to show that this is an upper bound.

Let $n = \max\{2(t - \overline{\alpha}' - 1) + 1, t\}$, and let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence that is not potentially P_3-graphic. Then $d_1 = 1$, so any realization of π has maximum degree 1 and Lemma 5.9 shows that G is a subgraph of its complement. Thus, $\pi \to (P_3, G)$, so $r_{pot}(P_3, G) \leq \max\{2(t - \overline{\alpha}' - 1) + 1, t\}$. Since $t > 2(t - \overline{\alpha}' - 1) + 1$ if and only if $\alpha'(\overline{G}) = t/2$, the result follows. □

In the above cases, that is for $H = K_2, 2K_2$, and P_3, the potential-Ramsey number $r_{pot}(H, G)$ is equal to the graph Ramsey number $r(H, G)$ for all G (see [29] for the graph Ramsey numbers).

Theorem 5.18 *Let G be a graph of order $t \geq 3$. Then*

$$r_{pot}(P_4, G) = \begin{cases} 2(t - \overline{\alpha}' - 1) + 1 & \text{if } \alpha'(\overline{G}) \leq \frac{t-3}{2} \\ t + 1 & \text{if } \alpha'(\overline{G}) > \frac{t-3}{2}. \end{cases}$$

Proof. Let π be a graphic sequence that is not potentially P_4-graphic. We begin by assuming that $\overline{\alpha}' \leq \frac{t-3}{2}$ and π has length $2(t - \overline{\alpha}' - 1) + 1$. Lemma 5.7 gives the desired lower bound on $r_{pot}(H, G)$, so we need only prove that $\overline{\pi}$ is potentially G-graphic.

Since π is not potentially P_4-graphic, Proposition 5.13 shows that either $d_2 = 2$ and $d_4 = 0$, or $d_2 \leq 1$. In the first case, π is uniquely realized by $K_3 \cup (2t - 2\overline{\alpha}' - 4)K_1$. The complement of this graph is $K_{2t-2\overline{\alpha}'-4} \vee 3K_1$, which contains a K_t. Thus, $\overline{\pi}$ is potentially G-graphic for any graph of order t. If $d_2 \leq 1$, then $d_3 \leq 1$ and Lemma 5.12 shows that $\overline{\pi}$ is potentially G-graphic.

Now suppose $\overline{\alpha}' \geq \frac{t-2}{2}$ and π has length $t+1$. If $\overline{\alpha}' > \frac{t-3}{2}$, then $2(t-\overline{\alpha}'-1)+1 \leq t+1 \leq r_{pot}(P_4, G)$, by Lemma 5.6. Lemma 5.9 shows that if $d_1 \leq 1$, then $\overline{\pi}$ is potentially G-graphic. If $d_2 \leq 1$, and there is some realization of π with no components isomorphic to K_2, then there are at least t independent vertices in this realization and we are done. On the other hand, if $d_2 \leq 1$ and there are components of order two in every realization of π, then we can apply Lemma 5.11. We have $q = 1$, $a \geq 2$, and $\overline{\alpha}' \geq \frac{t-2}{2} \geq \lfloor \frac{t-a}{2} \rfloor$. The observation that $|R| - t = 1$ completes the proof.

Otherwise $d_2 = 2$ and $d_4 = 0$, so the unique realization of π is $R = K_3 + (t-2)K_1$ which means $\overline{R} = 3K_1 \vee K_{t-2}$. Since $\overline{\alpha}' \geq \frac{t-2}{2}$, there are two nonadjacent vertices in G of degree at most $t-2$. Identify these two vertices with two of the vertices of degree $t-2$ in $\overline{R}$, and identify the remaining vertices of G with the vertices of degree $t+2$ in $\overline{R}$. This mapping shows that $G \subseteq R$, so $\pi \to (P_4, G)$ and $r_{pot}(P_4, G) = t+1$. □

Finally, we address $r_{pot}(K_3, G)$. We restate the characterization of potentially K_3-graphic sequences from Chapter 2 for reference.

Theorem 5.19 (Luo [92]) *Let $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $n \geq 3$. Then π is potentially K_3-graphic if and only if $d_3 \geq 2$ except for two cases: $\pi = (2^4)$ and $\pi = (2^5)$.*

First we will concern ourselves with the general case where $t \geq 5$, and then we will prove a result for smaller graphs.

Theorem 5.20 *If G is a graph of order $t \geq 5$, then*

$$r_{pot}(K_3, G) = \begin{cases} 2(t - \alpha'(\overline{G}) - 1) + 1 & \text{if } \alpha'(\overline{G}) \leq \frac{t-3}{2} \\ t + 2 & \text{if } \alpha'(\overline{G}) > \frac{t-3}{2} \text{ and } \delta(G) \geq \lceil \frac{t+1}{2} \rceil \\ t + 1 & \text{if } \alpha'(\overline{G}) > \frac{t-3}{2} \text{ and } \delta(G) < \lceil \frac{t+1}{2} \rceil. \end{cases}$$

Proof. Since $t \geq 5$, Lemma 5.6 shows that we only need to consider sequences of length at least 6. Lemmas 5.6, 5.7, and 5.8 give us the appropriate lower bounds in each case. We thus need to show that these lower bounds are also upper bounds. First suppose $\overline{\alpha}' \leq \frac{t-3}{2}$ and let π be a graphic sequence of length $2(t - \overline{\alpha}' - 1) + 1$ that is not potentially K_3-graphic. By Theorem 5.19, the third term in π must be at most 1, and Lemma 5.12 shows that $\pi \to (K_3, G)$.

Now consider the case $\overline{\alpha}' > \frac{t-3}{2}$. If $\delta(G) \geq \lceil \frac{t+1}{2} \rceil$, let $\pi = (d_1, \ldots, d_{t+2})$ be a K_3-free graphic sequence. Since $t + 2 \geq 7$, we must have $d_3 \leq 1$. This means that every realization R of π has the form $R_1 \cup R_2$, where $\Delta(R_1) \leq 1$ and R_2 is either empty (with $R_1 = R$), a star, two disjoint stars, or a double star. If there is a realization in which $\Delta(R_1) = 0$, then there are at least t independent vertices in R, so $\overline{R}$ contains K_t and $G \subseteq \overline{R}$.

If $\Delta(R_1) = 1$ for any realization of π, then we can apply Lemma 5.11 with $q' \geq 1$, $a \geq 2$, and $q \leq 2$. Since $a \geq 2$, we are in the case where $\overline{\alpha}' > \lfloor \frac{t-a}{2} \rfloor$. Since $|R| = t + 2$ and $q \leq 2$, we have $G \subseteq \overline{R}$. This concludes the case where $\delta(G) \geq \lceil \frac{t+1}{2} \rceil$.

If $\overline{\alpha}' > \frac{t-3}{2}$ and $\delta(G) < \lceil \frac{t+1}{2} \rceil$, then $t + 1$ is a lower bound on $r_{pot}(K_3, G)$. Let $\pi = (d_1, \ldots, d_{t+1})$ be a K_3-free graphic sequence. If $d_1 = 1$, then Lemma 5.9 implies that $\pi \to (K_3, G)$. If $d_1 > 1$ and $d_2 \leq 1$, then we can again use Lemma 5.11 with $q = 1$ and $a \geq 2$. Since $\overline{\alpha}' > \frac{t-3}{2}$, we only need to satisfy part (i) of the lemma, and since π has length $t + 1$, $|R| - t = 1 = q$ for any realization R of π.

Finally, consider the case where $\pi = (d_1, d_2, 1^r, 0^{t-r-1})$ and d_1 and d_2 are at least 2. By the Erdős-Gallai criteria (Theorem 1.3), we must have $d_1 + d_2 \leq r + 2$, and $d_1 + d_2$ must have the same parity as r. If $d_1 + d_2 = r + 2$, then the unique realization of π is a double star and isolated vertices. Let u_1 and u_2 be the vertices of degree d_1 and d_2, respectively, in a realization R of π. Obviously $u_1 \sim u_2$, and u_2 has at most $\lfloor \frac{r}{2} \rfloor \leq \lfloor \frac{t-1}{2} \rfloor$ neighbors other than u_1. Since $\delta(G) < \lceil \frac{t+1}{2} \rceil$, there is a vertex w in G with at least $\lceil \frac{t}{2} \rceil - 1 = \lfloor \frac{t-1}{2} \rfloor$ non-neighbors in G. Let $\phi : V(G) \to V(R)$ send w to u_2 and $d_2 - 1$ non-neighbors of w to $N_R(u_2) \setminus \{u_1\}$ and maps all other vertices to $N_R(u_1)$ and the isolated vertices of R arbitrarily. This mapping shows that $G \subseteq \overline{R}$.

If $d_1 + d_2 \leq r$, then there is a realization of π consisting of two (disjoint) stars, disjoint edges, and isolated vertices. Let $R = R_1 \cup R_2$ be such a realization of π, where R_2 is the subgraph consisting of the two stars, and R_1 is the remainder of the graph. We want to apply Corollary 5.10 to show that $G \subseteq \overline{R}$. To do this, we want to find subgraphs G_1 and G_2 of G such that G_2 has order $d_1 + d_2 = a$ and is a subgraph of $\overline{R_2}$, and $\alpha'(\overline{G_1}) \geq \frac{t-a-1}{2}$. If this can be done, we will have $|R_1| = |G_1| = t - a \geq \max\{2(t-a) - \alpha'(\overline{G_1}) - 1, t - a\}$.

To this end, let w be a vertex of minimum degree in G such that w has at least $\frac{t-1}{2} \geq d_2$ nonneighbors in G, let M be a maximum anti-matching in G, and let ϕ be a homomorphism that sends w to u_2 and d_2 nonneighbors of w to the neighborhood of u_2. This uses the endpoints of at most d_2 edges of M. Since $\overline{\alpha}' \geq \frac{t-2}{2}$, this leaves at least

$$\frac{t-2}{2} - d_2 = \frac{t-2-2d_2}{2} \geq \frac{t-d_1-d_2-2}{2} = \frac{t-a-1}{2}$$

edges of M where both endpoints have not yet been used. Since the vertices in the neighborhood of u_1 are independent, we can choose edges of M to map to edges of R_1, and then choose the vertices that map to $N_R(u_1)$ to be any vertices that are left after this choice. This gives us G_1 with $\alpha'(\overline{G_1}) \geq \frac{t-a-1}{2}$, so Corollary 5.10 gives us our result. □

Finally, we need to consider K_3 versus graphs of order less than 5. Graphs of order 3 and disconnected graphs of order 4 are addressed by Propositions 5.5, 5.15, 5.16 and 5.17.

Proposition 5.21 *The following are potential Ramsey numbers for K_3 versus connected graphs of order 4:*

- $r_{pot}(K_3, K_4) = 7$
- $r_{pot}(K_3, P_4) = 5$
- $r_{pot}(K_3, C_4) = 6$
- $r_{pot}(K_3, K_{1,3}) = 6$
- $r_{pot}(K_3, K_4 - e) = 6$
- $r_{pot}(K_3, K_4 - P_3) = 6$

Proof. The values of $r_{pot}(K_3, K_4)$ and $r_{pot}(K_3, P_4)$ follow from Theorem 5.2 and Theorem 5.18, respectively.

For the others, let $\pi = (d_1, \ldots, d_6)$ be a graphic sequence that is not potentially K_3-graphic. If $d_1 = d_6 = 1$, then π is realized by a matching of size 3 and $\overline{\pi}$ is realized by $K_6 - 3K_2$, which contains a copy of $K_4 - e$. If $d_1 > 1$, then $d_3 = 1$, and there are at least 4 independent vertices in any realization of π. Thus any realization of $\overline{\pi}$ contains K_4 and if G is any non-complete graph of order 4, $r_{pot}(K_3, G) \leq 6$.

To see that this is also a lower bound on $r_{pot}(K_3, G)$, consider $\pi = (2^5)$. Since $\overline{\pi} = (2^5)$, $\pi \not\rightarrow (K_3, G)$ if G is any of C_4, $K_{1,3}$, $K_4 - e$, or $K_4 - P_3$. Thus $r_{pot}(K_3, G) \geq 6$ in each of these cases, and we get our result. □

5.4 Potential-Ramsey numbers for cycles

In this section, we will establish $r_{pot}(C_s, C_t)$ for all values of t and s with $t \geq s \geq 3$. For the sake of comparison, here is what is known about the Ramsey number for cycles:

Theorem 5.22 (Károlyi and Rosta [67]) *Let $3 \leq s \leq t$ be integers. Then*

$$r(C_s, C_t) = \begin{cases} 6 & \text{if } s = t = 3 \text{ or } 4 \\ t + s/2 - 1 & \text{if } s, t \text{ are even} \\ \max\{t + s/2 - 1, 2s - 1\} & \text{if } t \text{ is odd, } s \text{ is even} \\ 2t - 1 & \text{otherwise.} \end{cases}$$

Our results begin with some special cases. When $s = 3$, we have already computed $r_{pot}(C_3, G)$ for all G. In particular, this yields $r_{pot}(C_3, C_3) = r_{pot}(C_3, C_4) = 6$ and $r_{pot}(C_3, C_t) = t + 1$ for $t \geq 5$.

When $s = 4$, we have:

Proposition 5.23 *For $t \geq 5$, $r_{pot}(C_4, C_t) = t + 1$. When $t = 4$, $r_{pot}(C_4, C_4) = 6$.*

Proof. To determine $r_{pot}(C_4, C_t)$, for $t = 4$ and $t \geq 6$, we can use Lemma 5.1. Chvátal and Harary [28] showed that $r(C_4, C_4) = 6$, and C_5 is a self-complementary graph of order 5 that does not contain C_4. Since $\pi(C_5) = (2^5)$ has only one realization, we see that $r_{pot}(C_4, C_4)$ is also 6.

Chartrand and Schuster [12] showed that $r(C_4, C_t) = t + 1$ for $t \geq 6$, with $K_{1,t-1}$ as the critical graph. The degree sequence of $K_{1,t-1}$ has no other realizations, and the graph does not contain C_4. The complement of $K_{1,t-1}$ is $K_{t-1} \cup K_1$, which does not contain C_t. So we again use Lemma 5.1 to see $r_{pot}(C_4, C_t) = t + 1$.

Unfortunately, $r(C_4, C_5) = 7$ [42], and we wish to show that $r_{pot}(C_4, C_5) = 6$, so we can no longer use Lemma 5.1. For the lower bound, we consider $\pi(K_{1,4}) = (4, 1^4)$, which is not potentially C_4-graphic and whose complement is not potentially C_5-graphic. Thus, we know $r_{pot}(C_4, C_5) \geq 6$.

Let π be any graphic sequence of length 6 that is not potentially C_4-graphic. By Theorem 2.14, we have the following possibilities for π:

- $d_4 \leq 1$. In this case, π is one of the following sequences: $(d_1, d_2, d_3, 1^3)$, $(d_1, d_2, d_3, 1^2, 0)$, $(d_1, d_2, d_3, 1, 0^2)$, or $(d_1, d_2, d_3, 0^3)$. In order for each of these to be graphic, we must have $d_2 \leq 3$ by the Erdős-Gallai Criterion. For each sequence, $\overline{\pi}$ is: $(4^3, 5 - d_3, 5 - d_2, 5 - d_1)$, $(5, 4^2, 5 - d_3, 5 - d_2, 5 - d_1)$, $(5^2, 4, 5 - d_3, 5 - d_2, 5 - d_1)$, or $(5^3, 5 - d_3, 5 - d_2, 5 - d_1)$, respectively. Analyzing each sequence, we see that $\overline{\pi}$ satisfies conditions (1) through (3) of Theorem 2.15, so they are all potentially C_5-graphic.

- $d_1 = 5$ and $d_2 \leq 2$. If this happens and we are not in the first case, then $\pi = (5, 2^4, 1)$ and $\overline{\pi} = (4, 3^4, 0)$. Again $\overline{\pi}$ satisfies the conditions of Theorem 2.15, so it is potentially C_5-graphic.

- $\pi = (2^6)$. In this case, $\overline{\pi} = (3^6)$, which does have a realization containing C_5, so $\pi \to (C_4, C_5)$.

Thus, $r_{pot}(C_4, C_5) = 6$. □

The proof of the main theorem in this section uses a standard technique in the degree-sequence literature called the edge-exchange. This is a generalization of the 2-switch described in Chapter 1. An *alternating circuit* in a graph G is a circuit C with edges $e_1 e_2 \cdots e_{2p}$ such that for even i, the edge $e_i \in E(G)$ and for odd i, the edge $e_i \notin E(G)$. Exchanging the edges for the nonedges (that is, removing the edges from G and adding the nonedges to G) yields a graph with the same degree sequence as G. This allows us to change realizations of a degree sequence to obtain desirable properties, but gives us a little more freedom than the standard 2-switch.

Theorem 5.24 *For $t \geq s \geq 3$, the potential-Ramsey number for cycles is $r_{pot}(C_s, C_t) = t + \lfloor \frac{s-1}{2} \rfloor$, except for $r_{pot}(C_4, C_4) = 6$, $r_{pot}(C_3, C_3) = 6$, and $r_{pot}(C_3, C_4) = 6$.*

Proof. Given the above propositions, we may assume $s \geq 5$. Let $k = \lfloor \frac{s-1}{2} \rfloor$. To see that $r_{pot}(C_s, C_t) \geq t + k$, consider the degree sequence $((k + t - 2)^k, k^{t-1})$, which has the unique realization $G = K_k \vee \overline{K}_{t-1}$. Any subgraph of order s in G must use at least $s - k$ vertices from the independent set, forcing there to be an independent set of order $s - k$ in the graph. However, $\alpha(C_s) = \lfloor \frac{s}{2} \rfloor < s - k$, so C_s is not a subgraph of G. The complement of G is $\overline{K}_k \cup K_{t-1}$, which does not contain C_t. So $r_{pot}(C_s, C_t) \geq t + k$.

Now we must show that $t + k$ is an upper bound on $r_{pot}(C_s, C_t)$. To that end, let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence of length $t + k$. First suppose that every realization of π is acyclic; then in particular π is not potentially C_3-graphic. Since $s \geq 5$, we know $t + k \geq 7$, so Theorem 2.13 implies that $d_i \leq 1$ for $i \geq 3$. Thus, there are at least t terms of value at most 1 in π, and in any realization of $\overline{\pi}$, there are at least t vertices of degree at least $t + k - 2$. The subgraph induced by the t vertices of highest degree in any realization of $\overline{\pi}$ thus has minimum degree at least $t - 2$. Since $t \geq 5$, we have $t - 2 \geq t/2$, and Dirac's Theorem (Theorem 2.7) implies that this subgraph is Hamiltonian. Thus, there is a cycle of order t in any realization of $\overline{\pi}$, so $\pi \rightarrow (C_s, C_t)$.

Now suppose no realization of π contains a cycle of order s. We consider the following cases.

Case 1: Some realization of π contains a C_{s+1}.

Let G be such a realization of π with cycle $C = v_1 v_2 \cdots v_{s+1}$. For each i , $1 \leq i \leq s + 1$, the edge $v_i v_{i+2}$ is not in G, because such an edge would create an s-cycle. If $v_i v_{i+3} \in E(G)$ for some i, we can exchange the edges $v_i v_{i+3}$ and $v_{i+1} v_{i+2}$ with the nonedges $v_i v_{i+2}$ and $v_{i+1} v_{i+3}$ to create a realization of π with an s-cycle. So we also have $v_i v_{i+3} \notin E(G)$ for any i.

Let $d = \max_{x \in V(G) \setminus V(C)} d_C(x)$. Suppose first that $d = 0$. If there exists an edge xy in $G \setminus C$, then $v_1 v_3 v_2 x y v_2 v_1$ is an alternating circuit in G. Exchanging the edges in this circuit creates a realization in which $v_1 v_3 v_4 \cdots v_{s+1}$ is an s-cycle. If there is no such edge, then $G \setminus C$ is a set of isolated vertices. Let $\{w_1, w_2, \ldots, w_{t+k-(s+1)}\} = V(G) \setminus V(C)$. Since the vertices that are not on C have no neighbors on C, we have a t-cycle in $\overline{G}$. If s is even, one such cycle is $w_1 w_2 \cdots w_{t-k-2} v_2 v_4 \cdots v_{2k+2} v_1 w_1$, and if s is odd, a possible cycle is $w_1 w_2 \cdots w_{t-k-2} v_2 v_4 \cdots v_{2k+2} v_3 w_1$.

Now suppose $d > 0$. If $xv_i \in E(G)$, then $xv_{i+3} \notin E(G)$ because then we would have an s-cycle $v_i x v_{i+3} v_{i+4} \cdots v_{i-1}$. Thus, there must be an index j such that $xv_j \in E(G)$ and $xv_{j+1} \notin E(G)$. We can then exchange the edges xv_j and $v_{j+1}v_{j+2}$ with the nonedges xv_{j+1} and $v_j v_{j+2}$, to get a realization of π containing an s-cycle.

Case 2: Some realization of π contains a C_{s+2}.

Let $C = v_1 v_2 \cdots v_{s+2}$ be an $(s+2)$-cycle in a realization G of π. By Case 1, we can assume that G does not contain a C_{s+1}. Then, for each index i, we again have that $v_i v_{i+2}$ and $v_i v_{i+3}$ are not edges in G. If there is an $x \in V(G) \setminus V(C)$ such that xv_i and xv_{i+3} are both edges in G, then $v_i x v_{i+3} v_{i+4} \cdots v_{i-1}$ is an $(s+1)$-cycle in G. So either there is an $x \in V(G) \setminus V(C)$ and an index j such that $xv_j \in E(G)$ and $xv_{j+1} \notin E(G)$, or no $x \in V(G) \setminus V(C)$ has neighbors in C. In the first case, we can exchange edges xv_j and $v_{j+1}v_{j+2}$ with nonedges xv_{j+1} and $v_j v_{j+2}$ to get an $(s+1)$-cycle in G.

In the second case, if there is an edge xy in $G \setminus C$, then $v_1 v_3 v_2 x y v_2 v_1$ is an alternating circuit in G, and exchanging the edges in this circuit yields an $(s+1)$-cycle $v_1 v_3 v_4 \cdots v_{s+2}$. If there are no edges in $G \setminus C$, then we can find a t-cycle in $\overline{G}$ as we did in Case 1: If s is even, we have $w_1 w_2 \cdots w_{t-k-2} v_2 v_4 \cdots v_{2k+2} v_{2k+4} w_1$, and if s is odd, we have $w_1 w_2 \cdots w_{t-k-2} v_2 v_4 \cdots v_{2k+2} v_1 w_1$.

Case 3: Some realization of π contains a C_p, where $p \geq s+3$.

Choose a realization G of π so that p is maximal, and assume (by Case 2) that G contains no C_{s+2}. We may assume $v_1 v_s \notin E(G)$, because that would create an

s-cycle. If $v_pv_{s+1} \in E(G)$, then we have the $(s+2)$-cycle $v_1v_2 \cdots v_{s+1}v_p$, so we can assume that this edge is also not in G. Thus, we can exchange the edges v_pv_1 and v_sv_{s+1} for the non-edges v_1v_s and $v_{s+1}v_p$ to get a realization of π with an s-cycle.

Case 4: Every realization of π has circumference at most $s-1$.

Let G be a realization of π containing a longest cycle $C = v_1 \cdots v_p$, where $p \leq s-1$ and is maximum over all realizations of π. Let H be the subgraph of G induced by $V(G)\backslash V(C)$. The next two claims, which develop the structure of H, are similar to those in [9].

Claim 5.1 *H has no cycles.*

Suppose $x_1 \cdots x_\ell$ is a cycle in H. If x_i is adjacent to v_j, then we must have $v_{j+1}x_i \notin E(G)$ and $v_{j+2}x_{i+1} \notin E(G)$, for in the first case, we get the cycle $v_1 \cdots v_jx_iv_{j+1} \cdots v_p$ and in the second case we get the cycle $v_1 \cdots v_jx_ix_{i+1}v_{j+2} \cdots v_p$. Each of these is a cycle of length $p+1$ in G, contradicting our hypothesis. Thus, we can exchange the edges v_jv_{j+1} and x_ix_{i+1} for the nonedges x_iv_{j+1} and $x_{i+1}v_{j+1}$ to get a realization of π with a cycle of length $p+\ell$, contradicting our maximal choice of G and p. If no vertex on a cycle in H has a neighbor on C, then we can do the same edge-exchange. Thus H must not contain any cycles.

Claim 5.2 *If $\Delta(H) > 1$, then the only nontrivial component of H is a star.*

First suppose that there are at least two vertices, x and y, in H with $d_H(x) > 1$ and $d_H(y) > 1$. If such x and y cannot be found in the same component of H, then they are each the center of a star, so there are vertices x' and y' such that xx' and yy' are edges of H but xy and $x'y'$ are not edges. We can then exchange the edges with the nonedges so that x and y are in the same component, and are in fact adjacent. Thus we can assume that $xy \in E(G)$.

Since H has no cycles, there are vertices $x' \in N_H(x) \setminus \{y\}$ and $y' \in N_H(y) \setminus \{x\}$ such that $x' \not\sim y'$ and $x' \neq y'$. Since C is maximal in G, there must be a vertex v_i on

C such that $x \not\sim v_i$ and $y \not\sim v_{i+1}$. Thus, $v_i x x' y' y v_{i+1} v_i$ is an alternating circuit in G; exchanging the edges and nonedges on this cycle yields a cycle of length $p+2$ in G, contradicting our maximal choice of G.

This means that there is at most one vertex of degree greater than 1 in H. Suppose that there are at least two nontrivial components in H. One of these components must be a single edge, say yy'. Let x be the vertex of degree greater than 1. If x misses consecutive vertices on C, say v_i and v_{i+1}, then $v_i x x' x'' x v_{i+1} v_i$ is an alternating circuit in G, where x' and x'' are neighbors of x in H. Exchanging edges on this alternating circuit yields a cycle of length $p+1$, again contradicting the choice of G. Clearly, x is not adjacent to consecutive vertices on C, else we would already have a longer cycle. So we may assume, without loss of generality, that $x \not\sim v_1$ but xv_2 and xv_p are edges in G. Now, if $y \not\sim v_2$, then $v_1 v_2 y y' x' x v_1$ is an alternating circuit in G, and exchanging the edges on this alternating circuit allows us to extend C by using the path $v_1 x v_2$. Similarly, if $y' \not\sim v_p$, then exchanging edges on the alternating circuit $yy'v_p v_1 x x' y$ allows us to extend C by using the path $v_p x v_1$. Thus, we must have $y \sim v_2$ and $y' \sim v_p$. However, this allows us to extend C by replacing the path $v_p v_1 v_2$ with the path $v_p y' y v_2$, once again contradicting our choice of G. Thus, we cannot have two nontrivial components in H.

Claim 5.3 *If $\Delta(H) > 1$, then $\pi \to (C_s, C_t)$.*

The previous claims show that if $\Delta(H) > 1$, then the only nontrivial component of H is a star. Let x be the single vertex of degree greater than 1 in H, and let $y_1, \ldots, y_\ell$ be the neighbors of x in H. If $x \sim v_i$ for any $v_i \in C$, then $v_{i+1} \nsim y_j$ for $1 \leq j \leq \ell$, and $v_{i+1} \nsim x$, because otherwise we would have a longer cycle in G. If x has two consecutive non-neighbors in C, say v_i and v_{i+1}, then $v_i v_{i+1} x y_1 y_2 x v_1$ is an alternating circuit in G; exchanging the edges and nonedges of this circuit creates a realization of π with a longer cycle. Thus, x must be adjacent to precisely every other vertex on C, which implies that p is even.

Suppose, without loss of generality, that x is adjacent to v_i for every odd i. If z is an isolated vertex in H, then $z \nsim v_{i+1}$ for any odd i, because this creates the alternating circuit $xv_{i+1}zxy_1y_2x$, whose edges we can exchange to create a realization of π with a longer cycle. Thus, if i is even then v_i has no neighbors in H. Also, $v_i \nsim v_{i+2}$ for any even i, because such an edge gives a cycle $v_{i-1}xv_{i+1}v_iv_{i+2}v_{i+3}\cdots v_{i-2}$ in G of length $p+1$. Now we can find a cycle of length $t+k-\frac{p}{2}$ in $\overline{G}$: if $z_1, z_2, \ldots z_m$ are the isolates in H, let $C' = v_2xv_4y_1\cdots y_\ell z_1\cdots z_m v_6v_8\cdots v_p$. Since $k-\frac{p}{2} \geq 0$, it is possible that C' is longer than necessary; we can shorten it to length t by removing vertices from the star or isolates. This establishes the claim.

Now we assume $\Delta(H) \leq 1$. If $|H| \geq t$, then we can find a t-cycle in $\overline{G}$ by taking any set of t vertices from H, and we are done. If not, then $|H| < t$ implies that $p > k$ (since $|H| = t+k-p$), which further implies that $|H| \geq t - \lceil\frac{s-1}{2}\rceil \geq \lfloor\frac{s-1}{2}\rfloor + 1 \geq \lceil\frac{s-1}{2}\rceil \geq \lceil\frac{p}{2}\rceil$. Our goal is to find a t-cycle in $\overline{G}$. We will do this by finding a cycle in $\overline{G}$ that alternates between vertices of C and vertices of H until it uses $\lceil\frac{p}{2}\rceil$ vertices of C. Since

$$\begin{aligned}
|H| + \left\lceil\frac{p}{2}\right\rceil &= t+k-p+\left\lceil\frac{p}{2}\right\rceil \\
&= t+k-\left\lfloor\frac{p}{2}\right\rfloor \\
&= t+\left\lfloor\frac{s-1}{2}\right\rfloor - \left\lfloor\frac{p}{2}\right\rfloor \\
&\geq t
\end{aligned}$$

adding further vertices from H will result in a cycle of length t.

We begin by examining the structure of H in relation to C. Assume C has an orientation in the direction of increasing indices. First note that no $w \in V(H)$ has two consecutive neighbors on C, for this would contradict the maximality of p. Call a vertex $v_i \in C$ *surrounded* if there are vertices $w_1, w_2 \in H$ such that $w_1 \sim v_i$ and

$w_2 \sim v_{i\pm1}$. (Note that $w_1 \sim w_2$ is possible.) In this case, we can exchange edges w_2v_{i-1} and w_1v_i with nonedges w_2v_i and w_1v_{i-1} to get a longer cycle using the path $v_iw_2v_{i+1}$. So there can be no surrounded vertices on C.

If some vertex $w \in V(H)$ has neighbors v_i and v_j on C, then $v_{i-1} \nsim v_{j-1}$, because such an edge would create the cycle $v_1C^+v_{i-1}v_{j-1}C^-v_iwv_jC^+v_1$, where C^- means we follow C against its orientation and C^+ means we follow C with its orientation. This cycle has length greater than p, so this contradicts our choice of G. Similarly, if $v_{i+1} \sim v_{j+1}$, the cycle $v_1C^+v_ixv_jC^-v_{i+1}v_{j+1}C^+v_1$ is a cycle of length $p+1$.

Among any three consecutive vertices on C, every pair of vertices in H has a common nonneighbor. Suppose z_1 and z_2 are any (possibly adjacent) vertices in H, and consider, without loss of generality, v_1, v_2, and v_3 on C. If z_1 is adjacent to both v_1 and v_3, then $z_2 \nsim v_2$, because then v_2 is surrounded. If z_1 misses a pair of consecutive vertices among v_1, v_2, and v_3, then since z_2 cannot be adjacent to consecutive vertices, z_2 must miss one of these. If $z_1 \sim v_2$, then z_2 cannot be adjacent to both v_1 and v_3 (or v_2 is surrounded), so again they have a common nonneighbor.

Let xy be an edge of H. If for some $v_i \in C$ we have $x \sim v_i$, then y is not adjacent to either v_{i+1} or v_{i+2}, because either of these would create a longer cycle in G. If $x \sim v_i$ while $y \nsim v_i$, we can exchange edges v_iv_{i+1} and xy with the nonedges xv_{i+1} and yv_i to get a longer cycle in G. So for every edge xy in H, $x \sim v_i$ implies $y \sim v_i$, and this implies that the endpoints of each edge in H are adjacent to at most every third vertex on C.

Suppose $xy \in E(H)$ such that $x, y \sim v_i$, and z is an isolate in H. If $z \sim v_{i+1}$, then $z \nsim v_i$, and exchanging edges yv_i and zv_{i+1} with the nonedges yv_{i+1} and zv_i creates a longer cycle going through xy. So $z \nsim v_{i+1}$. Similarly, $z \nsim v_{i-1}$.

Suppose x_1y_1 and x_2y_2 are edges in H, and suppose $x_1, y_1 \sim v_i$ for some i. Then $x_2, y_2 \nsim v_{i+1}$, because in this case exchanging the edges x_1y_1 and x_2y_2 with the nonedges x_1x_2 and y_1y_2 would create a longer cycle in G. Similarly, $x_2, y_2 \nsim v_{i+2}$.

If $x_2, y_2 \nsim v_i$, then again exchanging x_1y_1 and x_2y_2 for x_1x_2 and y_1y_2 will give us the edge x_1x_2 with only $x_1 \sim v_i$, which we have shown leads to a longer cycle in G. So x_2 and y_2 must also be adjacent to v_i; this is true for any edge in H, so every non-isolated vertex in H has the same neighborhood on C.

With this structure, we can now create a cycle C' in $\overline{G}$ of length $|H| + \lceil \frac{p}{2} \rceil$. Let $C = v_1v_2 \cdots v_p$. Suppose $\Delta(H) = 0$, and let $V(H) = \{w_1, \ldots, w_{|H|}\}$. Without loss of generality, we may assume w_1 and w_2 are such that $v_1 \nsim w_1$ and $v_1 \nsim w_2$. We begin C' with $w_1v_1w_2$, and add to C' by picking vertices from C greedily (according to the order of C) and alternating with vertices from H (always ending with a vertex in H). Suppose we have built C' up to the point where we have $C' = w_1v_1w_2 \cdots v_{m-1}w_l$ (so $v_{m-1}w_l$ is a nonedge of G). Let $H_l = H - \{w_1, \ldots, w_l\}$.

If there are at least two vertices in H_l, we proceed as follows. If $w_l \nsim v_m$ and some vertex w_{l+1} of H_l is also not adjacent to v_m, then we add v_mw_{l+1} to C'. Suppose that $w_l \nsim v_m$ but every vertex of H_l is adjacent to v_m. In this case, if w_l is also not adjacent to w_{m+1}, then we add $w_lv_{m+1}w_{l+1}$ to C', since no vertex in H_l can be adjacent to v_{m+1}. If on the other hand w_l is adjacent to v_{m+1}, then none of the vertices in H_l are adjacent to v_{m-1}. Here, we would replace w_l in C' with some z_1 from H_l, so that we have $v_{m-1}z_1v_{m+1}z_2$ to extend C', where z_2 is another vertex from H_l.

If $w_l \sim v_m$, let z_1 and z_2 be vertices of H_l. The vertices w_l and z_i (where i is either 1 or 2) must have a common nonneighbor in the set $\{v_{m-1}, v_m, v_{m+1}\}$, and since $w_l \sim v_m$, we know that w_l misses both v_{m-1} and v_{m+1}. If some z_i misses v_{m+1}, then we continue C' with the path $v_{m-1}w_lv_{m+1}z_i$. If every vertex of H_l is adjacent to v_{m+1}, then these vertices must miss both v_{m-1} and v_m. In this case, instead of using $v_{m-1}w_l$ in C', we continue the cycle with $v_{m-1}z_1v_mz_2v_{m+2}w_l$.

Thus, as long as there are at least two vertices remaining in H_l, we can continue building C' in this way. This construction ensures that at least every other vertex of C is included in C' – we never omit two consecutive vertices of C from C'. Also, if

$v_j w$ is the latest edge added to C', none of the vertices in $\{v_{j+1}, \ldots, v_p\}$ are in C'.

Now suppose $|H| > \lceil \frac{p}{2} \rceil$ and we have built C' with $\lceil \frac{p}{2} \rceil$ vertices from H and $\lceil \frac{p}{2} \rceil - 1$ vertices from C. We need to add one more vertex of C to C', and then we can collect $t - 2\lceil p/2 \rceil$ vertices of H to finish the cycle.

Let v_q be the last vertex of C that was added to C', and let $b = \lceil \frac{p}{2} \rceil$ (so the last edge of C' so far is $v_q w_b$). Since at least every other vertex of C must be used in C', we see that

$$q \leq 2(\left\lceil \frac{p}{2} \right\rceil - 1) - 1 = \begin{cases} p-3 & \text{if } p \text{ is even} \\ p-2 & \text{if } p \text{ is odd.} \end{cases}$$

If p is even, or if p is odd and $q \leq p-3$, then there are at least three consecutive vertices of C that aren't used in C' yet; thus, w_b and any other vertex in H must have a common non-neighbor among these three vertices. Adding this non-neighbor to C' gives us $\lceil \frac{p}{2} \rceil$ vertices of C and a last vertex in H. From here, the rest of H can be inserted into C' until C' is a cycle of length t in $\overline{G}$.

The remaining case to consider is if p is odd and $q = p-2$. If this happens, we must have skipped every possible vertex of C; that is, every v_i with i even is not used in C', and every v_j with j odd is used in C', up to $j = q = p-2$.

Now we see that the last edge in C' so far is $v_{p-2}w_b$. There is at least one vertex, say z, unused in H. If z and w_b have a common nonneighbor in $\{v_{p-1}, v_p\}$, we use that nonneighbor on a path from w_b to z and finish the cycle with vertices from H. Otherwise, if $w_b \nsim v_{p-3}$, and some $z \in H_b$ is not adjacent to v_{p-3}, then we continue C' with $w_b v_{p-3} z$, and finish with vertices of H. If every $z \in H_b$ is adjacent to v_{p-3}, then none of them is adjacent to v_{p-2}. In this case, instead of using $v_{p-2}w_b$ in C', we use $v_{p-2}z$. Thus we may assume $w_b \sim v_{p-3}$. If some $z \in H_b$ is adjacent to v_{p-2}, then v_{p-1} must be a nonneighbor of both w_b and z, so we can add $w_b v_{p-1} z$ to C', and finish the cycle with vertices of H. Finally we have the case where $z \nsim v_{p-2}$ and w_b and z have no common nonneighbors in $\{v_{p-1}, v_p\}$. We may assume that $w_b \sim v_{p-1}$

and $z \sim v_p$ (because if we instead have $w_b \sim v_p$ and $z \sim v_{p-1}$, we can 2-switch these edges to end in this case). Since v_{p-2} and v_p are then successors of neighbors of w_b, they are not adjacent. Thus, we may finish C' by taking the path $v_{p-2}v_pw_bz$ and then continuing with vertices of H.

Now we must consider the case where $|H| = \lceil \frac{p}{2} \rceil$. This implies that $p = s - 1$, s is even, and $s = t$. The creation of C' proceeds as above until we have added $\lceil \frac{p}{2} \rceil - 1$ vertices of H and $\lceil \frac{p}{2} \rceil - 2$ vertices of C to C'. Again, since at least every other vertex of C is added to C', the index, q, of the last vertex added to C' must be at most $2(\lceil \frac{p}{2} \rceil - 2) - 1 = p - 4$. Now we have $C' = w_1v_1 \cdots v_qw_{b-1}$, where $b = \lceil \frac{p}{2} \rceil$, and we need to add two more vertices of C to C', while ending back at w_1.

If w_{b-1} and w_b both miss v_{q+1}, then we continue C' with $w_{b-1}v_{q+1}w_b$. Then, there are three consecutive unused vertices (v_{q+2}, v_{q+3}, and v_{q+4}) among which w_b and w_1 must have a common non-neighbor. Adding this vertex to C' and returning to w_1 gives us the desired t-cycle. If both w_{b-1} and w_b miss v_{q+2}, and $w_b \sim v_{q+1}$, then we add $w_{b-1}v_{q+2}w_b$ to C'. Now, if w_b and w_1 have a common nonneighbor, say v', in $\{v_{q+3}, \ldots, v_p\}$, then we finish C' with $w_bv'w_1$. If not, we know that $q + 4 = p$ and we may assume that $w_b \sim v_{q+3}$ and $w_1 \sim v_{q+4}$. Since $w_b \sim v_{q+1}$, we know that v_{q+2} is not adjacent to v_{q+4}, so we finish C' with $v_{q+2}v_{q+4}w_bw_1$.

Next, suppose w_{b-1} and w_b miss v_{q+2} but $w_{b-1} \sim v_{q+1}$. If $w_b \nsim v_q$, then instead of using v_qw_{b-1} in C', we take $w_{b-2}v_qw_bv_{q+2}w_{b-1}$. We need one more vertex of C to finish C'; if w_{b-1} and w_1 have a common nonneighbor, v', in $\{v_{q+3}, \ldots, v_p\}$, then we finish C' with $w_{b-1}v'w_1$. If not, we know that $q + 4 = p$ and we may assume that $w_{b-1} \sim v_{q+3}$ and $w_1 \sim v_{q+4}$. Since w_{b-1} is adjacent to both v_{q+1} and v_{q+3}, we know that $v_{q+2} \nsim v_{q+4}$. Thus, we can end C' with $v_{q+2}v_{q+4}w_{b-1}w_1$.

We may therefore assume that $w_b \sim v_q$, and C' continues from w_{b-1} with $w_{b-1}v_{q+2}w_b$. Now we need one more vertex from C. If w_b and w_1 have a common nonneighbor in $\{v_{q+3}, \ldots, v_p\}$, then we use this vertex to finish C'. If not, we may once again assume that $w_b \sim v_{q+3}$ and $w_1 \sim v_{q+4}$. Since $w_b \sim v_q$, we see that $v_{q+1} \nsim v_{q+4}$. Exchange the edges $w_{b-1}v_{q+1}$ and w_1v_{q+4} with the nonedges $w_{b-1}w_1$ and $v_{q+1}v_{q+4}$. This yields the cycle $v_1C^+v_qw_bv_{q+3}C^-v_{q+1}v_{q+4}v_1$ of length $p+1$ in G, a contradiction. This means we must have the common nonneighbor we needed to finish the cycle.

Finally, suppose w_{b-1} and w_b have no common nonneighbor in $\{v_{q+1}v_{q+2}\}$. We may assume that $w_{b-1} \sim v_{q+1}$ and $w_b \sim v_{q+2}$ (or we can 2-switch to make this happen). If $w_1 \nsim v_{q+1}$, then we may end C' with $v_qw_{b-1}v_{q+3}w_bv_{q+1}w_1$. If $w_1 \nsim v_{q+2}$, then we end C' with $v_qw_bv_{q+3}w_{b-1}v_{q+2}w_1$.

Now we consider the case where $\Delta(H) = 1$. Let $x_1y_1, \ldots, x_ry_r$ be the edges of H and let $w_1, \ldots, w_\ell$ be the isolated vertices of H. As before, we will find a cycle, C', of length at least t in $\overline{G}$ by finding a cycle that uses $\lceil \frac{p}{2} \rceil$ vertices from C and the remaining vertices from H.

Let $C = v_1v_2 \cdots v_p$. We may assume that v_1 is such that $x_1 \nsim v_1$ and $x_1 \nsim v_2$, and we begin C' with $x_1v_1y_1v_2x_2$. If there is only one edge in H, then in $\{v_2, \ldots, v_p\}$, there must be a vertex that is not adjacent to both y_1 and w_1, so we use the lowest-indexed such vertex to return to H, then continue as in the $\Delta(H) = 0$ case. If $p < 5$, we already have $\lceil \frac{p}{2} \rceil$ vertices of C and we can finish C' by using the other vertices of H. If $p = 5$, then since edges of H can be adjacent to at most every third vertex on C, there must be some $v_j \nsim x_2$, and we add $x_2v_jy_2$, then finish C' with vertices of H.

If $p \geq 6$, then again since edges of H are adjacent to at most every third vertex on C, either v_3 or v_4 must be nonadjacent to x_2. We add $x_2v_jy_2v_{j+1}$ or $x_2v_jy_2v_{j+2}$, where $j = 3$ or 4 and we use the second option if $x_2 \sim v_{j+1}$, to continue C'. We continue this process, at each step adding the first vertex of C' that is available, until we have obtained $\lceil \frac{p}{2} \rceil$ vertices from C' and ended with a vertex from H.

If $2r \geq \lceil \frac{p}{2} \rceil$, we end this portion of the process with v_my_j in C' for some $m < p$ and $j < r$. Then, since every vertex in H has at most one neighbor in H, we can add vertices of H to C' until we have a cycle of length t, and we are done. If $2r < \lceil \frac{p}{2} \rceil$, then since $|H| \geq \lceil \frac{p}{2} \rceil$, there must be isolated vertices in H, and we have C' that ends with $v_m y_r$ at this point. Since we missed at most every third vertex of C, there must be at least three unused consecutive vertices on C, and in this set there must be a vertex v_j that is not adjacent to both y_r and w_1. We continue C' with $y_r v_j w_1$ and now proceed with adding vertices to C' as in the case where $\Delta(H) = 0$. Since fewer vertices of C can be missed when we are using edges of H (at most every third vertex, rather than at most every other), we will be able to find enough vertices of C to complete the cycle as desired.

This gives us a t-cycle in $\overline{G}$, so $\pi \to (C_s, C_t)$, and we have our result. □

Table 5.1 summarizes the general results given in this chapter. In the table, H is a fixed graph and G is a graph of order t with $t \geq |V(H)|$. The properties of G that affect the value of $r_{pot}(H, G)$ are given in the column under G.

Table 5.1: $r_{pot}(H, G)$ for small H and $r_{pot}(C_s, C_t)$

H	G	$r_{pot}(H, G)$
K_2	G	t
$2K_2$	K_t	$t+2$
	$G \neq K_t$	$t+1$
P_3	$\alpha'(\overline{G}) < t/2$	$2(t - \overline{\alpha}' - 1) + 1$
	$\alpha'(\overline{G}) = t/2$	t
P_4	$\alpha'(\overline{G}) \leq \frac{t-3}{2}$	$2(t - \overline{\alpha}' - 1) + 1$
	$\alpha'(\overline{G}) > \frac{t-3}{2}$	$t+1$
K_3	K_4	7
	P_4	5
	$C_4,\ K_{1,3},\ K_4 - e,\ K_4 - P_3$	6
	$\alpha'(\overline{G}) \leq \frac{t-3}{2}$	$2(t - \overline{\alpha}' - 1) + 1$
	$\alpha'(\overline{G}) > \frac{t-3}{2}$ and $\delta(G) \geq \lceil \frac{t-1}{2} \rceil$	$t+2$
	$\alpha'(\overline{G}) > \frac{t-3}{2}$ and $\delta(G) < \lceil \frac{t-1}{2} \rceil$	$t+1$
C_4	C_4	6
$C_s,\ s \geq 4$	$C_t,\ t \geq s,\ t \neq 4$	$t + \lfloor \frac{s-1}{2} \rfloor$

5.5 Potential-Ramsey numbers for all graphs of order at most 4

Inspired by the work of Chvátal and Harary in [29], we give in Table 5.2 a table of all potential-Ramsey numbers $r_{pot}(H, G)$ where H and G are each one of the ten graphs of order 4 that do not have isolated vertices.

Table 5.2: Small potential-Ramsey numbers

	K_2	P_3	$2K_2$	K_3	P_4	$K_{1,3}$	C_4	$K_{1,3}+e$	K_4-e	K_4
K_2	2	3	4	3	4	4	4	4	4	4
P_3		3	4	5	4	5	4	5	5	7
$2K_2$			5	5	5	5	5	5	5	6
K_3				6	5	6	6	6	6	7
P_4					5	5	5	5	5	7
$K_{1,3}$						6	6	6	7	8
C_4							6	6	7	8
$K_{1,3}+e$								6	7	8
K_4-e									7	8
K_4										8

The results in this paper yield the first five rows of this table, as well as the value for $r_{pot}(C_4, C_4)$. Theorems 5.2 and 5.3 give us $r_{pot}(K_4, K_4)$ and $r_{pot}(C_4, K_4)$, respectively. To determine the other values on this table, we make use of the following characterizations of potentially H-graphic sequences for small H.

Observation 5.25 *A nonincreasing graphic sequence $\pi = (d_1, \ldots, d_n)$ with $n \geq 4$ is potentially $K_{1,3}$-graphic if and only if $d_1 \geq 3$.*

Theorem 5.26 (Chen, Li [16]) *A nonincreasing graphic sequence $\pi = (d_1, \ldots, d_n)$ with $n \geq 4$ is potentially $K_{1,3}+e$-graphic if and only if $d_1 \geq 3$ and $d_3 \geq 2$.*

Theorem 5.27 (Eschen [41]) *Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence of length $n \geq 4$. Then π is potentially $(K_4 - e)$-graphic if and only if the following conditions hold:*

(a) $d_2 \geq 3$,

(b) $d_4 \geq 2$, *and*

(c) *if* $n = 5, 6$, *then* $\pi \neq (3^2, 2^{n-2})$ *and* $\pi \neq (3^6)$.

Theorem 5.28 (Luo [93]) *Let* $\pi = (d_1, \ldots, d_n)$ *be a nonincreasing graphic sequence of length* $n \geq 4$. *Then* π *is potentially* K_4*-graphic if and only if* $d_4 \geq 3$ *and* $\pi \neq (n-1, 3^s, 1^{n-s-1})$ *for each* $s = 4, 5$ *except the following sequences:*

- $n = 5$: $(4, 3^4)$, $(3^4, 2)$
- $n = 6$: (4^6), $(4^2, 3^4)$, $(4, 3^4, 2)$, (3^6), $(3^5, 1)$, $(3^4, 2^2)$
- $n = 7$: (4^7), $(4^3, 3^4)$, $(4, 3^6)$, $(4, 3^5, 1)$, $(3^6, 2)$, $(3^5, 2, 1)$
- $n = 8$: $(3^7, 1)$, $(3^6, 1^2)$.

Consider the sequence $\pi = (2^5)$. This sequence is not potentially H-graphic for any of C_4, $K_{1,3}$, $K_{1,3} + e$, $K_4 - e$, and K_4, and it is self-complementary. Thus, $r_{pot}(H, G) \geq 6$ if H and G are any of the graphs just listed. Recall that $r_{pot}(H, G) \leq r(H, G)$ for any graphs H and G. Since $r(K_{1,3}, K_{1,3}) = 6$ and $r(K_{1,3}, C_4) = 6$ [29] we see that $r_{pot}(K_{1,3}, K_{1,3}) = 6$ and $r_{pot}(K_{1,3}, C_4) = 6$ as well.

Similarly, the sequence $\pi = (2^6)$ is not potentially H-graphic for any of these graphs, and its complement, $\overline{\pi} = (3^6)$ is not potentially $(K_4 - e)$-graphic. Thus, $r_{pot}(H, K_4 - e) \geq 7$ for any of these H. Since $r(H, K_4 - e) = 7$ for $H = K_{1,3}, C_4$, and $K_{1,3} + e$, we see that the potential-Ramsey number in each of these cases is 7. To see that $r_{pot}(K_4 - e, K_4 - e) = 7$, consider the nonincreasing graphic sequence $\pi = (d_1, \ldots, d_7)$. By Theorem 5.27, we may assume that either $d_2 \leq 2$ or $d_2 \geq 3$ and $d_4 \leq 1$. If $d_2 \leq 2$, then in $\overline{\pi} = (\overline{d}_1, \ldots, \overline{d}_7)$ (which is rearranged to be nonincreasing) we must have $\overline{d}_6 \geq 4$, which means $\overline{\pi}$ is potentially $(K_4 - e)$-graphic. If $d_2 \geq 3$

and $d_4 \leq 1$, then $\overline{d}_4 \geq 5$, so $\overline{\pi}$ is also potentially $(K_4 - e)$-graphic, and we have our conclusion.

Now we determine $r_{pot}(K_{1,3} + e, H)$ for $H = K_{1,3}, C_4$, and $K_{1,3} + e$. If $\pi = (d_1, \ldots, d_6)$ is a nonincreasing graphic sequence that is not potentially $(K_{1,3} + e)$-graphic, then by Theorem 5.26, either $d_1 \leq 2$ or $d_3 \leq 1$. If $d_1 \leq 2$, then in $\overline{\pi}$ we see that $\overline{d}_6 \geq 3$, and if $d_3 \leq 1$, then $\overline{d}_4 \geq 4$. In either case, the sequence $\overline{\pi}$ satisfies the conditions to be potentially H-graphic for each H under consideration, so the potential-Ramsey number in each case is 6.

Finally we consider $r_{pot}(H, K_4)$ for $H = K_{1,3}, K_{1,3} + e$, and $K_4 - e$. For each such H, the sequence $\pi = (2^7)$ is not potentially H-graphic, and its complement, $\overline{\pi} = (3^7)$, is not potentially K_4-graphic. Thus, $r_{pot}(H, K_4) \geq 8$ in each case. For $n = 8$, we see that $\pi = (d_1, \ldots, d_n)$ is potentially K_4-graphic unless $d_4 \leq 2$ or π is one of the exceptions listed in Theorem 5.28. If $d_4 \leq 2$, then $\overline{d}_5 \geq 5$, and this is enough to guarantee that $\overline{\pi}$ is potentially H-graphic for each H we are considering. If π is one of the exceptional sequences, then $\overline{\pi}$ is one of $(6^3, 4^4, 0)$, $(6^2, 4^5, 0)$, $(7, 4^7)$, or $(7^2, 4^6)$, each of which is also potentially H-graphic for the given graphs. Thus $r_{pot}(K_4, H) = 8$ for $H = K_{1,3}, K_{1,3} + e$, and $K_4 - e$, completing the table.

6. Future Work

In Chapter 4, we discussed open problems in the area of σ-stability of graphs. In this chapter, we give problems related to other results presented in this dissertation.

6.1 Hypergraphic degree sequences

A hypergraph is a generalization of the graphs we have discussed so far, in which edges are no longer restricted to containing only two vertices; that is, a hypergraph H consists of a vertex set $V(H)$ and an edge set $E(H)$ where an edge can be any subset of $V(H)$. A hypergraph is called *k-uniform* or is a *k-graph* if every edge contains exactly k vertices, and *simple* if there are no repeated edges or loops (edges which contain a vertex more than once). Thus, the degree of a vertex in a simple hypergraph is the number of edges that contain it, as with standard graphs. A sequence π of nonnegative integers is *k-graphic* if there is a simple k-uniform hypergraph with π as its degree sequence.

In contrast to the field of graphic sequences, where there is a significant body of literature, much less is known about k-graphic sequences. In fact, the following theorem of Dewdney [31] is the only characterization of k-graphic sequences currently known.

Theorem 6.1 (Dewdney 1975) *Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers. π is k-graphic if and only if there exists a nonincreasing sequence $\pi' = (d'_2, \ldots, d'_n)$ of nonnegative integers such that*

1. *π' is $(k-1)$-graphic,*
2. *$\sum_{i=2}^{n} d'_i = (k-1)d_1$, and*
3. *$\pi'' = (d_2 - d'_2, d_3 - d'_3, \ldots, d_n - d'_n)$ is k-graphic.*

Recent work by this author and Behrens, et al. [6] has produced some new directions from which to approach problems relating to k-graphic sequences.

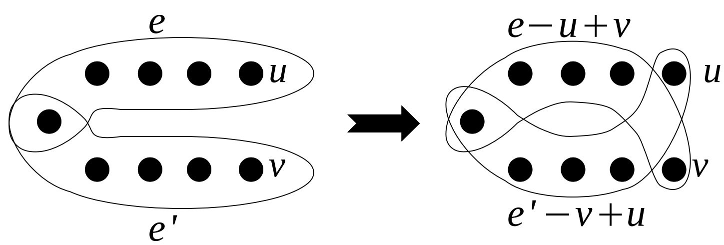


Figure 6.1: The edge-exchange $e \overset{u}{\underset{v}{\rightleftharpoons}} e'$

In particular, we showed that there is a family of edge-exchanges, similar to the 2-switch for 2-graphs, that can be used to move between realizations of a k-graphic sequence. Let e and e' be distinct edges in a k-graph G, and choose vertices $u \in e \setminus e'$ and $v \in e' \setminus e$. The operation $e \overset{u}{\underset{v}{\rightleftharpoons}} e'$ deletes the edges e and e' and adds the edges $e-u+v$ and $e'-v+u$ (where $e-u+v$ denotes the set $(e-\{u\})\cup\{v\}$); see Figure 6.1. Similar to the result of Petersen that showed that 2-switches suffice to transform any realization of a graphic sequence into any other, we showed that these edge-exchanges suffice to move between k-realizations of k-graphic sequences. However, in performing these edge-exchanges, it is sometimes necessary to create realizations of the sequence that are not simple k-graphs – they have repeated edges. This raises the following problem:

Problem 6.2 *Determine the minimal collection $\mathcal{Q}$ of edge exchanges such that for any k-graphic sequence π, there is a sequence of exchanges from $\mathcal{Q}$ that suffice to move between realizations of π using only simple k-graphs.*

Recently, Gu and Lai [52] proved a characterization of potentially r-edge-connected k-graphic sequences. Together with other results in [6], this shows that despite the difficulty in determining a simple characterization of k-graphic sequences, many other questions about k-graphic sequences can be answered.

One such question relates to a theorem of Kundu.

Theorem 6.3 (Kundu [72]) *Let $\pi = (d_1, \ldots, d_n)$ be a graphic sequence. There is a realization of π that contains a c-factor if and only if $\pi' = (d_1 - c, \ldots, d_n - c)$ is also graphic.*

The proof of this theorem uses only 2-switches. As such, it seems possible that with the family of edge-exchanges described here we may be able to prove a similar result on the existence of realizations of k-graphic sequences with c-factors.

I hope to study problems such as Problem 6.2 and Kundu's theorem for k-graphic sequences, as well as other potential problems for hypergraphic sequences, in the future.

6.2 Further exploration of potential-Ramsey numbers

The study of potential-Ramsey numbers is still in its infancy. As such, there are many questions that can be asked about these numbers.

One nice result in graph Ramsey theory is that if T is an n-vertex tree, then $r(T, K_t) = (n-1)(t-1)+1$, which was proved by Chvátal in 1977 [27]. I would like to determine $r_{pot}(T, K_t)$ for an n-vertex tree T. Many of our results on $r_{pot}(H, G)$ for small fixed H depend on the existence of a simple characterization of the potentially H-graphic sequences, which give us simple bounds on the exact values of some of the terms of a sequence. While it is easy to see when a graphic sequence is the degree sequence of a tree, the characterization does not say much about the values of specific terms. Thus, exploring $r_{pot}(T, K_t)$ may require some additional techniques.

In the same vein, determining $r_{pot}(mG, nH)$ for graphs G and H, where mG means m disjoint copies of G, may be much more difficult than determining $r_{pot}(G, H)$. Burr, Erdős, and Spencer [8] gave upper and lower bounds for $r(nG, nH)$ that depend only on n and the orders and independence numbers of G and H. If similarly simple results could be found for $r_{pot}(nG, nH)$ or even $r_{pot}(mG, nH)$, this would advance our understanding of the potential-Ramsey number.

While the potential-Ramsey number and Ramsey number are very far apart in some cases (such as cliques), it has been shown that many pairs of graphs have Ramsey numbers that are linear in the order of the graphs. Since we know that potential-Ramsey numbers are linear, it may be worthwhile to determine the potential-Ramsey numbers for such pairs of graphs, to compare these values to the Ramsey numbers.

As indicated in Chapter 5, we can also define the potential-Ramsey number for more than 2 colors. To define this, we must discuss what it means for graphic sequences to pack. Given graphic sequences $\pi_1, \ldots, \pi_k$ with $\pi_i = (d_1^{(i)}, \ldots, d_n^{(i)})$, we say that $\pi_1, \ldots, \pi_k$ *pack* if there exist edge-disjoint graphs $G_1, \ldots, G_k$ with vertex set $\{v_1, \ldots, v_n\}$ such that $d_{G_i}(v_j) = d_j^{(i)}$ for each i, and $d_{G_1 \cup \cdots \cup G_k}(v_j) = \sum_{i=1}^{n} d_j^{(i)}$. Note that the sequences $\pi_1, \ldots, \pi_k$ need not be in nonincreasing order, and in fact the order of the terms of the sequence may not be altered.

Now we define $r_{pot}(G_1, \ldots, G_k)$ to be the least integer n such that for any collection of graphic sequences $\pi_1, \ldots, \pi_k$, each of length n, that termwise sum to $n - 1$ and pack, there exist edge-disjoint graphs $F_1, \ldots, F_k$ that realize this packing and for some i, the graph F_i contains G_i as a subgraph.

The natural first question to ask in this vein is, what is $r_{pot}(K_3, K_3, K_3)$? Greenwood and Gleason showed that $r(K_3, K_3, K_3) = 17$ [51], but this is the only classical (complete graph) multicolor Ramsey number known. Given that 2-color potential-Ramsey numbers are perhaps easier to compute than 2-color Ramsey numbers, but degree-sequence packing problems are very difficult, it would be interesting to see if more progress can be made in the multicolor potential-Ramsey case.

REFERENCES

[1] R. Baber and J. Talbot, New Turán densities for 3-graphs. *Electron. J. Combin.* **19** (2012), no. 2, P22, 21 pp.

[2] D. Bauer, H.J. Broersma, J. van den Heuvel, N. Kahl, E. Schmeichel, Degree sequences and the existence of k-factors. *Graphs Combin.* **28** (2012), no. 2, 149–166.

[3] D. Bauer, H.J. Broersma, J. van den Heuvel, N. Kahl, E. Schmeichel, Toughness and vertex degrees. *J. Graph Theory* **72** (2013) no. 2, 209–219.

[4] M.D. Barrus, On 2-switches and isomorphism classes. *Discrete Math.* **312** (15) (2012) 2217–2222.

[5] M.D. Barrus, M. Kumbhat, and S.G. Hartke, Graph classes characterized both by forbidden subgraphs and degree sequences. *J. Graph Theory* **57** (2008) 131–148.

[6] S. Behrens, C. Erbes, M. Ferrara, S. Hartke, B. Reiniger, H. Spinoza and C. Tomlinson, New results on degree sequences of uniform hypergraphs. *Electron. J. Combin.* **20** (4) (2013), P14, 18pp.

[7] A. Blokhuis, A. Brouwer, T. Szönyi, and Z. Weiner, On q-analogues and stability theorems. *J. Geom.* **101** (2011), no. 1-2, 31–50.

[8] S.A. Burr, P. Erdős, and J. H. Spencer. Ramsey theorems for multiple copies of graphs. *Trans. Amer. Math. Soc.* **209** (1975), 87–99.

[9] A. Busch, M. Ferrara, S. Hartke and M. Jacobson, Ramsey-type numbers for degree sequences. To appear in *Graphs Combin.*

[10] A. Busch, M. Ferrara, S. Hartke, M. Jacobson, H. Kaul, and D. West. Packing of graphic n-tuples. *J. Graph Theory*, **70** (1):29–39, 2012.

[11] N. Bushaw and N. Kettle. Turán numbers of multiple paths and equibipartite trees. *Combin. Probab. Comput.* **20** (2011), 837–853

[12] G. Chartrand, and S. Schuster, On the existence of specified cycles in complementary graphs. *Bull. Amer. Math. Soc.* **77** (1971) 995–998.

[13] G. Chen, Potentially C_6-graphic sequences. *J. Guangxi Univ. Nat. Sci. Ed.* **28** (2003), 119–124.

[14] G. Chen, A note on potentially $K_{1,1,t}$-graphic sequences. *Australas. J. Combin.* **37** (2007), 21–26.

[15] G. Chen, M. Ferrara, R. Gould, and J. Schmitt, Graphic sequences with a realization containing a complete multipartite subgraph. *Discrete Math.* **308** (2008), no. 23, 5712–5721.

[16] G. Chen and X. Li, On potentially $K_{1,t} + e$-graphic sequences. *J. ZhangZhou Teach. Coll. Nat. Sci.* **20** (2007), 5–7.

[17] G. Chen, J. Li and J. Yin, A variation of a classical Turán-type extremal problem. *European J. Combin.* **25** (2004), 989–1002.

[18] G. Chen, R. Gould, F. Pfender and B. Wei, Extremal graphs for intersecting cliques. *J. Combin. Theory Ser. B* **89** (2003), 159–181.

[19] G. Chen and J.H. Yin, Potentially F_{2m+i}-graphic sequences. *Util. Math.* **85** (2011), 87–95.

[20] G. Chen and J.H. Yin, The smallest degree sum that yields potentially W_5-graphic sequences. (in Chinese) *J. XuZhou Normal University* **21** (2003), 5–7

[21] G. Chen, J.H. Yin and Y.M. Fan, Potentially ${}_kC_6$-graphic sequences. (in Chinese) *J. Guangxi Norm. Univ. Nat. Sci.* **24** (2006), 26–29.

[22] A. A. Chernyak, Zh. A. Chernyak, and R. I. Tyshkevich, On forcibly hereditary P-graphical sequences. *Discrete Math* **64** (1987) No. 2–3, 111–128.

[23] S.A. Choudum, Characterization of forcibly outerplanar graphic sequences. *Combinatorics and graph theory (Calcutta, 1980)*, pp. 203–211, *Lecture Notes in Math.*, 885, Springer, Berlin-New York, 1981.

[24] S.A. Choudum, On forcibly connected graphic sequences. *Discrete Math.* **96** (1991), no. 3, 175–181.

[25] M. Chudnovsky, P. Seymour, Rao's degree-sequence conjecture. To appear in *J. Combin. Theory Ser. B.*

[26] V. Chvátal, On Hamilton's ideals. *J. Combin. Theory Ser. B* **12** (1972), 163–168.

[27] V. Chvátal, Tree-complete graph Ramsey numbers. *J. Graph Theory* **1** (1977), 93.

[28] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs. II. Small diagonal numbers. *Proc. Amer. Math. Soc.* **32** (1972), 389–394.

[29] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs, III. Small off-diagonal numbers. *Pacific J. Math.* **41** (1972), 335–345.

[30] C.R.J. Clapham and D.J. Kleitman, The degree sequences of self-complementary graphs. *J. Combin. Theory Ser. B* **20** (1976), 67–74.

[31] A. Dewdney. Degree sequences in complexes and hypergraphs. *Proc. Amer. Math. Soc.*, 53(2):535–540, 1975.

[32] G.A. Dirac, Some theorems on abstract graphs. *Proc. Lond. Math. Soc.* **2** (1952), 69–81.

[33] Z. Dvořák and B. Mohar, Chromatic number and complete graph substructures for degree sequences. *Combinatorica* **33** (2013), no. 5, 513–529.

[34] J. Edmonds, Existence of k-edge connected ordinary graphs with prescribed degree. *J. Res. Nat. Bur. Stand. Sect. B* **68B** (1964) 73–74.

[35] P. Erdős, On the graph theorem of Turán (Hungarian). *Mat. Lapok.* **21** (1970), 249–251.

[36] P. Erdős, Z. Füredi, R. Gould and D. Gunderson, Extremal graphs for intersecting triangles. *J. Combin. Theory Ser. B* **64** (1995), 89–100.

[37] P. Erdős and T. Gallai, Graphs with prescribed degrees of vertices (Hungarian). *Mat. Lapok.* **11**, 264–274, 1960.

[38] P. Erdős, M.S. Jacobson, and J. Lehel, Graphs realizing the same degree sequence and their respective clique numbers. *Graph Theory, Combinatorics and Applications* (eds. Alavi, Chartrand, Oellerman and Schwenk), Vol. 1, 1991, 439–449.

[39] P. Erdős and A. Stone, On the structure of linear graphs. *Bull. Amer. Math. Soc.* **52** (1946), 1087–1091.

[40] P. Erdős and M. Simonovits, A limit theorem in graph theory. *Studia Sci. Math. Hungar.* **1** (1966), 51–57.

[41] E. Eschen and J. Niu, On potentially $(K_4 - e)$-graphic sequences. *Australas. J. Combin.* **29** (2004), 59–65.

[42] R.J. Faudree and R.H. Schelp, All Ramsey numbers for cycles in graphs. *Discrete Math.* **8** (1974) 313–329.

[43] M. Ferrara, Graphic sequences with a realization containing a union of cliques. *Graphs Combin.* **23** (2007), 263–269.

[44] M. Ferrara, R.J. Gould, and J. Schmitt, Graphic sequences with a realization containing a friendship graph. *Ars Combin.* **85** (2007), 161–171.

[45] M. Ferrara, R.J. Gould, and J. Schmitt, Using edge exchanges to prove the Erdős-Jacobson-Lehel conjecture. *Bull. Inst. Combin. Appl.* **56** (2009), 73–80.

[46] M. Ferrara, T. LeSaulnier, C. Moffatt, P. Wenger. On the sum necessary to ensure that a degree sequence is potentially H-graphic. Submitted.

[47] M. Ferrara and J. Schmitt, A general lower bound for potentially H-graphic sequences. *SIAM J. Discrete Math.* **23** (2009), 517–526.

[48] D. Fulkerson, A. Hoffman, and M. McAndrew. Some properties of graphs with multiple edges. *Canad. J. Math.*, 17:166–177, 1965.

[49] L. Gerencsér and A. Gyárfás, On Ramsey-type problems. *Ann. Univ. Sci. Budapest Eötvös Sect. Math.* **10**, 167–170.

[50] R.J. Gould, M.S. Jacobson, J. Lehel, Potentially G-graphical degree sequences. In: Y. Alavi et al. (Ed.), *Combinatorics, Graph Theory, and Algorithms*, Vol. 1, New Issues Press, Kalamazoo, Michigan, 1999, 451–460.

[51] R.E. Greenwood and A.M. Gleason, Combinatorial relations and chromatic graphs. *Canad. J. Math.* **7** (1955) 1–7.

[52] X. Gu and H-J. Lai, Realizing degree sequences with k-edge-connected uniform hypergraphs. *Discrete Math.* **313** (2013), no. 12, 1394–1400.

[53] A. Gyárfás, M. Ruszinkó, G.N. Sárközy, E. Szemerédi, Three-color Ramsey numbers for paths. *Combinatorica* **27** (2007) 35–69.

[54] A. Gyárfás, G. Sárközy, E. Szemerédi, Stability of the path-path Ramsey number. *Discrete Math.* **309** (2009), no. 13, 4590–4595.

[55] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe (German). *Vierteljschr. Naturforsch. Ges. Zürich* **88** (1943), 133–142.

[56] S.L. Hakimi, On the realizability of a set of integers as degrees of vertices of a graph. *SIAM J. Appl. Math*, **10** (1962), 496–506.

[57] P. L. Hammer and B. Simeone, The splittance of a graph. *Combinatorica* **1** (1981), 275–284.

[58] S. Hartke and T. Seacrest, Graphic sequences have realizations containing bisections of large degree. *J. Graph Theory* **71** (2012), no. 4, 386–401.

[59] V. Havel, A remark on the existence of finite graphs (Czech). *Časopis Pěst. Mat.* **80** (1955), 477–480.

[60] H. Hoogeveen and G. Sierksma. Seven criteria for integer sequences being graphic. *J. Graph Theory*, 15: 223–231, 1991.

[61] L. Hu and C. Lai, On potentially 3-regular graph graphic sequences. *Util. Math.* **80** (2009), 33–51.

[62] L. Hu and C. Lai, On potentially $K_5 - E_3$-graphic sequences. *Ars Combin.* **101** (2011), 359–383.

[63] L. Hu and C. Lai, On potentially $(K_5 - C_4)$-graphic sequences. *Ars Combin.* **99** (2011), 175–192.

[64] L. Hu and C. Lai, and P. Wang, On potentially $K_5 - H$-graphic sequences. *Czechoslovak Math. J.* **59** (134) (2009), no. 1, 173–182.

[65] Q. Huang, On potentially $K_5 - e$-graphic sequences. *J. XinJiang University* **22** (2005), 276–284.

[66] G. Isaak and D. West. The edge-count criterion for graphic lists. *Electron. J. Combin.* **17** (2010), No. 1, 5pp.

[67] G. Károlyi and V. Rosta, Generalized and geometric Ramsey numbers for cycles. Combinatorics and computer science (Palaiseau, 1997). *Theoret. Comput. Sci.* **263** (2001), no. 1–2, 87–98.

[68] P. Keevash, Shadows and intersections: stability and new proofs. *Adv. Math.* **218** (2008), 1685–1703.

[69] A.E. Kézdy and J. Lehel, Degree sequences of graphs with prescribed clique size. In: Y. Alavi et al. (Ed.), *Combinatorics, Graph Theory, and Algorithms*, Vol. 1, New Issues Press, Kalamazoo, Michigan, 1999, 535–544.

[70] D. Kleitman and D. Wang, Algorithms for constructing graphs and digraphs with given valences and factors. *Discrete Math.* **6** (1973) 79–88.

[71] M. Koren, Extreme degree sequences of simple graphs. *J. Combin. Theory Ser. B*, **15** (1973), 213–224.

[72] S. Kundu, The k-factor conjecture is true. *Discrete Math.* **6** (1973), 367–376.

[73] C. Lai, On potentially C_k-graphic sequences. *J. ZhangZhou Teach. Coll. Nat. Sci.* **11** (1997), 27–31.

[74] C. Lai, A note on potentially $K_4 - e$ graphical sequences. *Australas. J. Combin.* **24** (2001), 123–127.

[75] C. Lai, Characterize the potentially $K_4 - e$ graphical sequences (Chinese). *J. ZhangZhou Teach. Coll. Nat. Sci.* **15** (2002), 53–59.

[76] C. Lai, The smallest degree sum that yields potentially C_k-graphical sequences. *J. Combin. Math. Combin. Comput.* **49** (2004), 57–64.

[77] C. Lai, An extremal problem on potentially $K_{p,1,1}$-graphic sequences. *Discrete Math. Theor. Comput. Sci.* **7** (2005), 75–80.

[78] C. Lai, An extremal problem on potentially $K_m - C_4$-graphic sequences. *J. Combin. Math. Combin. Comput.* **61** (2007), 59–63.

[79] C. Lai, The smallest degree sum that yields potentially $K_{r+1} - Z$-graphical sequences. *Ars Combin.* **102** (2011), 65–77.

[80] C. Lai and L. Hu, Potentially $K_m - G$-graphical sequences: a survey. *Czechoslovak Math. J.* **59(134)** (2009), no. 4, 1059–1075.

[81] C. Lai and L. Hu, An extremal problem on potentially $(K_{r+1} - H)$-graphic sequences. *Ars Combin.* **94** (2010), 289–298.

[82] C. Lai and Y. Sun, An extremal problem on potentially $K_{r+1} - (kP_2 \cup tK_2)$-graphic sequences. *Int. J. Appl. Math. Stat.* **14** (2009), No. M09, 30–36.

[83] C. Lai and G. Yan, On potentially $K_{r+1} - U$-graphical sequences. *Util. Math.* **80** (2009), 233–244.

[84] J. Li and R. Luo, Potentially ${}_3C_l$-graphic sequences. *J. Univ. Sci. Technol. China* 29 (1999), 1–8.

[85] J. Li, R. Luo, and Y. Liu, An extremal problem on potentially ${}_3C_l$-graphic sequences. *J. Math. Study* **31** (1998), 362–369.

[86] J. Li and Z. Song, An extremal problem on the potentially P_k-graphic sequences. The International Symposium on Combinatorics and Applications (W.Y.C. Chen et. al., eds.), Tanjin, Nankai University 1996, 269–276.

[87] J. Li and Z. Song, The smallest degree sum that yields potentially P_k-graphical sequences. *J. Graph Theory* **29** (1998), 63–72.

[88] J. Li, Z. Song, and R. Luo, The Erdős-Jacobson-Lehel conjecture on potentially P_k-graphic sequences is true. *Science in China, Ser. A*, **41** (1998), 510–520.

[89] J.Li and J. Yin, A variation of an extremal theorem due to Woodall. *Southeast Asian Bull. Math.* **25** (2001), 427–434.

[90] J. Li and J. Yin, An extremal problem on potentially $K_{r,s}$-graphic sequences. *Discrete Math.* **260** (2003), 295–305.

[91] M. Liu and C. Lai, On potentially $K_{2,2,1,1}$-graph graphic sequences. *Util. Math.* **85** (2011), 45–63.

[92] R. Luo, On potentially C_k-graphic sequences. *Ars Comb.* **64** (2002) 301–318.

[93] R. Luo and M. Warner, On potentially K_k-graphic sequences. *Ars Comb.* **75** (2005) 233–239.

[94] W. Mantel, Problem 28, soln. by H. Gouwentak, W. Mantel, J.Teixeira de Mattes, F. Schuh and W.A. Wythoff. *Wiskundige Opgaven* **10** (1907), 60–61.

[95] M. Molloy and B. Reed, *Graph Colouring and the Probabilistic Method*. Springer-Verlag, Berlin, 2002.

[96] D. Mubayi, A hypergraph extension of Turán's theorem. *J. Combin. Theory Ser. B* **96** (2006), no. 1, 122–134.

[97] D. Mubayi, Structure and stability of triangle-free set systems. *Trans. Amer. Math. Soc.* **359** (2007), 275–291.

[98] D. Mubayi and O. Pikhurko, A new generalization of Mantel's theorem to k-graphs. *J. Combin. Theory Ser. B* **97** (2007), no. 4, 669–678.

[99] V. Nikiforov, Stability for large forbidden subgraphs. *J. Graph Theory* **62** (2009), no. 4, 362–368.

[100] V. Nikiforov, R.H. Schelp, Cycles and stability. *J. Combin. Theory Ser. B* **98** (2008) 69–84.

[101] J. Petersen, Die Theorie der regularen Graphen. *Acta Math.* **15** (1891),193–220.

[102] O. Pikhurko, An analytic approach to stability. *Discrete Math.* **310** (2010), no. 21, 2951–2964.

[103] O. Pikhurko, A note on the Turán function of even cycles. *Proc. Amer. Math. Soc.* **140** (2012), no. 11, 3687–3692.

[104] O. Pikhurko and A. Taraz, Degree sequences of F-free graphs. *Electron. J. Combin.* **12** (2005), R69, 12pp.

[105] S. Radziszowski, Small Ramsey numbers. *Electron. J. Combin.* **1** (1994), Dynamic Survey 1, 94 pp.

[106] F.P. Ramsey, On a problem of formal logic. *Proc. Lond. Math. Soc.* **30** (1930) 264–286.

[107] A.R. Rao, An Erdős-Gallai type result on the clique number of a realization of a degree sequence. (unpublished)

[108] A.R. Rao and S.B. Rao, On factorable degree sequences. *J. Combin. Theory Ser. B* **13** (1972), 185–191.

[109] S.B. Rao, Characterization of forcibly line-graphic degree sequences. *Util. Math.* **11** (1977), 357–366.

[110] S.B. Rao, A survey of the theory of potentially P-graphic and forcibly P-graphic degree sequences. In *Combinatorics and Graph Theory, Proc. Int. Conf., (Calcutta 1980)*, pp. 417–440, *Lecture Notes in Math., 885*, Springer, Berlin-New York, 1981.

[111] S.B. Rao, Towards a theory of forcibly hereditary P-graphic sequences. In *Combinatorics and Graph Theory, Proc. Int. Conf., (Calcutta 1980)*, pp. 441–458, *Lecture Notes in Math., 885*, Springer, Berlin-New York, 1981.

[112] N. Robertson and P. Seymour, Graph Minors. XX. Wagner's Conjecture, *J. Combin. Theory, Ser. B* **92** (2004), 325–357.

[113] M. Simonovits, A method for solving extremal problems in graph theory, stability problems. In: *Theory of Graphs, Proc. Colloq., Tihany*, 1966, Academic Press, New York, 1968, pp. 279–319.

[114] J. H. Spencer, Ramsey's theorem – A new lower bound. *J. Combin Theory Ser. A*, **18** (1975) 108–115.

[115] A. Tripathi and H. Tyagi, A simple criterion on degree sequences of graphs. *Discrete Appl. Math.* **156** (2008) 3513–3517.

[116] A. Tripathi and S. Vijay, A note on a theorem of Erdős and Gallai. *Discrete Math.* **265**, 417–420, 2003.

[117] P. Turán, Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok* **48** (1941), 436–452.

[118] R. I. Tyshkevich, A. A. Chernyak, and Zh. A. Chernyak, Forcibly p-graphic sequences. *Dokl Akad Nauk BSSR* **29** (8) (1985), 677–680, 763 (in Russian).

[119] Y. Wang, On potentially W_6-graphic sequences. *OR Trans.* **13** (2009), no. 4, 39–44.

[120] D.L. Wang and D.J. Kleitman, On the existence of n-connected graphs with prescribed degrees ($n \geq 2$). *Networks* **3** (1973), 225–239.

[121] Y. Wang and J.H. Yin, A characterization for a sequence to be potentially $\{K_{r+1} - E(P_2), K_{r+1}^{-2}\}$-graphic. *Util. Math.* **89** (2012), 129–139.

[122] D. West, *Introduction to Graph Theory*, 2nd. ed., Prentice Hall, Upper Saddle River, NJ, 2001.

[123] Z. Xu and L. Hu, Characterize the potentially $K_{1,4} + e$ graphical sequences. *J. ZhangZhou Teach. Coll. Nat. Sci.* **55** (2007), 4–8.

[124] Z. Xu and C. Lai, On potentially $K_6 - C_5$- graphic sequences. *Util. Math.* **86** (2011), 3–22.

[125] J.H. Yin, Degree sequences of graphs containing a cycle with prescribed length. *Czechoslovak Math. J.* **59**(134) (2009), no. 2, 481–487.

[126] J.H. Yin, A characterization for a graphic sequence to be potentially C_r-graphic. *Sci. China Math.* **53** (2010), no. 11, 2893–2905.

[127] J.H. Yin, A Rao-type characterization for a sequence to have a realization containing a split graph. *Discrete Math.* **311** (2011), 2485–2489.

[128] J.H. Yin, A short constructive proof of A. R. Rao's characterization of potentially K_{r+1}-graphic sequences. *Discrete Appl. Math.* **160** (2012), 352–354.

[129] J.H. Yin, A Havel-Hakimi type procedure and a sufficient condition for a sequence to be potentially $S_{r,s}$-graphic. *Czechoslovak Math. J.* **62** (137) (2012), no. 3, 863–867.

[130] J.H. Yin, The characterization for a graphic sequence to have a realization containing $K_{1,1,s}$. *Graphs Combin.* **28** (2012), no. 4, 585–595.

[131] J.H. Yin and G. Chen, On potentially $K_{r_1,r_2,\dots,r_m}$-graphic sequences. *Util. Math.* **72** (2007), 149–161.

[132] J.H. Yin, G. Chen and G.L. Chen, On potentially ${}_kC_l$-graphic sequences. *J. Combin. Math. Combin. Comput.* **61** (2007), 141–148.

[133] J.H. Yin, G. Chen, and J. Schmitt, Graphic sequences with a realization containing a generalized friendship graph. *Discrete Math.* **308** (2008), 6226–6232.

[134] J.H. Yin and J.S. Li, An extremal problem on the potentially $K_{r,s}$-graphic sequences. *Discrete Math.* **260** (2003), 295–305.

[135] J.H. Yin and J.S. Li, Two sufficient conditions for a graphic sequence to have a realization with prescribed clique size. *Discrete Math.* **301** (2005), 218–227.

[136] J. H. Yin and J.S. Li, Potentially $K_{r_1,r_2,\dots,r_l,r,s}$-graphic sequences. *Discrete Math.* **307** (2007), 1167–1177.

[137] J. H. Yin and J.S. Li, A variation of a conjecture due to Erdős and Sós. *Acta Math. Sin.* **25** (2009), no. 5, 795–802.

[138] J. H. Yin and J.S. Li, Potentially K_{r+1}^{-p}-graphic sequences. *Ars Combin.* **90** (2009), 405–409.

[139] J. H. Yin and J.S. Li, Variations of two classical Turán-type extremal results. *Ars Combin.* **97A** (2010), 485–497.

[140] J. Yin, J. Li and G. Chen, The smallest degree sum that yields potentially ${}_3C_l$-graphic sequences. *Discrete Math.* **270** (2003), 319–327.

[141] J. Yin, J. Li and G. Chen, A variation of a classical Turán-type extremal problem. *European J. Combin.* **25** (2004), no. 7, 989–1002.

[142] J. Yin, J. Li and G. Chen, The smallest degree sum that yields potentially $K_{2,s}$-graphic sequences. *Ars Combin.* **74** (2005), 213–222.

[143] J. H. Yin, J.S. Li, and W.Y. Li, Potentially $K_{1,1,t}$-graphic sequences. *Ars Combin.* **104** (2012), 481–488.

[144] J. H. Yin, J.S. Li, and R. Mao, An extremal problem on the potentially $K_{r+1}-e$-graphic sequences. *Ars Combin.* **74** (2005), 151–159.

[145] J.H. Yin and Y. Wang, A characterization for a graphic sequence to have a realization containing a desired cycle. *Util. Math.* **86** (2011), 79–86.

[146] J.H. Yin and Y. Wang, A characterization for a sequence to be potentially $K_{r+1} - e$-graphic. *Acta Math. Appl. Sin. Engl. Ser.* **29** (2013), no. 4, 787–792.

[147] M-X. Yin, The smallest degree sum that yields potentially $K_{r+1} - K_3$-graphic sequences. *Acta Math. Appl. Sin. Engl. Ser.* **22** (2006), 451–456.

[148] M-X. Yin and J-H. Yin, On potentially H-graphic sequences. *Czechoslovak Math. J.* **57** (132) (2007), no. 2, 705–724.

[149] M-X. Yin and J-H. Yin, The smallest degree sum that yields potentially $K_{r_1,r_2,\ldots,r_l,2,s}$-graphic. *Util. Math.* **78** (2009), 165–173.

[150] M-X. Yin and J-H. Yin, A characterization on potentially $K_6 - E(K_3)$-graphic sequences. *Ars Combin.* **111** (2013), 193–206.

[151] M-X. Yin, J-H. Yin, C. Zhong, and F. Yang, On the characterization of potentially $K_{1,1,s}$-graphic sequences. *Util. Math.* **85** (2011), 129–141.

[152] M-X. Yin, J-H. Yin, Y. Wang, and C. Zhong, A characterization for a graphic sequence to be potentially $K_{2,s}$-graphic. *Util. Math.* **82** (2010), 25–31.

[153] M-X. Yin, C. Zhong, and F. Yang, A note on the characterization of potentially $K_{1,1,s}$-graphic sequences. *Ars Combin.* **93** (2009), 275–287.

[154] I. E. Zverovich, Proof of a theorem of Rao on forcibly planar degree sequences. *Vestsi Akad Navuk BSSSR Ser Fiz-Mat Navuk* No. 4, (1987), 113 (in Russian).

[155] I. E. Zverovich, Forcibly 3-colorable degree sequences. *Vychisl Sistemy* No. 119, Voprosy Algoritm Anal Strukturn Inform. (1987) 34–42, 113–114.

[156] I. E. Zverovich and O. P. Urbanovich, 2-matroid graphs and power sequences. *Vestsi Akad Navuk BSSR Ser Fiz-Mat Navuk* No. 3 (1990), 35–39, 124.

[157] I. E. Zverovich and V. E. Zverovich, Contributions to the theory of graphic sequences. *Discrete Math.* **103** (1992), 293–303.