THE TENT MAP, THE HORSESHOE AND THE PENDULUM:
THE GEOMETRY OF CHAOS CONTROL

by

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ABSTRACT

When the space of a dynamical system is continually stretched and folded, a horseshoe structure may develop. The horseshoe is a guarantee of chaotic behavior, and we study the suspension of a horseshoe map as a model for chaotic dynamics of a periodically driven system. Consideration of the dynamics of a suspended horseshoe and its perturbation leads to a general geometrical approach to the control of chaos in low dimensional periodically driven systems by the method of stable subspace targeting. We show how OGY control is a special case of the general method, and show why in experimental situations certain modifications must be made to OGY in order for it to work. We develop and implement two new types of chaos control based on these considerations, control by time proportioned perturbations (TPP) and control by capture and release (CR).

This abstract accurately represents the contents of this candidates thesis. I recommend its publication.

Signed  William Briggs

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1. Introduction

Chaotic solutions exist for many systems, from maps like the logistic map to continuous time systems described by differential equations, to real world systems. Many interesting and important physical systems have chaotic dynamics over certain parameter ranges, and it is sometimes possible to control these systems to obtain improved performance. A prime example is the control of a multimode laser[1], where the stability regime of a YAG laser has been extended by an order of magnitude by the application of small perturbations to the DC bias level of the laser pump.

Following the publication of results of Ott, Grebogi and Yorke[2] (OGY) on the control of chaotic systems, experimental control of several physical systems was reported[3]. OGY control works by applying a small perturbation to a system parameter based on information taken from a map, which, in an experimental system, is some sort of surface of section (SOS) map. The control signal remains constant over the period between mappings, and is recalculated at each SOS. The goal of the perturbation is to place the system state on the stable subspace of an unstable periodic orbit (UPO) in one iteration of the map. Experimentalists found that although strict application of the OGY algorithm was suitable for some simple systems, in order to achieve tight control it was sometimes necessary to delay the application of control, and that sometimes the control perturbation must be turned off before the next SOS was taken.

I show why delay of perturbation and change of perturbation length are sometimes necessary in an experimental situation, and extend OGY control to the continuous time systems by means of stable subspace targeting. I develop these results by looking at the geometry of the suspension of the Smale horseshoe, a simple geometrical construction exhibiting chaotic dynamics. By considering the interaction of a suspended horseshoe and its perturbation, I show how a continuous spectrum of control rules based on perturbation magnitude, control on time, and delay time may be established. The spectrum of limit cycle control will be made plausible from a geometrical point of view, and later developed as a numerical procedure for control of continuous time periodically forced experimental systems.
In the final two chapters of this report, I develop the mathematics necessary for the implementation of two new continuous-time chaos control schemes, control by capture and release (CR) and control by time proportioned perturbations. I then use these control schemes to control a numerical model of the vertically forced pendulum with damping.
2. Chaotic Maps and their Control

2.1. The tent map

Consider the chaotic tent map, a difference equation defined by:

\[
\begin{align*}
    x_{n+1} &= 2x_n \quad \text{for } 0 \leq x_n \leq \frac{1}{2} \quad (2.1) \\
    x_{n+1} &= 2(1 - x_n) \quad \text{for } \frac{1}{2} < x_n \leq 1. \quad (2.2)
\end{align*}
\]

This map can be used to describe the stretching and folding of a one dimensional elastic band of length one. Each time the map is iterated the band stretches to twice its length and folds into itself. Points in \([0, \frac{1}{2}]\) will double in magnitude and map to points in \([0, 1]\), while points in \((\frac{1}{2}, 1]\) will be stretched out to twice their magnitude and folded into \([0, 1]\). Figure 1 illustrates the result of this mapping.

![Figure 2.1.1. The tent map takes the unit interval linearly into itself.](image)

The diagonal line in the graph of Figure 2.1.2 is the identity line \(x_{n+1} = x_n\) and its intersection with the tent gives the fixed points \(\frac{2}{3}\) and 0. We can trace the forward iteration of a point in the unit interval by the method of graphical analysis: Start with an initial value \(x_0\) (the seed) on the \(x_n\) axis and draw a vertical line from this point to the
graph of the function. Then draw a horizontal line from the function to
the identity line. The \( x_n \) coordinate of this point is \( x_1 \). We can iterate
the map by repeating this process, producing what is known as a web
diagram.

\[
\begin{array}{c}
\text{Figure 2.1.2. The web diagram illustrates graphically the result of iteration of the map.}
\end{array}
\]

The tent map is two to one except at 1, which has only \( \frac{1}{2} \) as its first
preimage. Every \( x \) in \([0,1]\) of the form \( x = \frac{p}{2^n} \), \( p, n \) integer, eventually
maps under forward iteration to the fixed point \( x = 0 \). Solving \( x_{n+1} = x_n \)
shows that the point \( x = \frac{1}{3} \) is the only other fixed point. There are,
however, an infinite number of periodic points of the form \( x = \frac{p}{q} \), where
\( p \) and \( q \) are integers and \( q \) is odd, like the period two points \( \frac{2}{5} \) and \( \frac{4}{5} \).
It can be shown that any fraction in lowest terms whose denominator
contains an odd number as a factor is a preimage of a point in a periodic
orbit. There are periodic points of all periods, and the set of irrationals in \([0, 1]\) make up the set of points with chaotic orbits.

**Definition 2.1.** (Taylor and Toohey)\(^4\) A dynamical system on a topological space is chaotic if every pair of non-void open subsets share a periodic orbit.

Since points of the form \(\frac{p}{q}\), \(q\) odd are periodic points and dense in the unit interval, we can prove the map is chaotic there.

**Theorem 2.2.** The tent map is chaotic on \([0, 1]\).

**Proof.** Consider the open balls \(B(\varepsilon_j, x_j)\) and \(B(\varepsilon_k, x_k)\), \(x_j, x_k\) arbitrary points in \((0, 1)\), and \(\varepsilon_i\) chosen so that the balls are subsets of the unit interval. Let \(B(\varepsilon_i, x_i)_n\) and \(B(\varepsilon_i, x_i)_m\) be the \(n\)th forward and \(n\)th backward iterate of \(B(\varepsilon_i, x_i)\) respectively. Under the tent map there will be an \(n\) such that \(B(\varepsilon_j, x_j)_n\) completely covers \(B(\varepsilon_k, x_k)\), and such that a preimage \(B(\varepsilon_k, x_k)_{-n} \subset B(\varepsilon_j, x_j)\) of \(B(\varepsilon_k, x_k)\) is a single open interval. Iterate \(B(\varepsilon_k, x_k)\) forward \(m\) times until it completely covers \(B(\varepsilon_j, x_j)\), and so that a preimage \(B(\varepsilon_j, x_j)_{-m} \subset B(\varepsilon_k, x_k)\) of \(B(\varepsilon_j, x_j)\) is a single open interval. There is then a continuous mapping from \(B(\varepsilon_k, x_k)_{-n}\) to \(B(\varepsilon_k, x_k)_m\) and, by Brouwer's fixed point theorem for one dimension, there is a fixed point \(x^*\) for the \(n + m\) times iterated map \(F^{n+m} : B(\varepsilon_k, x_k)_{-n} \to B(\varepsilon_k, x_k)_m\). But \(B(\varepsilon_k, x_k)_{-n} \subset B(\varepsilon_j, x_j)\), so \(x^* \in B(\varepsilon_j, x_j)\). Thus there is an \(n + m\) periodic point \(x^*\) whose orbit is in common with both \(B(\varepsilon_j, x_j)\) and \(B(\varepsilon_k, x_k)\), and the tent map is chaotic.

Irrational numbers are the points in chaotic orbits, and the fact that any irrational number \(r\) will map to another irrational, and that \(r\) will never be repeated gives an intuitive feeling for the nature of these chaotic orbits. The set of all periodic points that participate in orbits of period \(n\) can be computed exactly. Say that \(x_n\), the \(n\)th iterate of \(x_0\), is in bin 0 if it lies in \([0, \frac{1}{2}]\), and that it is in bin 1 if it lies in \((\frac{1}{2}, 1]\). We can construct a doubly infinite string of symbols \(\ldots \bar{s}_2 s_1 s_0 \bar{s}_1 s_2 \ldots\), where \(s_n\) is the bin number of \(x_n\), that represents the history of a point.
$x_0$ under the map. The left or right shift of the binary point corresponds to forward and backward iteration of $x_0$. For example, the orbit of $\frac{2}{9}$ is $\frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \ldots$ so its symbol sequence is $..001001.001001...$. Shifting the binary point to the left or right produces the backward or forward bin sequence, called the *itinerary*. Each itinerary is unique and corresponds to one particular orbit, be it periodic or chaotic.

Now suppose we wanted to construct all orbits of period 2. There are four possible strings that repeat after two cycles: $...0000.0000..., \ldots1111.1111..., \ldots0101.0101...$ and $...1010.1010...$. If $x$ is the initial value and it is in bin 0, its value is to be multiplied by 2. If $x$ is in bin 1 its value is to be subtracted from 1 and then multiplied by 2. Thus the period 2 point with orbit represented by $...0101.0101...$ has initial value $x = 2(1 - (2x)) = \frac{2}{3}$. The period 2 orbit represented by $...1010.1010...$ starts on $x = 2(2(1 - x)) = \frac{4}{5}$. If we compute the points corresponding to $...000.000...$ and $...111.111...$ we get 0 and $\frac{2}{3}$ respectively, the period one points. Points of higher period are computed in the same way, by compounding the algebraic operators in 2.1 based on the itinerary.

All periodic points of the tent map are unstable. Irrational points, no matter how close to a point in a periodic orbit, will move exponentially away from the rational points of this orbit under the action of the map. This exponential departure, along with our ability to make small perturbations in a control parameter will allow us to control the chaos.

To implement control we need an accessible parameter that will change the action of the map in a uniform way. If we change the height of the tent smoothly we change the mapping of points in $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$ in a continuous fashion, so we write the equations of the tent map as

$$
\begin{align*}
    x_{n+1} &= (2 + \delta)x_n & \text{for } 0 \leq x_n \leq \frac{1}{2} \\
    x_{n+1} &= (2 + \delta)(1 - x_n) & \text{for } \frac{1}{2} < x_n \leq 1
\end{align*}
$$

with $\delta$ set initially to 0. Now we can control the height of the tent by changing $\delta$. If some iterate $x$ of the map were to land near the preimage of a periodic point $x^*$, we could perturb $\delta$ in a way that would cause the next iterate to land on $x^*$. Figure 2.1.3 illustrates the technique.
applied to a period 2 point $\frac{2}{5}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{tent_map.png}
\caption{The tent map is controlled around an unstable period two orbit by changing the height of the tent.}
\end{figure}

An iterate of a chaotic orbit falls within $\varepsilon$ of $\frac{4}{5}$, the preimage of $\frac{2}{5}$. We increase or decrease $\delta$ depending on which side of $\frac{4}{5}$ $x$ falls, and the next iterate lands squarely on $\frac{2}{5}$. Then we reset $\delta$ to 0, and the orbit is now periodic with period 2. Because the map is ergodic (for any $\varepsilon$ ball around some point $x$ and for any point $x_k$ in this interval, there is an $N$ such that $|x_{k+N} - x| < \varepsilon$), we can be assured of an iterate of any point in a chaotic orbit eventually coming as near as we like to the periodic point of interest. Suppose we want to control about a periodic point $x > \frac{1}{2}$, and the orbit has landed in the control region $B(\varepsilon, x_{-1})$, the epsilon ball around the preimage of $x$. The control parameter $\delta$ is computed as follows:
If $x > \frac{1}{2}$ is the target point and $x_{-1}$ the current iterate, we want

$$(2 + \delta)(1 - x) = 2(1 - x_{-1})$$

so

$$
\delta = 2 \frac{1 - x_{-1}}{1 - x} - 2. $$

Using this $\delta$ for one iteration gets us on the periodic orbit, and we then reset $\delta$ to 0. A similar calculation will get us to the periodic point if $x_k < \frac{1}{2}$.

Just as we can direct the orbit of the tent map from any initial condition to any periodic (or chaotic) orbit by a one-time application of a perturbation of the tent height, we can direct a trajectory of the tent map along any sequence of $n$ periodic orbits, coming as close as we like to any point in $[0, 1]$, as periodic points of the tent map are dense in $[0, 1]$. We merely compute the sequence of perturbations $\delta_k$ necessary to take the orbit to the target points $x_{t_k}$, $k = 1, 2...n$ and the controlled tent map becomes

$$x_{k+1} = (2 + \delta_k)x_k \text{ for } 0 \leq x_k \leq \frac{1}{2}$$

$$x_{k+1} = (2 + \delta_k)(1 - x_k) \text{ for } \frac{1}{2} < x_k \leq 1.$$  \hfill (2.6)

$$x_{k+1} = (2 + \delta)(1 - x)$$

While we can vary the height of the tent to take any point in $[0, 1]$ to any other in one iterate, the size of the perturbation $\delta$ required to get from $x_{t_k}$ to $x_{t_{k+1}}$ goes to $\infty$ as $x_{t_k} \to 0$ or $x_{t_k} \to 1$.

We specify a maximum perturbation $\delta_{\text{max}}$ which determines the size of our control region $B(\varepsilon, x_{-1})$, the $\varepsilon$ ball around $x_{-1}$. This point is the preimage of the periodic point we are targeting. There is a trade-off: if we want to use only a small perturbation, it may take a long time before the orbit lands in the control region, especially if the preimage of the point we are trying to control is near 1 or 0. If we want to control quickly we can consider the whole unit interval our control region and force the chaotic orbit onto the target point in one iteration, but at the expense of needing a possibly infinite $\delta$. In a physical control situation,
there will certainly be a limit to the size of $\delta_{\text{max}}$, but here we choose to limit ourselves to small control perturbations for the sake of elegance.

We can reduce the size of the perturbations necessary to get from one place to another by using the sensitivity to initial conditions of a chaotic map. Under iteration the map will come as close as we like to any point, so by computing a clever sequence of corrections that takes us on side-trips between target points, we can minimize some sort of a cost function. If we are currently stuck in a periodic orbit and want to go to another, we apply a small perturbation for one iterate of the map to knock us out of the periodic orbit and onto a chaotic one. Then we rely on the chaos of the system to eventually bring us into the control region around a point in the periodic orbit of interest.

Suppose we were limited to very small perturbations. Are there faster ways of getting from one region of the map to another, riding its dynamics, so to speak? We know the dynamics of the tent map well: any interval entirely inside $[0, 1/2]$ or $(1/2, 1]$ will be expanded by a factor of two and mapped one to one into the unit interval. Segments containing $1/2$ have sets of points that map two to one into the interval because of the change in dynamics at $1/2$. Backward iteration of an interval is up a binary tree: at each branch we have two possibilities for our previous position. The interval also shrinks by a factor of two, so the $n$th backward iterate of a line segment of length $L$ is a set of $2^n$ segments of length $2^{-n}L$, distributed in some fashion over the interval.

Now if we had a target point $x^*$ and we iterated $B(\varepsilon, x_{-1}^*)$ backwards $n$ times, we would have $2^n$ small regions that map into our target region, the ball around the preimage of $x^*$. The forward iterates of a point $x$ chosen at random would have a far greater chance of landing within $\varepsilon$ of one of the $k$th preimages of $B(\varepsilon, x_{-1}^*)$ than of landing in $B(\varepsilon, x_{-1}^*)$. Of course, for each $k$, the preimage of $B(\varepsilon, x_{-1}^*)$ requires a different $\varepsilon$, depending on the distance of the preimage from $1/2$, since for a given maximum allowable perturbation the region of controllability shrinks as we approach 0 or 1. Figure 4 shows the range of possible preimages of a target point $x$ under the restriction of the control parameter to $\delta_{\text{min}} \leq \delta \leq \delta_{\text{max}}$. We can see that points $x \in [x_{\text{min}}, x_{\text{max}}]$ can all be
made to map to $\tilde{x}$ by the appropriate choice of $\delta$, but points arbitrarily close to 0 or 1 require arbitrarily large perturbations to map to $\tilde{x}$ in one iterate. We can fix $\delta$ and choose $\varepsilon$ to be the $\min$ of all $\varepsilon$ for a limited set of preimages. When a forward iterate of $x$ comes within $\varepsilon$ of $B(\varepsilon, x_{-1}^{n})$, where the subscript $-k$ indicates the $kth$ backward iterate of $B(\varepsilon, x_{-1})$, we vary $\delta$ to coax it into $B(\varepsilon, x_{-1}^{n})$ and let the dynamics of the map take us to $B(\varepsilon, x_{-1})$. We then apply the control again so that we land right on $x^*$. 

Figure 2.1.4. The height of the tent map can vary from $\delta_{\min}$ to $\delta_{\max}$, and the effect of this parameter change on the controllable region around a preimage of $\tilde{x}$ varies inversely with the distance of the preimage from 0 or 1.

Another way to increase our efficiency is to use the other periodic orbits of the map as routes to the final destination $x^*$. Locating periodic points is quite simple, and by a judicious choice of a limited set of periodic orbits we can cover most of the interval with epsilon balls.
about each point in each orbit. We can vary \( \delta \) to cause any initial point to move onto a periodic orbit from our set in a few iterations, and by choosing our set of orbits so that each epsilon ball contains the periodic point of its two neighboring balls, we can also jump from orbit to orbit by varying \( \delta \). There will be a path through these orbits that will lead us to \( x \).

If we are controlling a physical system or simulating the tent map on a real computer, then experimental noise or round-off error will drive our computed orbit away from the true periodic orbit we are trying to reach, and the control would have to be reapplied whenever we drifted too far from our goal.

Now we can control a tent map, one in which the periodic orbits are not too difficult to compute. What if we wished to control the logistic map? Computing high period orbits, even period 5 for instance, becomes very difficult. However, the OGY algorithm doesn’t require exact knowledge of the dynamics or the periodic points of the system under study. All we need is an approximate location of the periodic point we want to stabilize, and the approximate dynamics about that point. We can iterate any map we like, or collect experimental data from a physical system, and if there are periodic points and ergodicity, we can use the OGY method to stabilize the UPOs.

To locate periodic points of the logistic map we could use a Newton’s method to find the zeroes of the compounded map, but in preparation for the control of a physical system for which we have no model, we try the following method. Iterate the equation on a computer and store several thousand iterates in an array. Then check for pairs of points \( x_n, x_{n+k} \) for which \( |x_{n+k} - x_n| < \varepsilon \) for some small \( \varepsilon \), and \( k \) is the prime period of the point we are seeking. A linear least squares fit to the data can be made, followed by a linear least squares fit to a line orthogonal to the first. The intersection of these lines will be the center of gravity of the data points, an approximation of the position of the period \( k \) point. As the map is ergodic, we can expect a chaotic orbit to approach arbitrarily close to any periodic point as \( n \) approaches infinity.

Once the periodic point \( x^* \) has been located, we can determine the local dynamics by noting the rate of escape of points in the epsilon
neighborhood of $x^*$. Then we apply the control in the same way as before, reapplying when necessary.

Once we lock in to a periodic orbit using the control, we can determine the position of the periodic point $x^*$ more accurately by moving $x^*$ experimentally and minimizing the average value of $\delta$ required to keep the control locked in. Once we have fine tuned the position of the periodic point, we can recompute the local dynamics to get a more accurate estimate. After only a few applications of this procedure, we can fine tune the position of the periodic points and the local dynamics to the limit of the accuracy of our machine. In this way we can control any unimodal chaotic map with almost the same ease as we can the tent map.

2.2. The baker's map

Now let us move up one dimension to a two-dimensional tent map. This map is sometimes called the two-dimensional baker's map. Imagine a square sheet of rubber that stretches without deformation. We stretch the sheet until it is twice its length, then cut it in half and flop the right half over the left, as illustrated in Figure 2.2.1.

![Figure 2.2.1. This two dimensional transformation is sometimes called the baker's map.](image-url)
This action is equivalent to one iteration of the map

\[
\begin{align*}
    x_{n+1} &= 2x_n & \text{for } 0 < x_n < \frac{1}{2} \\
    y_{n+1} &= \frac{1}{2} y_n \\
    x_{n+1} &= 2(1 - x_n) & \text{for } \frac{1}{2} < x_n < 1 \\
    y_{n+1} &= 1 - \frac{1}{2} y_n
\end{align*}
\]  

(2.8)

The physical description of the map did not take into account the delicate matter of fitting surfaces together after a fold, or the structure of the left and right sides after an infinite number of iterates. We could let the map fold in a two-to-one manner, but for reasons of symmetry, we will take a different course. In order that the map not be two-to-one, we require that the first iterate fit together like a puzzle, that is, require the square to be open on the top right side and closed on the left. The halves will fit together as in Figure 2.2.2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure222}
\caption{Here we see the condition that allows the map to fit together in a one-to-one fashion: the left half of the top is closed and the right half is open for all iterates.}
\end{figure}

The top of the square is now what was the bottom right, so in order for the next iterate to fit together as did the last, the bottom of the square must have had this structure:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure223}
\caption{The right hand side of the bottom of the square must have the structure of the right half of this figure in order that the top will always have the structure illustrated in Figure 2.2.2.}
\end{figure}
Furthermore, as the bottom left half of the square becomes the entire bottom on the next iterate, and we require the top to fold on itself in a one-to-one fashion each time, the bottom left half must have a scaled down structure like that of the bottom right. The left half of the bottom left half must also have this scaled down structure, as must its left half, etc.

Figure 2.2.4. The bottom of the square must have this structure overall in order that the right half of the bottom will have the structure of the right half of Figure 2.2.3 at each iterate.

A similar structure appears on the left and right sides upon successive iteration as a result of the Dedekind cut we must make at each iteration. Referring to 2.8 we see that the first cut and fold makes the top left boundary open and the bottom half closed, while the right hand boundary is closed. The next iterate finds the right hand side the same and the left hand side has a closed bottom half, while the top half is closed on its upper portion and closed on its lower portion. Repeated iteration gives the structure we required of the top and bottom boundaries, and we can see that reverse iteration of the map gives the same structure to the top and bottom edges of the square as forward iteration gave to the left and right edges.

This map has the same dynamics in the $x$ direction as forward iteration of the one-dimensional tent map, and the dynamics in the $y$ direction are the dynamics of reverse iteration of the one-dimensional tent map. Reverse iteration of the tent map causes segments to contract, and if we iterate the unit segment $0 \leq y \leq 1$, $x^*$ where $x^*$ is the $x$ coordinate of a point in a periodic orbit, the segment will approach a periodic point as a limit. Take for example the unit segment $0 \leq y \leq 1, x = \frac{2}{9}$. Iteration of the two-dimensional tent map will cause this segment to cycle in the $x$ direction between $\frac{2}{9}$, $\frac{4}{9}$ and $\frac{8}{9}$, while in the $y$ direction the segment will contract and approach $y = \frac{8}{9}$, $\frac{4}{9}$ and $\frac{2}{9}$ in the limit. The points $(\frac{2}{9}, \frac{8}{9})$, $(\frac{4}{9}, \frac{4}{9})$ and $(\frac{8}{9}, \frac{2}{9})$ are therefore a period three orbit.
Figure 2.2.5. This figure shows a period three orbit of the baker’s map. In order to control this map, we only have to control the x direction.

There is something new in this map: there is a contracting as well as an expanding direction. Chaotic and periodic orbits are dense in the unit square under this map, so we can still hop from orbit to orbit to get where we want to go. What’s more, to implement OGY control, we only have to perturb \( \delta \), the x direction control parameter. The contraction of the map in the y direction automatically brings us into the two dimensional periodic orbit we are aiming for.

This map shares many of the dynamical properties of maps made from dissipative chaotic three-dimensional flows. There are expanding and contracting directions, unstable periodic orbits and chaotic orbits. However, maps of more complex dynamical systems are usually not so polite as to have their contracting and expanding directions perpendicular to each other and linear. Furthermore, this map is not dissipative, so it doesn’t have the fractal structure of the continuous time systems we will examine later.

We can build a three-dimensional dynamical system whose Poincaré section exhibits the dynamics of the tent map, and whose pseudo flow is similar in behavior to general chaotic flows in three dimensions. First,
imagine a long skinny rectangular solid (see Figure 2.2.6). Squash it gradually along its length so that at one end it is twice as wide as it is high. Now split it up the middle from the flatter end almost to the square end. Twist the split ends so that the two top faces come together. Now form it into a loop, twist it 90 degrees and connect the two ends.

Figure 2.2.6. The baker’s map can be easily suspended to give a smooth pseudo-flow.

This construction gives a continuous evolution in three dimensions between the steps pictured in Figure 2.2.1 of the two-dimensional tent map. The periodic points are now continuous periodic orbits and chaotic points are now chaotic orbits. Once again, the periodic and chaotic orbits are dense in the space, which is now three-dimensional. We can consider the mapping at the cross-section that is the unit square to be the intersection of a flow with a plane perpendicular to the flow, a Poincaré section. A Poincaré section of a three-dimensional flow is used in most OGY control of real physical systems.

Controlling the three-dimensional baker’s map is just as easy as controlling the two-dimensional baker’s map: we look at the two-dimensional
Poincaré section of the flow, and change $\delta$ in the same fashion as before in order to coax the flow onto a periodic point of the two-dimensional map. Then, every time the orbit pierces the Poincaré plane, we check for divergence from the periodic point and reapply the control as necessary. In a physical situation or a real computer simulation this reapplication of the control is necessary because of noise and uncertainty of the actual dynamics of the system.

2.3. The Lozi map

The Lozi[5] map is a two-dimensional piecewise linear map whose dynamics are similar to those of the more familiar Hénon[6] map. The Lozi map, like the Hénon map, is an affine transformation that has a chaotic invariant set for certain parameter combinations.

**Definition 2.3.** A set $A$ is called an attractor of a map if whenever an initial point $x_0$ is chosen close enough to $A$, the distance between the $k$th iterate $x(k)$ and the set $A$ goes to 0 as $k \to \infty$.

One formulation for the Lozi map is

\[
\begin{align*}
x_{n+1} &= (1 + \rho) + \alpha y_n + \beta |x_n| \\
y_{n+1} &= -x_n,
\end{align*}
\] (2.9)

where $\alpha$ determines the contraction, $\beta$ the stretching and folding, and $\rho$ is a parameter that shifts a region to the left or right. A geometrical view of the action of the Lozi map on a planar region is shown in Figure
The action of the Lozi map may be broken down into separate components.

The action of the map on the unit square may be broken down into its components:

a. the original square is flipped about the horizontal axis by the \(-x_n\) term,

b. the square is compressed to a rectangle by the \(ay_n\) term,

c. the rectangle is folded into a chevron by \(b|x_n|\),

d. the chevron is shifted up by \(1 + \rho\),

e. the chevron is rotated about the origin by 90° due to the switching of variables.
Figure 2.3.2. The basins of attraction of the Lozi map consist of the set of points whose orbit under the map is asymptotic to a subset of the plane. There are two attractors, the point at infinity, whose basin is colored white, and the Lozi attractor, whose basin is shown in black. The Lozi attractor itself is the white area inside the dark basin. Like the Lozi attractor, the basin of attraction has a fractal structure.

Most points in the plane move further and further from the attractor under repeated applications of the mapping. However, the points in a region called the basin of attraction collapse onto an attracting set (see Figure 2.3.2), which in this case is a strange attractor (see Figure 2.3.3), a fractal object that is an invariant set of the mapping.
Figure 2.3.3. This figure shows several thousand iterates of an initial condition inside the basin of attraction for the Lozi attractor. The orbit appears to be chaotic, and in fact, the Lozi map has been shown to have a strange attractor with a hyperbolic structure (it is the union of an infinite set of saddles).

These three deformations (stretching, folding and rotating) are responsible for the chaotic motion of individual points on the attractor. Almost any two points that are initially nearby are stretched away from each other, only to be folded and rotated back nearby, but in a different layer. There is an infinite number of layers, and there are layers infinitesimally close to each other. Points jump in what appears to be an irregular manner all over the attractor as the map is iterated. The Lozi map has a set of stable "manifolds" of periodic points that is dense in the basin of attraction, but as these sets are piecewise linear and non-differentiable, they are not technically manifolds.

The state vector is the current point $X_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$. An orbit is the path of the point in the plane under successive iterations of the map.
We can write the Lozi map in matrix notation,

\[
\begin{bmatrix}
    x_{n+1} \\
    y_{n+1}
\end{bmatrix} = \begin{bmatrix}
    1 + \rho & \beta \\
    0 & -1
\end{bmatrix} + \begin{bmatrix}
    \alpha \\
    0
\end{bmatrix} \begin{bmatrix}
    x_n \\
    y_n
\end{bmatrix} \text{ for } x_n < 0 \quad (2.10)
\]

\[
\begin{bmatrix}
    x_{n+1} \\
    y_{n+1}
\end{bmatrix} = \begin{bmatrix}
    1 + \rho & -\beta \\
    0 & -1
\end{bmatrix} + \begin{bmatrix}
    \alpha \\
    0
\end{bmatrix} \begin{bmatrix}
    x_n \\
    y_n
\end{bmatrix} \text{ for } x_n \geq 0
\]

where two cases account for the absolute value in the original formulation.

As the map is iterated for some initial condition \(X_0\) in the attractor, the orbit will fall in the left or right half of the plane, designated \(L\) and \(R\) respectively, and for each iterate, the map applied will depend on whether \(X_n \in L\) or \(X_n \in R\). The orbit of the path can be specified to any degree of accuracy simply by knowing enough of its left-right history. The itinerary of a point is its \(L-R\) history, written as a doubly infinite sequence such as

\[...LLRLLLRL. LLRLLLRL...
\]

The first digit to the right of the binary point is \(L_1\), the current state of the system. Upon forward iteration of the map the current state will follow the itinerary. The itinerary specifies a periodic orbit uniquely in this map, and a right or left shift of the binary point gives the forward iterates or backward iterates of the map.

A periodic point with period \(k\) is a point \(X^*\) whose image under \(k\) mappings is again \(X^*\). For the set of parameters that make the Lozi map chaotic, all of these periodic points are unstable. Upon iteration, any state vector not precisely on a periodic point will wander away. The Lozi attractor is the closure of the dense set of all periodic points. There is an infinite number of periodic itineraries, and the periodic orbits that they define fill a fractal region of space. There are also non-periodic itineraries that correspond to chaotic orbits, but we can find a periodic itinerary that matches the itinerary of the chaotic orbit to as many places as we desire. Therefore, orbits whose itinerary is non-repeating (chaotic) lie in the closure of the open set of periodic orbits.
Consider a 5-periodic point $X^*$ with itinerary

$$\ldots LLRLRRLLRRL RRRLLRRLRRRL \ldots$$

Which point exactly does this itinerary represent? To answer this question, we rewrite the equations and in a more convenient form. Letting

$$X_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad U = \begin{bmatrix} 1 + \rho \\ 0 \end{bmatrix}, \quad L = \begin{bmatrix} \beta & \alpha \\ -1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} -\beta & \alpha \\ -1 & 0 \end{bmatrix}$$

(2.11)

gives us

$$X_{n+1} = U + LX_n \text{ for } X_n \in L$$
$$X_{n+1} = U + RX_n \text{ for } X_n \in R,$$

or

$$X_{n+1} = F(X_n).$$

Now we can easily express multiple iterates of the map in terms of its itinerary. For example, five iterates of the initial system state $X_n$ whose forward itinerary is $LLRLR$ are

$$X_{n+1} = U + LX_n$$
$$X_{n+2} = U + LU + LLX_n$$
$$X_{n+3} = U + RU + RLU + RLLX_n$$
$$X_{n+4} = U + RU + RRU + RRLU + RLLRX_n$$
$$X_{n+5} = U + LU + LRU + LRRU + LRRLU + LRRLLX_n$$

where $K$, the full itinerary matrix, is the product of the matrices $L$ and $R$ associated with the reverse of the itinerary (which is $LRRL$ above). The partial itinerary matrix $T$ is the sum of the identity $I$ and the first

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\( k-1 \) partial products of \( K \), which is \( I + L + LR + LRR + LRRL \) above.

A periodic point of period \( k \) is a fixed point (period one point) of the \( k \) times iterated map

\[
X_{n+k} = TU + KX_n.
\]

Solving for the period \( k \) point \( X^* \) we obtain

\[
X^* = [I - K]^{-1}TU.
\]

As an example, we can find the period-3 point whose itinerary is \( LLR \). We give \( \rho, \alpha, \) and \( \beta \) simple rational values for which the Lozi map has a strange attractor, \( \rho = 0, \alpha = \frac{1}{2}, \) and \( \beta = \frac{7}{4} \). A period-3 point is a fixed point of the map

\[
X_{n+3} = U + RU + RLU + RLLX_n
\]

where

\[
K = RLL = \begin{bmatrix}
\frac{-7}{4} & \frac{1}{2} \\
-1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\frac{7}{4} & \frac{1}{2} \\
-1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\frac{7}{4} & \frac{1}{2} \\
-1 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
\frac{-343}{64} & \frac{-57}{16} \\
\frac{-83}{16} & \frac{1}{2} \\
\end{bmatrix}
\]

\[
T = I + R + RL = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
+ \begin{bmatrix}
\frac{-7}{4} & \frac{1}{2} \\
-1 & 0 \\
\end{bmatrix}
+ \begin{bmatrix}
\frac{-7}{4} & \frac{1}{2} \\
-1 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
\frac{-69}{16} & \frac{-3}{2} \\
\frac{1}{2} & \frac{1}{2} \\
\end{bmatrix}
\]

Solving for the period three point we have

\[
X^* = \left[ \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix} - \begin{bmatrix}
\frac{-343}{64} & \frac{-57}{16} \\
\frac{16}{11} & \frac{1}{2} \\
\end{bmatrix} \right]^{-1} \left[ \begin{bmatrix}
\frac{-69}{16} & \frac{-3}{2} \\
\frac{1}{2} & \frac{1}{2} \\
\end{bmatrix} \right] \begin{bmatrix}
1 \\
0 \\
\end{bmatrix} = \begin{bmatrix}
\frac{-68}{157} & \frac{12}{471} \\
\frac{12}{471} & \frac{1}{2} \\
\end{bmatrix}
\]

Since the Lozi map is a well-defined geometrical transformation that maps a region of the plane into the attractor by successive stretches and folds, we expect characteristic rates of stretching and folding. For the infinite number of unstable periodic points in the attractor, these
directions of stretching and folding are locally linear and vary over the attractor.

Let's look at the characteristic directions and rates near a period one point. From the point of view of the fixed point $X^*$ the state vector is $X_n = \Delta X_n + X^*$. The next iterate of the map is $X_{n+1} = \Delta X_{n+1} + X^*$. As is the usual practice for control of a two-dimensional map, we expand in a Taylor series about the fixed point $X^*$ (this is unnecessary in the linear case, but the idea will be used for nonlinear systems later). We have

$$\Delta X_{n+1} + X^* = F(X^*) + \left[ \begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array} \right] \Delta X_n + O(\Delta X_n^2)$$

In the case of the Lozi map, the higher order terms in the Taylor series vanish. Since $X^*$ is the fixed point of the once iterated map $F(X^*) = X^*$, we have

$$\Delta X_{n+1} = \left[ \begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array} \right] \Delta X_n = A \Delta X_n,$$  

where $A$ is the Jacobian of the map.

Here is a simple example using the parameter values $\rho = 0$, $\alpha = \frac{1}{2}$, and $\beta = \frac{7}{4}$. Because of the absolute value in the Lozi map, the Jacobians will be different in the left and right halves of the plane:

$$A_L = J(f_L(x,y), g_L(x,y)) = \left[ \begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array} \right]$$  

$$= \left[ \begin{array}{cc} \frac{\partial f}{\partial x}(1 + \frac{1}{2}y + \frac{7}{4}x) & \frac{\partial f}{\partial y}(1 + \frac{1}{2}y + \frac{7}{4}x) \\ \frac{\partial g}{\partial x}(-x) & \frac{\partial g}{\partial y}(-x) \end{array} \right]$$  

$$= \left[ \begin{array}{cc} \frac{7}{4} & \frac{1}{2} \\ -1 & 0 \end{array} \right]$$

$$A_R = J(f_R(x,y), g_R(x,y)) = \left[ \begin{array}{cc} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array} \right]$$  

$$= \left[ \begin{array}{cc} \frac{\partial f}{\partial x}(1 + \frac{1}{2}y - \frac{7}{4}x) & \frac{\partial f}{\partial y}(1 + \frac{1}{2}y - \frac{7}{4}x) \\ \frac{\partial g}{\partial x}(-x) & \frac{\partial g}{\partial y}(-x) \end{array} \right]$$  

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Suppose that the state vector is \( X_n = \begin{bmatrix} \frac{5}{13} \\ \frac{6}{13} \end{bmatrix} \). Then

\[
\Delta X_n = X_n - X^* = \begin{bmatrix} \frac{5}{13} \\ \frac{6}{13} \end{bmatrix} - \begin{bmatrix} \frac{4}{13} \\ \frac{13}{13} \end{bmatrix} = \begin{bmatrix} \frac{1}{13} \\ \frac{1}{13} \end{bmatrix}.
\]

The Jacobian \( A_R \) applied to \( \Delta X \) gives us the change from the point of view of the fixed point. This change must equal the change given by the Lozi map applied to \( X_n \), when viewed from the fixed point. We should have

\[
\Delta X_{n+1} = A_R \Delta X_n = U + RX_n - X^*
\]

Indeed, we get

\[
\begin{bmatrix} \frac{-7}{4} & 1 & 2 \\ -1 & 0 & \frac{13}{13} \end{bmatrix} \begin{bmatrix} \frac{1}{13} \\ \frac{13}{13} \end{bmatrix} = \begin{bmatrix} \frac{5}{13} \\ \frac{13}{13} \end{bmatrix},
\]

and

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{-7}{4} & 1 & 2 \\ -1 & 0 & \frac{13}{13} \end{bmatrix} \begin{bmatrix} \frac{5}{13} \\ \frac{13}{13} \end{bmatrix} - \begin{bmatrix} \frac{4}{13} \\ \frac{13}{13} \end{bmatrix} = \begin{bmatrix} \frac{5}{13} \\ \frac{13}{13} \end{bmatrix}
\]

Usually the Jacobian is used this way to estimate the local linearized dynamics near the fixed point of a nonlinear system, but the Lozi map is already linear (except for the affine part \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)), so the Jacobian exactly describes the local dynamics near a periodic point.

Near a fixed point the direction in which the map stretches space is called the *unstable direction* and the direction in which the transformation contracts space is called the *stable direction*. The stable and unstable directions correspond to the *stable and unstable sets* near the fixed point. These directions are given by the eigenvectors of the Jacobian \( A \). The eigenvalues associated with the eigenvectors give the amount of stretch or compression of the space along these characteristic directions. The fact that the Lozi attractor has a hyperbolic structure
guarantees that all of the unstable periodic points on the Lozi attractor are saddles, that is, they have attracting and repelling directions.

Figure 2.3.4. This figure illustrates four different types of saddles. The numbers next to the points indicate the initial point and its next three iterates. The upper left saddle, a hyperbolic saddle, has eigenvalues $0 < \lambda_s < 1$, $\lambda_u > 1$. The upper right saddle, a flip-out saddle, has eigenvalues $0 < \lambda_s < 1$, $\lambda_u < -1$. The lower left saddle is a flip-in saddle, with eigenvalues $-1 < \lambda_s < 0$, $\lambda_u > 1$. The bottom right saddle is a flip saddle with eigenvalues $-1 < \lambda_s < 0$, $\lambda_u < -1$.

We designate $\rho$ as the control parameter and replace it with a function $\rho(x, y)$ that will change the dynamics of the Lozi map for one iteration whenever the system state comes within a certain distance of the periodic point we wish to stabilize. This temporary change in the dynamics is calculated so that the system state will land on the stable manifold of the periodic point on the next iterate. We have chosen $\rho = \rho(x, y)$ to be our parameter because of the specific way a change in $\rho$ affects the local dynamics. To calculate the perturbation that will direct the orbit onto the stable manifold we need to know something of the local dynamics of the periodic point we want to stabilize.
There is an attractor for \( \alpha = \frac{-7}{4}, \beta = \frac{1}{2}, \rho = 0 \). Let us stabilize the period one point whose itinerary is \(...RRR.RRR...\). Earlier we found

\[
X_{n+1} = U + RX_n,
\]

so for the periodic orbit \(...RRR.RRR...\)

\[
\begin{bmatrix}
x_{n+1} \\
y_{n+1}
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{-7}{4} & \frac{1}{2} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}.
\]

The fixed point of this once iterated map is found by solving

\[
X^* = [I - K]^{-1} TU,
\]

where \( T = I \) and \( K = \begin{bmatrix} \frac{-7}{4} & \frac{1}{2} \\ -1 & 0 \end{bmatrix} \). Numerically the fixed point is

\[
X^* = \left[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{-7}{4} & \frac{1}{2} \\ -1 & 0 \end{bmatrix} \right]^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{13} \\ \frac{13}{13} \end{bmatrix}.
\]

We will perturb the dynamics near this point to steer the orbit onto the stable manifold.

The local dynamics are determined by the eigenvalues \( \lambda_u, \lambda_s \) and the eigenvectors \( e_u, e_s \) of \( A \). For reasons that will become apparent shortly we will also need the left eigenvectors \( f_u \) and \( f_s \). These vectors are orthogonal to \( e_u \) and \( e_s \) and are normalized such that

\[
\begin{align*}
f_u \cdot e_s &= f_s \cdot e_u = 0 \\
f_u \cdot e_u &= f_s \cdot e_s = 1
\end{align*}
\]
Figure 2.3.5. The right and left (contravariant) eigenvectors are shown.

The eigenvalues of $A_R$ are $\lambda_u = \frac{-7-\sqrt{17}}{8}$, $\lambda_s = \frac{-7+\sqrt{17}}{8}$, while the associated eigenvectors are

$e_u = \begin{bmatrix} \frac{7+\sqrt{17}}{8} \\ -1 \end{bmatrix}$, $e_s = \begin{bmatrix} \frac{7-\sqrt{17}}{8} \\ 1 \end{bmatrix}$, $f_s = \begin{bmatrix} \frac{4}{\sqrt{17}+7} \\ \frac{\sqrt{17}-7}{2\sqrt{17}} \end{bmatrix}$, $f_u = \begin{bmatrix} -\frac{4}{\sqrt{17}+7} \\ \frac{\sqrt{17}+7}{2\sqrt{17}} \end{bmatrix}$.

We also need to know how the position of the fixed point, its associated stable and unstable manifolds, and thus the local dynamics, change upon a parameter perturbation. For example, if we change $\rho$ from $0$ to $\frac{1}{8}$ then the fixed point $X^*$ changes from $\begin{bmatrix} \frac{4}{13} \\ \frac{9}{26} \end{bmatrix}$ to $\begin{bmatrix} \frac{9}{26} \\ \frac{9}{26} \end{bmatrix}$, for a change $\Delta X = \begin{bmatrix} \frac{1}{26} \\ 0 \end{bmatrix}$. Therefore the change in the location of the fixed point with change in parameter $\rho$ is $\frac{\partial}{\partial \rho} X^* = \begin{bmatrix} \frac{4}{13} \\ \frac{4}{13} \end{bmatrix}$.

Let $\delta \rho$ be some small change in $\rho$. Then the location of the perturbed fixed point upon this change will be $X^* + \delta \rho \frac{\partial}{\partial \rho} X^*$ (see Figure 2.3.5).
2.3.6).

For convenience of notation we write $g = \frac{\partial}{\partial \rho} X^*$ so that $\delta \rho g = \delta \rho \frac{\partial}{\partial \rho} X^*$.

In one iteration a phase point $X_n$ evolves to $X_{n+1}$. Relative to the fixed point $X^*$ these points may be represented by $\Delta X_n = X_n - X^*$ and $\Delta X_{n+1} = X_{n+1} - X^*$. Our goal is to push the state variable onto the stable manifold of the desired periodic orbit in one iteration under a perturbation of the dynamics. Once on the stable manifold, the perturbation is turned off and the natural dynamics will draw the orbit into the fixed point. Figure 2.3.7 illustrates how this process looks schematically.
Figure 2.3.7. The schematic above illustrates the basic principle of OGY control. In 2.3.7a, we see the current system state with an arrow pointing to its next iterate. In 2.3.7b the system has been perturbed by an amount calculated to send the system state to the stable manifold of the unperturbed system in one iterate of the map. In 2.3.7c the perturbation has been turned off, as the system state is now on the unperturbed stable manifold. The stable dynamics will henceforth evolve the system state toward the periodic orbit as in 2.3.7d.

We will vary the parameter \( \rho \) to achieve this perturbation, so we write the local dynamics from the point of view of the shifted fixed point. Using a prime (') to indicate the quantities in the perturbed system, the state vectors are now

\[
\Delta X_n' = \Delta X_n - \delta \rho g
\]

and

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\[ \Delta X'_{n+1} = \Delta X_{n+1} - \delta \rho g. \]

We can treat the dynamics at the shifted fixed point as we did in 2.13 and write
\[ \Delta X'_{n+1} = A' \Delta X'_n, \]
where \( A' \) is the Jacobian at the perturbed fixed point. Assuming a small perturbation \( \delta \rho \), we may approximate \( A' \) by \( A \) and write
\[ \Delta X_{n+1} - \delta \rho g = A[\Delta X_n - \delta \rho g]. \]  
(2.14)
In this case, the parameter change affects only the side-to-side shift of the attractor and not the matrix \( A \), so \( A' = A \).

A state vector that is on the stable manifold is orthogonal to the left unstable eigenvector. Therefore the requirement that the next state vector be on the stable manifold of the unshifted fixed point can be written
\[ f_u \cdot \Delta X_{n+1} = 0. \]

We must now express \( \delta \rho \) as a function of the eigenvectors and eigenvalues of the system and current position \( \Delta X_n \). Rewriting 2.14 as
\[ \Delta X_{n+1} = \delta \rho g + A[\Delta X_n - \delta \rho g] \]  
(2.15)
and dotting both sides of this equation with \( f_u \) yields
\[ 0 = f_u \cdot \delta \rho g + f_u \cdot A[\Delta X_n - \delta \rho g]. \]
We wish to solve for \( \delta \rho \), so to this end we write \( A \) in terms of its components via the representation (see Appendix A)
\[ A = \lambda_u e_u^T f_u + \lambda_s e_s^T f_s \]  
(2.16)
Substituting 2.16 into 2.15, we obtain
\[ 0 = \delta \rho f_u \cdot g + f_u \cdot [\lambda_u e_u f_u + \lambda_s e_s f_s][\Delta X_n - \delta \rho g]. \]
\[ \delta \rho = \frac{\lambda_u f_u \cdot \Delta X_n}{\lambda_u - 1 \cdot f_u \cdot g}, \quad (2.17) \]

which gives the control perturbation we need based on the current system state as seen from the unperturbed fixed point and the precalculated quantities \( \lambda_u, f_u \) and \( g \).

Let's try this scheme on the example discussed above. Substituting the values of \( \lambda_u, f_u, \Delta X \) and \( g \) we found earlier into 2.17, we obtain

\[ \delta \rho = \frac{-7 - \sqrt{17}}{8} \cdot \frac{4\sqrt{17} - 7}{2\sqrt{17}} \cdot \Delta X_n \]

Whenever an iterate of the map enters a small box around the fixed point (whose size is limited by the extent of the local stable manifold), we change the value of the parameter by \( \delta \rho \) for one iteration. Suppose an iterate of the map lands on \( X_n = \left[ \frac{3}{13}, \frac{-2}{13} \right] \), so \( \Delta X_n = X_n - X^* \). Then our control rule says we need to apply a perturbation

\[ d\rho = \frac{7\sqrt{17} + 17}{32} \cdot \frac{4\sqrt{17} - 7}{2\sqrt{17}} \cdot \frac{-1}{13} \cdot \frac{2}{13} = \frac{-17 - 15\sqrt{17}}{104\sqrt{17}}. \]

A quick calculation shows that

\[ \begin{bmatrix} 1 + \frac{-17 - 15\sqrt{17}}{104\sqrt{17}} \\ 0 \end{bmatrix} + \begin{bmatrix} -7/4 \\ -1 \end{bmatrix} \begin{bmatrix} 3/13 \\ -3/13 \end{bmatrix} = \begin{bmatrix} 39\sqrt{17} - 17 \\ -104\sqrt{17} \end{bmatrix} \]
which, when translated to the fixed point, is on the stable manifold of \( X^* = \left[ \frac{4}{13}, \frac{-4}{13} \right] \). The control process for this example is depicted in Figure 2.3.8 below.

![Figure 2.3.8](image)

**Figure 2.3.8.** This figure illustrates the shifting of the manifolds necessary for control of the Lozi map, along with the measured quantities necessary to calculate the proper shift.

An iterate of the map \( X_n \) lands near the fixed point \( X^* \) (its manifolds are the dashed lines). The correction is applied, shifting the fixed point (and its manifolds) to \( X^* + \delta \rho X^* \) (solid lines). Under these dynamics the point \( X_n \) maps to \( X_{n+1} \). The correction is turned off, and the system state is on the stable manifold of the unshifted fixed point. The unperturbed dynamics now act on the system state under successive iterations to carry it along the stable manifold into the fixed point.

We can check this by dotting our answer minus the fixed point with the unstable left eigenvector:
We can control higher periodic points similarly. Control of a single point in a periodic orbit guarantees control of the other points in the orbit. The method of control is essentially the same as one would use to control the Hénon map, except that the control would have to be applied continually to the Hénon map to account for the non-linearity of the stable and unstable manifolds associated with the fixed points.

Formulation of the OGY control rule is not difficult, even for physical systems for which no mathematical model is known. A phase space may be built from a time series (a list of successive measurements of some property of the system) by the method of delay coordinate embedding [7], and the fixed points and local linearized dynamics may be determined by a combination of the methods of closest approach, least squares fit, and Newton’s method.
3. The Horseshoe

3.1. The Smale horseshoe

We have seen that piecewise linear chaotic systems are easy to control. The analysis of the dynamics of nonlinear systems, both maps and flows, is made simpler by observing the correspondence of their dynamics with those of an abstract chaotic structure, the Smale horseshoe. The horseshoe is a fundamental chaotic object. The dynamics of the invariant set of the Smale horseshoe map mimic, in abstract, the dynamics of a generic chaotic map in the plane; that is, the horseshoe contains unstable periodic orbits and chaotic orbits and their stable and unstable sets, as does the generic chaotic map. We will investigate the dynamics of the horseshoe, and show that there are similar dynamics in the Hénon map, and that a continuous interpolation between the invariant set of the horseshoe map (along with its stable and unstable structures) and its first iterate, provides a model for the dynamics of a chaotic flow in less abstract systems. We begin with a few definitions from Devaney[9].

A system is structurally stable if every nearby system has essentially the same dynamics. Structural stability is an important property for a chaotic system to have if we wish to control its unstable periodic orbits (UPOs). To control a chaotic system we must perturb a system parameter, and we require that the perturbed system have similar topological structure to the unperturbed system. It would be impossible to use capture-and-release control (C&R), for instance, if the UPO we are trying to stabilize disappeared upon a small perturbation (see section 5.3).

**Definition 3.1.** Let \( f \) and \( g \) be two maps. The \( C^0 \) distance between \( f \) and \( g \) is given by

\[
d_0(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|.
\]

The \( C^r \) distance is given by

\[
d_r(f, g) = \sup_{x \in \mathbb{R}} (|f(x) - g(x)|, |f'(x) - g'(x)|, \ldots, |f^r(x) - g^r(x)|)
\]
Definition 3.2. Let $f : A \to A$ and $g : B \to B$ be two maps. Then $f$ and $g$ are said to be topologically conjugate if there exists a homeomorphism $h : A \to B$ such that $h \cdot f = g \cdot h$.

Definition 3.3. Let $f : J \to J$. Then $f$ is said to be $C^r$ structurally stable on $J$ if $\exists \varepsilon > 0$ such that whenever $d_r(f, g) < \varepsilon$ for $g : J \to J$, it follows that $f$ is topologically conjugate to $g$.

The classic Smale horseshoe is formed as follows. Consider the stadium shaped region $D$ in Figure 3.1.1 below. Squash $D$ in the vertical direction by a factor of $\delta < \frac{1}{2}$ and stretch it in the horizontal direction by a factor of $\rho \geq 2$. Bend the region over in the shape of a horseshoe and lay the horseshoe over $D$ in the manner shown. Now repeat this process, stretching and folding the horseshoe again by the same factors and laying it in the stadium.

![Diagram of the Smale stadium](image)

Figure 3.1.1. The Smale stadium consists of a rectangular central region fitted with semi-circular endcaps.
Figure 3.1. 2. The Smale stadium, squashed, bent, and placed back inside itself.

Call the action of the map $F$. There is a unique attracting fixed point $p$ in $D_1$ since $F$ is a contraction mapping, and as $D_2$ is mapped to $D_1$, all points in $D_1 \cup D_2$ tend to $p$ under iteration, that is, $\lim_{n \to \infty} F^n(q) = p$ for all $q \in D_1 \cup D_2$. Furthermore, any point $r \in S$ whose image is not in $S$ for all $n$ obeys $\lim_{n \to \infty} F^n(r) = p$. We are interested in the points $s \in S$ such that $F^n(s) \in S$ for all $n$.

Consider the two segments of $S$ that are mapped back inside $S$ by the horseshoe map $F$. Call these $H_0$ and $H_1$, and their preimages $V_0$ and $V_1$ (see Figure 3.1.3). Since $F : S \to S$ is a linear map, it preserves horizontal and vertical lines in $S$. The width of $V_0$ and $V_1$ are $\rho$ and the height of $H_0$ and $H_1$ are $\delta$, and if $h$ is any horizontal line segment in $S$ whose image under $F$ is also in $S$, then the length of $F(h)$ is $\rho h$. Likewise if $v$ is any vertical line segment in $S$ whose image under $F$ is...
also in $S$ then the length of $F(v)$ is $\delta v$.

Figure 3.1.3. The intersection of $S$ and $F(S)$ gives $H_0$ and $H_1$, and the preimages of these horizontal strips are $V_0$ and $V_1$.

Suppose $F^n(s) \in S \forall n > 0$. Then $s$ must be in $V_0 \cup V_1$, $F(s) \in V_0 \cup V_1$, $F^2(s) \in V_0 \cup V_1$, ..., for all points not in $V_0 \cup V_1$ map to $D_1 \cup D_2$. Thus, we have that $s \in F^{-n}(V_0 \cup V_1)$ for all $n > 0$. The inverse image of any vertical strip of width $w$ in $V_0$ or $V_1$ that extends from the bottom to the top of $S$ is a pair of strips of width $\frac{1}{\rho}w$, one in $V_1$ and one in $V_0$, that extend from the bottom to the top of $S$. The inverse image $F^{-1}$ of $V_0 \cup V_1$ is a set of four rectangular strips of width $\frac{1}{\rho^2}w$, two in $V_0$ and two in $V_1$ (see Figure 3.1.5), the inverse image of $F^{-1}(V_0 \cup V_1)$ is a set of eight vertical strips of width $\frac{1}{\rho^3}w$, etc. Therefore $\lim_{n \to \infty} F^{-n}(V_0 \cup V_1)$ is the product of a Cantor set with a vertical interval. Any point $s \in S$ such that $F^n(s) \in S \forall n > 0$ must be in this set which we label $\Lambda_+$. By the same type of reasoning, we see that if a point $s \in S$ such that $F^{-n}(s) \in S \forall n > 0$, then it must belong to a product of a Cantor set with a horizontal interval, and we label this set $\Lambda_-$. Any point $s \in S$
such that $F^n(s) \in S \forall n$ must be in the intersection $\Lambda = \Lambda_+ \cap \Lambda_-.$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.1.4}
\caption{The second iterate of the Smale horseshoe map is shown above.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.1.5}
\caption{Shown is the intersection of the horizontal and vertical strips after the first two iterates of the map. In the limit the intersection is a Cantor set.}
\end{figure}

Just as we did with the tent map, we may define a symbol sequence on $\Lambda$. The sequence is doubly infinite and is written as

$$...s_{-2}, s_{-1}, s_0, s_1, s_2, ...,$$  \hspace{1cm} (3.1)

where the $s_j$ are 0 or 1 depending on which vertical strip $V_0$ or $V_1$ $s$ is in at the $j$th forward iterate of the map, and the $s_{-j}$ are 1 or 0 depending
on which horizontal strip \( s \) is in on the \( j \)th iterate of the map's inverse. The sequence \( \ldots s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots \) uniquely defines a point in \( \Lambda \), and the left or right shift of the binary point gives the backward or forward iteration of that point respectively. This shift map is a model for the dynamics of \( s \) under \( F \) restricted to \( \Lambda \).

As \( F \) is topologically conjugate to the shift on the symbol sequence, we can define a metric on \( F \) by

\[
d[(s), (t)] = \sum_{i=-\infty}^{\infty} \frac{|s_i - t_i|}{2^i}
\]

where \( (s) = \ldots s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots \).

**Definition 3.4.** Consider a set \( Q \) and a mapping \( F : Q \to Q \). Two points \( p_1 \) and \( p_2 \) are forward asymptotic if \( F^n(p_1), F^n(p_2) \in Q \) \( \forall n \geq 0 \) and \( \lim_{n \to \infty} |F^n(p_1) - F^n(p_2)| = 0 \).

**Definition 3.5.** Two points \( p_1 \) and \( p_2 \) are reverse asymptotic if \( F^n(p_1), F^n(p_2) \in Q \) \( \forall n \leq 0 \) and \( \lim_{n \to -\infty} |F^{-n}(p_1) - F^{-n}(p_2)| = 0 \).

Points in any vertical segment in \( \Lambda_+ \) are forward asymptotic, and points in \( \Lambda_- \) are reverse asymptotic. We can now formally define the stable and unstable set of a point \( s \) in \( \Lambda \).

**Definition 3.6.** The stable set \( W^s \) of \( s \) is the set of points \( t \) that are forward asymptotic to \( s \), or \( W^s(s) = \{ t \mid |F^n(t) - F^n(s)| \to 0 \text{ as } n \to \infty \} \) and the unstable set \( W^u \) of \( s \) is the set of points \( t \) that are reverse asymptotic to \( s \), \( W^u(s) = \{ t \mid |F^{-n}(t) - F^{-n}(s)| \to 0 \text{ as } n \to \infty \} \).

Consider a fixed point \( \bar{s} = \ldots 111.111... \in \Lambda \). Its stable set contains not only the vertical segment \( l_s \) in which it resides, but also any segment \( l \) that maps into \( l_s \). Thus the stable set of \( \bar{s} \) consists of \( \bigcup_k F^{-k}(l_s) \). The unstable set of \( \bar{s} \in \Lambda \) is different in form. Let \( l_u \) be the horizontal
segment in which $s$ resides. Forward iteration of the map will stretch and fold $l_u$, giving the structure in Figure 3.1.6.

![Diagram with $W_s$ and $W_u$ regions]

*Figure 3.1.6. The stable and unstable sets of the point $s$ are shown in this figure.*

Periodic points are dense in the horseshoe, and orbits are *topologically transitive* and have *sensitive dependence on initial conditions*. These conditions are the signature of chaos, and a system that can be shown to have a horseshoe, can be proven to be *chaotic*, at least on a subset of its attractor, according to the definition of Devaney [9, Robert Devaney, Chaotic Dynamical Systems, Addison-Wesley, 1989].

**Definition 3.7.** $f : J \rightarrow J$ is said to be topologically transitive if for any pair of open sets $U, V \in J$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.

**Definition 3.8.** $f : J \rightarrow J$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in J$ and any neighborhood $N$ of $x$ there exists $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

**Definition 3.9.** Let $V$ be a set. $f : V \rightarrow V$ is said to be chaotic on $V$ if
1. $f$ has sensitive dependence on initial conditions
2. $f$ is topologically transitive
3. Periodic points are dense in $V$

Devaney's definition is equivalent to that of Taylor and Toohey (Def. 2.9) when $V$ is a uniform Hausdorff space. We can see that the horseshoe is chaotic by considering the symbolic dynamics.

- **Density of periodic points**: We must exhibit an orbit that converges to an arbitrary point $s = \ldots s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots$. Let $r = \ldots r_0, r_0, \ldots r_n, s_0, s_0, \ldots s_n, s_0, \ldots s_n \ldots$ be the sequence that repeats the first $n$ symbols of $s$. Then $d[r, s] \leq \frac{1}{2^n}$, and $r \rightarrow s$.

- **Topological transitivity**: We must exhibit a point that comes arbitrarily close to every other point in $\Lambda$. Consider the symbol string $s^* = \ldots 011 010 001 000, 11 10 01 00, 1 0 0 1, 00 01 10 11, 000 001 010 011 \ldots$ formed by concatenating all possible permutations of strings of length $k$, $k = 1, \ldots, n$ (the commas merely delimit the groups of permutations of strings of length $k$). Then for some shift $\sigma^*$ the itinerary will agree with that of any point in $\Lambda$ to the precision we desire.

- **Sensitive dependence on initial conditions**: Let $r, s \in \Lambda$ and have identical symbol strings in the first $n$ places to the right of the binary point and for the first $m$ places to the left. Then iteration forward or backward will eventually shift out the identical strings and the orbits will diverge.

To sum up the action of the horseshoe map, we note:

- The horseshoe mapping of the Smale stadium has an invariant set $\Lambda$ that is the product of two Cantor sets.
- Dynamics on $\Lambda$ are chaotic under the map.
• There are unstable periodic orbits of all periods, and periodic points have stable sets that consist of the vertical line segment in which they reside and the set of all segments that map to the vertical segment.

• The stable and unstable sets of points in $\Lambda$ are orthogonal.

3.2. The Hénon map and its horseshoe structure

The horseshoe structure arises naturally in dissipative chaotic maps of the plane and Poincaré maps of chaotic flows. Consider the Hénon map

\[
\begin{align*}
    x_{n+1} &= a + by_n - x_n^2 \\
    y_{n+1} &= x_n
\end{align*}
\]

where $a$ and $b$ are parameters. The Hénon map is invertible, dissipative and has chaotic dynamics for certain values of $a$ and $b$. A few thousand iterates of the Hénon map are shown in Figure 3.2.7.

![Figure 3.2.7. Several thousand iterates of the Hénon map are shown for parameters $a = 2, b = -\frac{1}{4}$. This figure and Figure 3.2.8 were produced by the program Dynamics by James Yorke.](image)
This map has a fixed point $X^*$ at approximately (.922, .922) and it also has a set of points that are forward asymptotic to $X^*$. This set of points consists of the points in the intersection of the stable and unstable manifolds of $X^*$. Figure 8 shows the Hénon attractor and part of the stable manifold $E^s$ of the period one point (fixed point) $X^*$.

![Figure 3.2.8](image)

*Figure 3.2.8. Shown is the stable manifold of a period-one point of the Hénon map and the Hénon attractor.*

The intersection of the unstable manifold $E^u$ of $X^*$ with its stable manifold is the analog of the set of points in the intersection of the stable set $W^s$ and the unstable set $W^u$ of a period one point in $\Lambda$, the invariant set of the Smale horseshoe. This set $\Lambda_{X^*}$ is defined by $E^s_x \cap E^u_{X^*} = \Lambda_{X^*}$.

Let us follow a segment of the stable manifold $E^s_{X^*}$, as it is iterated forward. Consider a segment $l_0 \in E^s_x$ with endpoints $X_a, X_b \in \Lambda_{X^*}$ and containing no other points of $\Lambda_{X^*}$. Call the set of all segments so defined $L_{X^*}$. As only two of the points of $l_0$ are in $\Lambda_{X^*}$, the segment must map only to other segments like itself, that is, the forward orbit.
of \( l_0 \) is in \( L_{X^*} \). Under the action of the map \( X_a \) and \( X_b \), will approach \( X^* \) arbitrarily closely, and the length of \( l_0 \rightarrow 0 \) as \( n \rightarrow \infty \).

As in the horseshoe, the unstable manifold of the Hénon attractor \( A \) is the same for periodic points of all periods and for points in chaotic orbits. Each periodic point has its own stable manifold, and the same argument as above can be made for each of these stable manifolds, namely, any point in a segment \( l_k \in L_{X^*} \), where \( X^*_j \) is the \( j \)th point in the periodic orbit \( i \) will approach \( X^*_j \) arbitrarily closely as \( n \rightarrow \infty \). The set \( S_p \) of all stable manifolds for all periodic points in the Hénon attractor is dense in the basin of attraction of the attractor. The closure of the set \( S_p \) is the entire basin, and the set that closes \( S_p \) is the set of stable manifolds \( S_c \) of the chaotic orbits in \( A \).

**Definition 3.10.** Let \( \Lambda \) be an invariant set for a discrete dynamical system defined by \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \). A hyperbolic structure for \( \Lambda \) is a continuous invariant direct sum decomposition \( T_\Lambda \mathbb{R}^n = E^u_\Lambda \oplus E^s_\Lambda \) with the property that there are constants \( C > 0 \), \( 0 < \lambda < 1 \) such that:

1. if \( v \in E^u_\Lambda \), then \( |Df^{-n}(x)v| \leq C \lambda^n |v| \);
2. if \( v \in E^s_\Lambda \), then \( |Df^n(x)v| \leq C \lambda^n |v| \).

The invariant set \( \Lambda \) of the Smale horseshoe has a hyperbolic structure. Essentially, a hyperbolic structure implies that all the periodic points are of saddle type. The Hénon map may be shown to have a horseshoe structure; that is, it may be shown to contain a chaotic set and unstable points of all periods. This set is a product of Cantor sets, and lives in the unstable manifold of the Hénon attractor. However, we do not know whether all the points that appear to be in the attractor are part of a hyperbolic structure (there are good reasons to suppose that they are not[10]). This means that what appears to be chaotic motion over the entire connected unstable manifold of the Hénon attractor may instead be a chaotic transient preceding asymptotic approach to any one of an infinite number of stable periodic orbits of arbitrarily high period. We take the view that although we don’t know for sure whether the apparent chaotic evolution of an orbit over an attractor is
really chaotic or just a chaotic transient, for the purpose of developing a control strategy there is no difference. Whether we have a true chaotic orbit or one of very long period, our goal is to stabilize low period unstable orbits, and to navigate through the attractor along these orbits. In either a simulation on a computer or in a physical experiment, chaotic orbits and orbits of very large period will appear to be the same.

3.3. The suspension of the horseshoe

Our purpose in considering the horseshoe map is to use its suspension as an abstract representation of the dynamics of the flow of a chaotic dynamical system. The dynamics of a Poincaré map of a flow determine the dynamics of the flow itself; that is, if the Poincaré map has a periodic point of period $n$, the flow has a periodic orbit of period $n$, and if the Poincaré map has a chaotic orbit, the flow has a chaotic orbit. The stable and unstable manifolds of the Poincaré map of a chaotic flow act very much like the stable and unstable sets of the horseshoe, and with this in mind, we suspend the horseshoe in a way that forces the stable and unstable sets of the suspension to act like the stable and unstable manifolds of a flow. Consideration of the geometry of this suspension leads to the construction of a general framework in which control by stable subspace targeting takes place.

We provide some definitions:

**Definition 3.11.** Let $\varphi$ be a flow on a manifold $M$ with vector field $X$ and suppose that $\Sigma$ is a submanifold of $M$ of codimension 1 that satisfies

1. Every orbit of $\varphi$ meets $\Sigma$ for arbitrarily large positive and negative times.
2. If $x \in \Sigma$ then $X(x)$ is not tangent to $\Sigma$.

Then $\Sigma$ is said to be a *global cross-section* of the flow.
Definition 3.12. Let $y \in \Sigma$ and $\tau(y)$ be the least positive time for which $\varphi_{\tau(y)}(y) \in \Sigma$. The Poincaré map for $\Sigma$ is defined to be $P(y) = \varphi_{\tau(y)}(y)$, $y \in \Sigma$.

As with the baker’s map, the horseshoe can be suspended to give a three-dimensional flow. While the baker’s map required some care to ensure that it was not two to one, the suspension of the horseshoe map is straightforward, and its generalization can model the actual dynamics of a physical system rather well.

Consider the physical description of the iterative process that leads to the Smale horseshoe. We take a stadium $D$ composed of a central rectangular region $S$ and two endcaps $D_1$ and $D_2$, stretch it linearly in the horizontal direction, while squashing it linearly in the vertical direction, fold it over and reinsert it into its original boundaries in such a way that only points in $S$ get mapped back into $S$. This insures a structurally stable dynamical system consisting of the points $s \in \Lambda$, and the mapping $F$ restricted to $\Lambda$. The itinerary $S(s) = (\ldots s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots)$ is a doubly infinite sequence of symbols 1 and 0 that give a record of the travels of $s$ under $F$. Take $\Lambda$ and its stable and unstable sets $W^s$ and $W^u$ and call their union $G$. We cross $G$ with the circle $C$ to get $G \times C$, and stretch and fold $G \times C$ over itself in such a way that after one circuit of the circle vertical lines in $V_0$ and $V_1$ have mapped into $V_0$. 
and $V_1$.

Figure 3.3.1. We may connect the line segments in the stable set of the horseshoe thereby changing them to manifolds.
The ends of the stable sets are connected in this figure to make the stable sets into stable manifolds. When the horseshoe is stretched and folded, the manifolds unbend as the straight segments join.

The unstable set is a differentiable manifold, and we can make the stable set a differentiable manifold by including in it the paths traced by points in the right hand side of the top edge of $\Lambda_+$ as it stretches and folds over to rejoin the left hand side, as well as the paths traced out by the complimentary points on the bottom of $\Lambda_+$ (see Figures 3.3.1 and 3.3.2).

We can fold other horseshoes in continuous time that have properties superficially similar to the apparent stretching and folding of other dynamical systems. We emphasize that just because the stretching and folding seems similar, there is no guarantee that a horseshoe structure persists through an infinite number of stretch and fold operations. We have already seen the similarity between the stable and unstable sets of the Smale horseshoe and those of the Hénon attractor. In section 4.1 we will look at the stretching and folding of the pendulum attractor, and compare the pendulum dynamics to those of the suspended horseshoe. Depending on the values of parameters in the pendulum equation, the pendulum may have a strange attractor corresponding to chaotic motion that includes excursions over the top of the pivot, or chaotic motion for which the pendulum bob never goes over the top. In either case, there exist unstable periodic orbits whose manifolds are flip saddles, like the flip saddles of the Smale horseshoe. In the course
of one system cycle, the manifolds associated with a period one orbit in either the pendulum or the suspended Smale horseshoe undergo a half twist before rejoining, forming a Mobius band.

The rotation of the manifolds of the flip saddle are what make possible capture and release control, and a knowledge of the movement of the manifolds as an attractor goes through its cycle of stretching and folding allow us to design general control rules beyond the map based rules of OGY.

3.4. The control of the horseshoe

Like the baker’s map, the horseshoe map can be controlled. We imagine that we have two parameters that control the geometry of the horseshoe; \( \delta < \frac{1}{2} \) which controls how much the stadium is flattened, and \( \rho \) which controls the stretching. We fix \( \delta \) and allow \( \rho \) to vary so that the position of the stable set \( W^s \) of \( \Lambda \) shifts position as \( \rho \) varies. The change in position of the stable set is continuous with change in \( \rho \), as long as the endcaps \( D_1, D_2 \) do not map to the central rectangular region \( S \). Suppose an iteration of the horseshoe map brings a system state \( s \) within \( \varepsilon \) of the state \( s^* \), the state we wish to control. There will be a parameter value \( \rho(s^*, s) \) for which the dynamics of the map carry \( s \) to \( l_{s^*} \), the vertical segment containing \( s^* \), in one iteration. Successive iteration of the map brings any point in \( l_{s^*} \) as close to \( s^* \) as we like. Figure 3.4.1 below is a sequence showing how \( l_{s^*} \) approaches \( s^* \) when \( s^* \) is the period one point ...1, 1, 1.1, 1, 1.... The points shown are members
of the invariant set of the mapping.

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Figure 3.4.1. Here, the orbit of a line \( l_8 \) in the stable set of a period one point with itinerary \( ...111.111... \) is shown under three iterations of the horseshoe map.

In the same way we can target a preimage of \( l_8^* \), and again, successive iteration will bring the segment as close as we like to \( s^* \). If our perturbation is limited to a small \( \varepsilon \), we may choose a set of periodic orbits \( \{s_1, s_2, ..., s_n\} \), each of which we can target by the method above. We can restrict the number of preimages of \( l_{s_n} \) we will consider in our targeting procedure, noting that the accuracy of our calculation of the
position of the preimages of \( l_{s_k} \) decreases as we go further back in time. If we choose a sufficient number of orbits, an \( \varepsilon \) region around any vertical line in the stable set of one particular orbit will contain a member of the stable set of another orbit in the set \( \{ s_1, s_2, \ldots s_n \} \). Then, by threading our way through the preimages of the \( l_{s_k}, k = 1, \ldots n \) we can move from one orbit to another in an efficient fashion.

We can define an operator that evolves local behavior continuously in the horseshoe. Consider the suspension of the horseshoe \( G \times C \), that is, the union of the stable and unstable sets of the invariant set \( \Lambda \) of the horseshoe mapping \( F \) crossed with the circle \( C \). Define a periodic folding operator \( P(\tau), \tau = t \mod 2\pi \) such that \( P(\tau)(G \times C) \) gives the Poincaré section of \( G \times C \) at time \( \tau \). Then \( P(0)(G \times C) = P(2\pi)(G \times C) \) and \( P(\tau)(G \times C) = P(\tau \pm 2\pi n)(G \times C) \). We noted earlier that a piece \( l_k \) of the stable set \( \Lambda \) of the horseshoe maps to \( l_{k+1} \) after one iterate, and to \( l_{k+n} \) after \( n \) iterates. The discrete time operator \( P(2\pi) \) will be designated \( P \), and \( P^n \) designates \( n \) applications of \( P \). Therefore \( P(2\pi) l_k = P l_k = l_{k+1} \), and \( P^n l_k = l_{k+n} \). Similarly, the unfolding operator \( P^{-1}(\tau), \tau = -t \mod -2\pi \) is defined so that \( P(\tau)(G \times C) \) is the Poincaré section of \( G \times C \) at time \( \tau \).

The operator \( P(\tau) \) takes an initial segment \( l \) and evolves it smoothly from one iterate of the horseshoe map to the next. If the initial segment \( l_0 \) was in the stable set of a period one point, for instance, and \( l_0 \) was inside the attractor, the map would take \( l_0 \) to \( l_1 \), the segment one layer in and on the other side of the saddle. When the continuous time operator is applied to \( l_0 \), as \( \tau \) goes from 0 to \( 2\pi \), \( P(\tau)l_0 \) traces out a twisting ribbon that becomes narrower each cycle as it approaches the periodic orbit. \( P(\tau)l_0 \) may even braid the ribbon. The fold operator evolves periodic orbits along their flow, and the set of periodic orbits is an infinite set of intertwined links. The orbits of a continuous time dynamical system can link and knot in exceedingly complex ways. It has recently been shown, for instance, that the set of periodic orbits of the Lorenz attractor link in every possible way (the set of links of periodic orbits contains representatives of every tame knot isotopy class)[11].

Let \( \Lambda(\rho) \) be the invariant set of the suspension of the horseshoe,
where the dependence of \( \Lambda \) on the stretching parameter \( \rho \in -\delta \leq 0 \leq \delta \) has been made explicit, and let \( \Lambda(0) \) be the unperturbed system. The bound \( \delta \) is such that the end-caps of the stadium never map into \( S \). Then \( \Lambda(\rho) \) is the set of interlinked periodic orbits of all periods, and \( P(\tau)\Lambda(0) \) is the continuous time system in which points \( s \in \Lambda \) get mapped to their images \( Ps \in \Lambda \) at \( \tau = 2n\pi, \ n = 1, 2, ... \). We choose to stabilize a periodic point \( s' \). Suppose we monitor the evolution of an initial condition \( s \in \Lambda(0) \) until it is close enough to \( l_{s'} \), the vertical segment in which \( s' \) resides, so that a perturbation \( \rho(s, s') \in -\delta \leq 0 \leq \delta \) for one iteration would serve to place \( s \) in \( l_{s'} \). We apply this perturbation for one iterate and then turn it off. The system state \( s \) is then in \( l_{s'} \), and it approaches \( s' \) as \( t \to \infty \). This is the discrete time control described above. We will look later at the meaning of control in continuous time from the perspective of control of the suspended horseshoe.

The set \( \Lambda \) has a hyperbolic structure, so all of its periodic points under \( F \) are saddles. It is not difficult to see that a segment \( l \in W^s \) is flipped over upon iteration if it is in the right side of \( S \), and retains its orientation upon iteration if it is in the left half of \( S \). This implies that the local stable sets of periodic orbits whose itineraries contain \( n \) experience \( n \) flips, so a saddle will be a flip saddle if \( n \) is odd, and a regular saddle if \( n \) is even.

Note that even though a periodic point of a Poincaré section may not have a flip saddle, if its itinerary contains a 1 then the continuous time periodic orbit has a flip. Consider the \( n \) periodic orbit \( P^n s' = P(0)s' = s' \) where \( s_k = 1 \) for some integer \( k \leq n \). Then \( P^{k-1}s' \) is in the right half of \( S \), and so must flip upon iteration or during the continuous time evolution from \( P(2\pi(k - 1))s' \) to \( P(2\pi(k - 1) + 2\pi)s' \). Later we will show how \( P \) twists perturbed saddles relative to each other in time, allowing us to establish a framework in which to formulate control by capture and release.
4. The Dynamics of the Pendulum

4.1. Folding the horseshoe

Chaos in the driven pendulum may be viewed as resulting from a stretching and folding of an invariant subset of the full phase space. This stretching and folding appears to produce an attractor with a horseshoe structure in the Poincaré section. If we start with a carefully chosen set of initial conditions in the x, y phase plane and evolve them forward by \( \omega t \) we expect to find that they are stretched, folded and mapped inside the boundaries of the initial set. This horseshoe structure is the hallmark of chaos, and it guarantees periodic points of all periods as well as those of infinite period. The pendulum equation has not to my knowledge been shown to have a horseshoe, but its seemingly chaotic behavior and suggestive folding seem to indicate that it likely does. Although the classic Smale horseshoe preserves line segments, linearity is not a necessity for the formation of a chaotic invariant set.

Recall that the horseshoe map \( f \) has an invariant Cantor set \( \Lambda \) such that

(a) \( \Lambda \) contains a countable set of periodic orbits of arbitrarily long period.

(b) \( \Lambda \) contains an uncountable set of unbounded nonperiodic motions.

(c) \( \Lambda \) contains a dense orbit.

Moreover, any sufficiently \( C^1 \) close map \( \tilde{f} \) has an invariant Cantor set \( \tilde{\Lambda} \) with \( \tilde{f}|_{\tilde{\Lambda}} \) topologically equivalent to \( f|_{\Lambda} \).

We therefore proceed with confidence constructing horseshoes whose dynamics we liken to the dynamics of the vertically driven pendulum. We establish a framework for the allowable types of horseshoes with the following definitions, lemmata and a theorem.

**Definition 4.1.** A vertical curve \( x = v(y) \) is a curve for which

\[
0 \leq v(y) \leq 1, \quad |v(y_1) - v(y_2)| \leq \mu |y_1 - y_2| \text{ in } 0 \leq y_1 \leq y_2 \leq 1
\]

for some \( 0 < \mu < 1 \).
Definition 4.2. A horizontal curve \( y = h(x) \) is a curve for which
\[
0 \leq h(x) \leq 1, \quad |h(x_1) - h(x_2)| \leq \mu |x_1 - x_2| \quad \text{in} \quad 0 \leq x_1 \leq x_2 \leq 1
\]
for some \( 0 < \mu < 1 \).

Definition 4.3. A vertical strip \( V \) is defined by
\[
V = \{(x, y) | x \in [v_1(y), v_2(y)]; \; y \in [0, 1]\}
\]
where \( v_1(y) < v_2(y) \) are non-intersecting vertical curves.

Definition 4.4. A horizontal strip \( H \) is defined by
\[
H = \{(x, y) | y \in [h_1(x), h_2(x)]; \; x \in [0, 1]\}
\]
where \( h_1(x) < h_2(x) \) are non-intersecting horizontal curves.

Definition 4.5. The width of a vertical or horizontal strip is defined as
\[
d(V) = \max_{y \in [0,1]} |v_2(y) - v_1(y)|, \quad d(H) = \max_{x \in [0,1]} |h_2(x) - h_1(x)|
\]

Lemma 4.6. If \( V^1 \supset V^2 \supset V^3 \ldots \) is a sequence of nested vertical (or horizontal) strips and if \( d(V^k) \to 0 \) as \( k \to \infty \) then \( \cap_{k=1}^{\infty} V^k \) def \( V^\infty \) is a vertical (or horizontal) curve.

Lemma 4.7. A vertical curve \( v(y) \) and a horizontal curve \( h(x) \) intersect in precisely one point.

Hypothesis 1: Let \( \mathcal{S} \) be the set \( \{1, 2, \ldots, N\} \) and let \( H_i, V_i \) for \( i \in \mathcal{S} \) be disjoint horizontal and vertical strips and let \( f(H_i) = V_i, i \in \mathcal{S} \).

Hypothesis 2: \( f \) contracts vertical strips and \( f^{-1} \) contracts horizontal strips uniformly. Let \( v_1, v_2 \in V_i \) be any two vertical curves bounding a vertical substrip \( V'_i \subseteq V_i \). Then \( f(V'_i) \cap V_j \) is a vertical strip and
\[
d(f(V'_i), V_j) \leq vd(V'_i)d(V_j)/d(V_i)
\]
for some $v \in (0, 1)$ and $i, j \in \mathcal{S}$. Similarly, letting $h_1, h_2 \in H_i$ be any two horizontal curves bounding a horizontal substrip $H' \subseteq H_i$, then $f^{-1}(H'_i) \cap H_j$ is a horizontal strip and $d(f^{-1}(H'_i) \cap H_j) \leq vd(H_j)$

**Theorem 4.8.** [10] If $f$ is a two-dimensional homeomorphism satisfying Hypotheses 1 and 2 then $f$ possesses an invariant set $\Lambda$, topologically equivalent to a shift $\sigma$ on $\sum$, the set of bi-infinite sequences of elements of $\mathcal{S}$.

This theorem shows that a wide range of different nonlinear foldings can produce horseshoe structures, and if, in addition, the map $f$ is a $C^r$ diffeomorphism with $r \geq 1$, then (with a couple of additional assumptions) $\Lambda$ is hyperbolic.

Numerical evidence indicates that there are stable and unstable manifolds of periodic points in the Poincaré section of the driven pendulum that act like the stable and unstable manifolds of $\Lambda$. The stretching and folding of the pendulum attractor can be mimicked by a stretching and folding of the manifolds of the suspension of a suitable horseshoe map.

Let us consider how the dynamics of the vertically driven pendulum equation (see Appendix B for a physical derivation) stretch and fold a subset of the phase space with the passage of time. We examine the dynamics of the system of first order ODEs

$$
\dot{x} = y \\
\dot{y} = -\rho y - \sin x(1 - A \cos \omega t)
$$

where $\dot{x}$ is the angular velocity of the pendulum shaft, $\dot{y}$ the angular acceleration, $\rho$ the damping factor, $A$ the amplitude of the driving term and $\omega$ the drive frequency. We will examine the dynamics in the phase plane of $x$ and $y$.

The first ODE $\dot{x} = y$ guarantees that a phase point $(x, y)$ moves in the $x$ direction in direct proportion to the value of the $y$ component. Points in the upper half plane will move to the right and points in the lower half plane will move to the left. In the second ODE, the $-\rho y$
term forces points to move towards the $x$ axis, and the periodic part of $\sin x(1 - \cos \omega t)$ reinforces and counteracts the $-\rho y$ term by causing points to move away from the $x$ axis. Since the effect of the periodic term is modulated by $\sin x$, points near $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ will be driven upward and downward with more vigor than those near $\pm \pi$ or 0. This periodic action produces a fold that is pulled from left to right on the upper half plane, and from right to left on the lower half plane. Figure 4.1.1 is a plot of $y = \sin x(1 - \cos \omega t)$. In this plot, the $y$ axis is vertical, the $x$ axis goes from upper left to lower right, and the $t$ axis goes from front to back.

![Plot](image)

(4.1)

Figure 4.1.1. The plot above shows the periodic term responsible for the stretching dynamic of the pendulum equation.

The sequence in Figure 4.1.2 shows the evolution of the Poincaré section of the pendulum through one period of the drive cycle. The stretching and folding is clearly evident. As the attractor stretches and folds, the stable manifolds of the unstable periodic orbits stretch and
fold also.

Figure 4.1.2. The sequence above shows 16 successive Poincaré sections of the vertically driven pendulum when the damping is large enough so that the bob never goes over the top.

We can picture the stretching and folding in terms of horseshoes, but with an important difference. The Smale horseshoe construction maps part of the phase space out of the region of interest, resulting in an invariant set that is a product of Cantor sets. The basin of attraction of the pendulum equation is the whole phase space, and the folds at the ends of the attractor (see Figures 4.1.2 and 4.1.4) get mapped back into the central region corresponding to \( S \) in the Smale stadium. These internal folds are continually stretched out and folded deep into the interior of the attractor. Since no region is mapped out of the space, as in the Smale horseshoe, the pendulum attractor is the product of a Cantor set and a curve, rather than the product of two Cantor sets. If
the pendulum contains a horseshoe structure, it is non-linear horseshoe, and it is most likely a product of Cantor sets imbedded in the attractor rather than the attractor itself.

We may construct pendulum-like horseshoes and make an analysis of their dynamics. In some of the gross features, the pendulum horseshoes have similar dynamics to those of the pendulum. In the horseshoe of Figure 4.1.3, a symbol sequence of three symbols would suffice to describe the dynamics on the invariant set, a Cantor set formed by the intersection of the nestings of three vertical and three horizontal strips (as the Smale horseshoe was formed of two). There is a fixed point in the center of the rectangular region, and the outer layers get mapped successively closer to the horizontal segment in the center of the rectangular region. Note the similarity to the folding of the sequence of Poincaré sections of the pendulum. Furthermore, there are two fixed points in the end caps. If the mapping included a rotation by 180° the fixed points in the end caps would be periodic of period two, as in the pendulum attractor.

Figure 4.1.3. This particular folding resembles the folding of the pendulum attractor, and the action of the folding on the invariant set is topologically conjugate to a shift on three symbols.
Figure 4.1.4. *The invariant set of the pendulum horseshoe.*

In the mapping of Figure 4.1.5, we can see that successive iterations will eventually map the end cap into the linear body of the attractor. This situation is more similar to the dynamics of the pendulum and other real world attractors. There are folds within folds within folds, and the dynamics become extremely difficult, if not impossible, to analyze using symbolic dynamics. Nevertheless, if this folding generates a horseshoe structure, then whether the remainder of the attractor is populated with periodic orbits of extremely high period, or infinite period orbits, the horseshoe contributes chaotic dynamics on its invariant
set, and long chaotic transients for orbits that pass nearby.

\[ l \]

\textbf{Figure 4.1.5.} This stretching and folding is like that of the pendulum, but because the folds are eventually mapped into the body of the attractor, hyperbolicity cannot easily be established. The inclusion of the endcaps in the body of the attractor essentially destroy the nested sequences of vertical strips, as the folds are neither vertical nor horizontal and could create tangencies between the stable and unstable manifolds. When tangencies exist, there exists the possibility for an infinite number of stable orbits in the attractor.

When we look at a sequence of Poincaré sections of the pendulum taken at successive phases of the forcing term, we see a stretching and folding of the stable and unstable manifolds that bears a striking resemblance to the stretching and folding of the stable and unstable sets of the suspended horseshoe. Specifically, line segments \( l \) whose endpoints are in the invariant set \( \Lambda_p \) of the pendulum are mapped to other segments with endpoints in the set \( \Lambda_p \). We may imagine the continuous time evolution of a segment \( l \) as the unfolding of a ribbon that writhes through the torus in which the pendulum solution lives, eventually finding its way in between layers of the attractor, getting closer and closer
to the periodic orbit while becoming narrower and narrower.

A segment $I_s$ in the stable manifold of a periodic orbit $X^*$ eventually comes as close as we like to $X^*$, and it is this fact that we count on to make control by stable subspace targeting possible. We seek to perturb the system so that beginning with any state $X_0$ we can find a perturbation or sequence of perturbations so that $X$ evolves to $I_s \in E^s_{X^*}$, a line segment in the stable set of $X^*$. The dynamics of the evolution of $I_s$ then assure that the orbit will approach that of $X^*$ asymptotically. The following sequence of pictures shows a line segment in the stable set of the straight down steady state of the pendulum as it evolves through one drive cycle. Notice how the segment, originally "outside" the attractor is folded into the "inside" of the attractor. Thereafter, its fate is to go deeper and deeper into the "interior", alternating sides as it approaches the steady state. The stable straight down state is the center manifold of a flip saddle. It is easy to see the rotation of the manifolds in this picture. The figures are in sequence of increasing drive phase angle from top left to bottom right, and the final frame the attractor is in its original position, except that the stable manifold that started as a loop is now folded into the attractor. If we viewed the entire stable manifold of the unstable steady state, we would also see the loop that replaces the one folded inside.
Figure 4.1.3. These seven Poincaré sections taken at equally spaced intervals through the drive cycle illustrate how the stable manifold is folded into the attractor during the course of a cycle.
The dynamics of the stable manifold above are repeated over the entire body of the attractor for all the periodic orbits, that is, pieces of stable manifolds "outside" the attractor unfold and are compressed "inside" the attractor. Once "inside" they migrate deeper and deeper into the fractal layers, sometimes approaching a periodic point asymptotically, and sometimes mapping back out, only to be swallowed up again.

When we fold a horseshoe in the pattern of the folding of a pendulum attractor, we find that the dynamics along the stable and unstable sets of the horseshoe seem to mimic some of the dynamics of the vertically driven pendulum if the suspension includes a half twist to make a Mobius band.

![Diagram](image)

Figure 4.1.4. The pendulum horseshoe above corresponds to an attractor that has no points surrounding the origin. We can see that points near the origin get stretched out and there is no reinjection of the points in the fold to the central region. In a physical pendulum, this condition would correspond to an attractor with minimum period two. The attractor consists of a folded region that coils through the torus once in two drive cycles, piercing the Poincaré plane in two separate places. If we were to watch a movie of the evolution of the attractor’s Poincaré section with change in drive phase, we would see two folded regions that circled the origin, changing places once per cycle. Physically, this corresponds to a pendulum that always crosses the straight down position with positive velocity.
Figure 4.1.5. The pendulum horseshoe above corresponds to a pendulum attractor that has points near the origin. The folds reinject an orbit that has been stretched out of the central region back into the region near the origin. This situation corresponds to a pendulum that can have zero velocity at the origin. There is then a path to the steady state along a stable manifold of the attractor.

Figure 4.1.6 shows the unstable manifold of the pendulum attractor along with a portion of the stable manifolds of three periodic points. Notice the similarity of the loops in the stable manifold to those of the suspended horseshoe. The folding in Figure 4.1.6 corresponds to the folding of the pendulum horseshoe in Figure 4.1.5, as there is a
reinjection region at 0, 0.

Figure 4.1.6. Shown are the unstable manifold of all points of all periods (the $Z$ shaped region) and a portion of the stable manifolds of three separate periodic points. The stable manifolds of all the unstable periodic points are dense in the basin of attraction. The origin and the period two points are all flip saddles.

The rotation of the manifolds of the flip saddle are what make possible capture and release control, and a knowledge of the movement of the manifolds as an attractor goes through its cycle of stretching and folding allow us to design general control rules beyond the map based rules of OGY.
4.2. Bifurcation and orbit structure

The pendulum ODE has no analytical solutions in terms of simple functions, but numerical integration of this equation reveals rich dynamics, both chaotic and non-chaotic. Some analytic methods are useful in establishing approximate characteristics and locations of periodic orbits, but the method of OGY is model-independent, and for the rest of the paper we will assume that we have access to a data set only. Solutions to the driven pendulum equation live naturally in a solid torus of rectangular cross section. A slice through the torus perpendicular to the minor axis reveals the familiar phase plane of the unforced pendulum. This phase plane is now a Poincaré section, the map made by taking a cross section of a flow at some phase $\phi$ of the drive. Along the minor axis of the torus we plot the drive phase, which is $2\pi$-periodic. The cross section is scaled in the vertical direction to fit the largest angular velocity attained on the attractor, and is $2\pi$-periodic in the horizontal direction, corresponding to the periodic nature of over the top rotation.

![Diagram](image)

*Figure 4.2.1.* The dynamics of the pendulum live inside a torus of rectangular cross section. The left and right boundaries of the rectangle are periodic, but the top and bottom must be unbounded.

As any one of the three parameters $\rho$, $\omega$, or $A$ is varied, and for
certain sets of fixed values of the other two parameters, the motion of the pendulum goes through a series of period doubling bifurcations to chaos. We will consider only the case of variation of the damping parameter $\rho$, as variation of the other parameters produce similar changes, except in the details. With high enough damping (and in a specific range of fixed values for $\omega$ and $A$), the steady state $\theta = 0$ of the pendulum is straight down, even while being driven. This state is stable and its basin of attraction is the entire space, save a set of measure zero attracted to the unstable inverted state. As damping is decreased, a new stable solution appears, a period-two orbit, where the pendulum swings back and forth once every two drive cycles. There are now two basins of attraction, one for each solution. Paired with this stable subharmonic period-two solution is an unstable period-two solution. These solutions appear together and split from each other as the damping is decreased.

\begin{center}
\begin{tabular}{|c|c|}
\hline
$+$ & $+$ \\
\hline
$+$ & $+$ \\
\hline
\end{tabular}
\end{center}

\textit{Figure 4.2.2. The figure above illustrates the center, the straight down stable state, and at the sides, the stable and unstable period two orbits.}

\begin{center}
\begin{tabular}{|c|c|}
\hline
$+$ & $+$ \\
\hline
$+$ & $+$ \\
\hline
\end{tabular}
\end{center}

\textit{Figure 4.2.3. The stable and unstable period two orbits have separated as the damping decreases still more.}
As the stable and unstable solutions separate, the basin of the stable straight down state shrinks. Finally, the basin of the straight down stable state blinks out of existence, and the only stable solution is the period-two orbit. The unstable period-two solution has merged with the stable period one solution and the basin of the period two solution is the entire space, except for a set of measure zero (the stable set of the \( \theta = 0 \) solution). As damping is decreased still further, the symmetric period-two solution bifurcates to two stable asymmetric period-two orbits, where swings to one side alternate between two heights. The symmetric period-two solution remains as an unstable orbit. The basins of attraction of the two asymmetric states (which are mirror images) are fractally entwined in each other. A section through the basin plane would reveal a Cantor structure of rather high dimension. As the stable solutions separate from each other, the dimension of this fractal decreases.

Upon further decrease in damping, the stable period-two solutions bifurcate to period-four, period-four to period-eight, etc. The stable states of the pendulum keep bifurcating, but with smaller and smaller changes in the parameter until a point \( \rho_{\text{crit1}} \) is reached where an infinitesimal change in parameter produces a state where only an orbit of infinite period is stable. This is the onset of chaos.

As we decrease the damping still more, this attracting infinite period orbit approaches \( \pi \) in the \( \theta \) direction, which corresponds to the straight up position. As the pendulum goes over the top, the attractor collapses suddenly to two period one orbits, whirling continually over the top either clockwise or counterclockwise. This sudden collapse of a chaotic attractor is called a crisis, and it occurs whenever the basin of attraction of a chaotic orbit collides with the basin of a stable periodic orbit as a parameter is varied. More generally, a crisis occurs whenever basins collide.

We continue to decrease the damping and witness another period doubling bifurcation, but now as \( \rho_{\text{crit2}} \) is approached we have a stable infinite period orbit that can go over the top. As damping is decreased still further, the attractor increases in size, apparently occupying more of the phase space. It is this change in the size of the attractor, along
with the attendant shift in the position of the unstable periodic orbits therein embedded that makes OGY control possible.

Figure 4.2.4. The plot above is a bifurcation diagram. Damping is decreasing from left to right, and angular position is along the vertical axis.

The structure of this attracting orbit is quite complex, yet embedded in this structure are the ghosts of all its previous behaviors, the unstable periodic orbits. These periodic orbits in sum determine the structure of the attracting set. When the attracting set is chaotic and has a fractal structure, it is known as a strange attractor. As an orbit moves through phase space, it is attracted to a saddle orbit along the stable manifold, only to be driven off along the unstable manifold as it approaches the saddle. This occurs continually as the orbit moves through the varying
influences of the saddles, which are dense in the attractor.

**Figure 4.2.5.** The Poincaré section of the pendulum in the non-over-the-top mode.

**Figure 4.2.6.** The Poincaré section of the pendulum in the over-the-top mode.
5. Control of Continuous Time Systems

5.1. Control by OGY

Theoretically there is an infinite number of unstable saddles in the attractor of the chaotic pendulum, but in practice the lower period ones are easiest to control. The dynamics in the Poincaré plane is all we need to establish control by the method of Ott, Grebogi and Yorke, and we can use the Poincaré map to establish the linearized dynamics about any periodic point of interest.

Now let us suppose we want to stabilize one of these periodic orbits, and all that we have is a data set of length \( N \) composed of the successive positions \( \theta \) and velocities \( \dot{\theta} \) taken at some phase \( \phi \) of the drive. We form the vector \( \xi_i(\phi, \rho) = \begin{bmatrix} \theta_i(\phi, \rho) \\ \dot{\theta}_i(\phi, \rho) \end{bmatrix} \), which we denote \( \xi_i \), where the dependence on the phase \( \phi \) at which the Poincaré section is taken, and the damping \( \rho \) is made explicit. For this section, however, we will ignore the dependence on \( \phi \) and \( \rho \).

How do we locate these unstable periodic orbits from the data alone? For the sake of demonstration, let us suppose we want the pendulum to whirl clockwise over the top, a period one orbit of the flow, or equivalently a fixed point of the Poincaré map. There is only one periodic orbit of this form embedded in the attractor, and since we are limited to the preexistent unstable limit cycles by the nature of the algorithm, this is the orbit for which we must settle.

Define some small distance \( \varepsilon \) and locate all pairs of points \( (\xi_j, \xi_{j+1}) \) in the data set whose Euclidean distance apart is less than \( \varepsilon \) (if we were seeking points of period \( n \) we would look for pairs of points \( (\xi_j, \xi_{j+n}) \)). We locate the center of mass of these points, which is a good approximation to the position of \( \xi_0 \), the period one point we seek. Now let us compute the linearized dynamics near this point. We seek a \( 2 \times 2 \) matrix \( A \) that when applied to some point \( \xi_i \) near the periodic point gives the next iterate. One especially simple way is as follows: For the unperturbed system, write

\[
A [\xi_j, \xi_k] = [\xi_{j+1}, \xi_{k+1}],
\]

72
where the column vectors $\xi_j$, $\xi_k$ are two experimental points near the saddle and $\xi_{j+1}$, $\xi_{k+1}$ are their first iterates. Solving for $A$ we get

$$A = [\xi_{j+1}, \xi_{k+1}]^{-1} [\xi_j, \xi_k].$$

We can average the $A$'s obtained from several instances of close encounter, but there are two possible sources of error.

- Our assumption of local linearity must hold throughout the domain we choose as close encounters, otherwise our local transition matrix $A$ will be in error.

- If an iterate of the Poincaré map approaches the fixed point too closely, we may obtain an inaccurate estimate of the dynamics due to the error in our approximation of the location of the fixed point. Clearly, the set of points used to estimate the fixed point’s location should be close to the true fixed point if we want an accurate estimate of the fixed point.

Some familiarity with the dynamics is necessary to make an intelligent choice, and in practice, I used my judgement to determine the size of the data set used to locate the fixed point, and in the choice of close encounters used to obtain $A$.

As in the case of control of the Lozi map, we need the eigenvalues $\lambda_u$, $\lambda_s$ and the eigenvectors $e_u$, $e_s$ of $A$. For convenience, we normalize the lengths of these vectors so that they are unit vectors. We also need the left, or contravariant, eigenvectors $f_u$ and $f_s$. These vectors are orthogonal to $e_u$ and $e_s$, and will be defined by

$$f_u \cdot e_s = f_s \cdot e_u = 0$$
$$f_u \cdot e_u = f_s \cdot e_s = 1.$$

Now consider what happens as a parameter of the system (damping, in this case) is varied slightly. As mentioned above, the attractor changes size with variation of damping, and the location of the fixed points moves with the attractor. Let $\rho$ be the damping parameter, and
\( \delta \rho \) be some small change in \( \rho \). Then the location of the new fixed point upon this change will be \( \delta \rho \ g = \delta \rho \frac{\partial}{\partial \rho} \xi_0 \).

During one drive cycle a phase point \( \xi_i \) evolves to \( \xi_{i+1} \). Relative to the fixed point \( \xi_0 \) these points may be represented by

\[
\begin{align*}
\Delta \xi_i &= \xi_i - \xi_0 \\
\Delta \xi_{i+1} &= \xi_{i+1} - \xi_0.
\end{align*}
\]

We are interested in varying the parameter \( \rho \) to achieve a change in the next iterate, so we write the equation to express the dynamics from the point of view of the shifted fixed point. The state vectors are now

\[
\begin{align*}
\Delta \xi_i - \delta \rho g \\
\Delta \xi_{i+1} - \delta \rho g;
\end{align*}
\]

so

\[
\Delta \xi_{i+1} - \delta \rho g = A[\Delta \xi_i - \delta \rho g],
\]

where we have assumed that the local linearized dynamics of the perturbed system are near enough to those of the unperturbed system so that the use of \( A \) for both the unperturbed and the perturbed system is justified.

If the parameter \( \rho \) is allowed to vary, the position of the unstable limit cycles \( \xi_u \) in the attractor will vary, as will the position of the attractor itself in the phase space. Our goal is to push the state variable onto the stable manifold of a desired periodic orbit. Once there, the dynamics will draw us into the fixed point.

The requirement that the next system state be on the stable manifold of the unshifted fixed point may be written as

\[
f_u \cdot \Delta \xi_{n+1} = 0
\]

We desire an expression equating \( \delta \rho \) to a function of the eigenvalues and eigenvectors of \( A \) and the current position. Rewriting 5.1 as
\[ \Delta \xi_{i+1} = \delta \rho g + A[\Delta \xi_i - \delta \rho g] \]

and dotting by \( f_u \) yields

\[ 0 = f_u \cdot \delta \rho g + f_u \cdot A[\Delta \xi_i - \delta \rho g]. \]  \hspace{1cm} (5.2)

We wish to solve for \( \delta \rho \), the perturbation of the damping that puts us on the stable manifold, so to this end we write \( A \) in terms of its components via the transformation

\[ A = \lambda_u e_u^T f_u + \lambda_s e_s^T f_s. \]  \hspace{1cm} (5.3)

Substituting 5.3 into 5.2, we obtain

\[
\begin{align*}
0 &= \delta \rho \mathbf{f}_u \cdot \mathbf{g} + \mathbf{f}_u \cdot [\lambda_u \mathbf{e}_u \mathbf{f}_u + \lambda_s \mathbf{e}_s \mathbf{f}_s][\Delta \xi_i - \delta \rho \mathbf{g}] \\
0 &= \delta \rho \mathbf{f}_u \cdot \mathbf{g} + [\lambda_u \mathbf{f}_u \cdot \mathbf{e}_u \mathbf{f}_u + \lambda_s \mathbf{f}_u \mathbf{e}_s \mathbf{f}_s][\Delta \xi_i - \delta \rho \mathbf{g}] \\
0 &= \delta \rho \mathbf{f}_u \cdot \mathbf{g} + \lambda_u \mathbf{f}_u \cdot \Delta \xi_i - \lambda_u \delta \rho \mathbf{f}_u \cdot \mathbf{g},
\end{align*}
\]

so that

\[ \delta \rho = \frac{\lambda_u \mathbf{f}_u \cdot \Delta \xi_i}{\lambda_u - 1 \mathbf{f}_u \cdot \mathbf{g}}, \]

which is the control law we seek.

Whenever an iterate of the Poincaré map enters a small box around the fixed point (determined by either a physical restriction of an experimental parameter or by the limit to the approximation of linearity near the fixed point), we change the value of the parameter by \( \delta \rho \) for one Poincaré cycle. The shifted and unshifted attractors are shown in
Figure 5.1.1 below.

![Figure 5.1.1](image)

Figure 5.1.1. In these panels, the lobes containing the period one orbit are shown. The damping in the left panel is greater than that in the right panel. As the lobes contract or expand, the location of the fixed point changes.

Figure 5.1.2. This figure shows a computer screen image of the Poincaré section of the attractor and the data extracted from the dynamics.
Figure 5.1.3 below shows a plot of the time series of control of a physical pendulum by the method of Ott, Grebogi and Yorke[12].

![Figure 5.1.3](image)

*Figure 5.1.3. The picture illustrates the control of a physical pendulum by the OGY method. Control is alternated between period one and period two.*

5.2. General geometric picture of control

We have seen how the stretching and folding of an attractor brings points in the stable manifold of a periodic orbit closer and closer to the orbit upon each fold. We have also seen that some manifolds twist as the attractor stretches and folds, specifically in the sequence of 16 Poincaré sections of the pendulum attractor. We can use this knowledge to gain control of a chaotic system in less than one fundamental period, that is, we can establish control before the next SOS iteration is taken. The simplest of these procedures is control by capture and release (CR), which works only for flip saddles.

Flip saddles arise in period doubling bifurcations. The illustrations below show how a flip saddle is formed.
Figure 5.2.1. The stable period one orbit before bifurcation.

Figure 5.2.2. The stable period one orbit has become unstable, bifurcating to stable period two.

The first illustration shows a period one orbit and its piercing of a SOS map. A parameter is changed until the period one orbit loses stability, and a stable period two appears. One of the stable directions of the period one orbit has become unstable and is now serving also as a stable manifold of the period two orbit. Note that a half twist in the common manifold is necessary in order to form a period two orbit. Paths near the periodic orbit will follow the twisting manifolds and thus alternate sides in the surface of section map.

Now suppose we move in a frame that follows the period two orbit. For this orbit to bifurcate to period four, it must undergo a half twist in this moving frame. As the orbit bifurcates, the now unstable period
one and period two orbits must still be connected by their common manifold. For a new half twisted manifold to form connecting the period four orbit and its unstable progenitor, the manifold connecting the unstable period two and period one orbit must also fold, not once, but an infinite number of times. This scenario is responsible for the folded fractal structure of the strange attractor. Thus, a bifurcation to period four in a continuous time system implies an infinitely folded manifold. A cross section through this manifold will have a fractal structure.

Suppose we perturb a period one flip saddle, and look at the relationship between the perturbed and unperturbed manifolds. As both the perturbed and unperturbed manifolds undergo a half twist over the course of a drive cycle, the stable manifolds (or their linearized idealizations) must coincide at some point. Imagine the hands of two clocks side by side. One can see that at some time during the hour, the hands are colinear.

Figure 5.2.3. This figure show an idealized period one flip orbit and its manifolds.

We simplify the picture somewhat if we agree to move in the reference frame of the unperturbed manifold. Then the perturbed manifolds...
will be seen to orbit around us without twisting. Compare the right hand side of Figure 5.2.4 with the sketch in Figure 5.2.5.

Figure 5.2.4. The perturbed and unperturbed manifolds of a flow are shown on the left, and their Poincaré section on the right.

Figure 5.2.5. In this figure we are moving in the frame of the unperturbed manifolds. The perturbed manifolds will be seen to orbit us. The actual orbit will be more elliptical than circular in the case of the pendulum.
Capture and release control works by perturbing the system in such a way as to place the current system state on the stable manifold of the perturbed system, and waiting until the perturbed and unperturbed manifolds are nearly colinear (we are seldom blessed with linear manifolds, so they can only be approximately colinear). At this point the perturbation is turned off, and the system evolves along the stable manifold of the unperturbed system. No knowledge of the eigenvalues of the system is needed to implement CR control. We need only discover how long it takes for the manifolds to become colinear, but we do need a continuously adjustable control parameter.

Figure 5.2.6. This figure illustrates the method of capture and release. The system is perturbed so that the current state is on the stable manifold of the perturbed periodic point. The perturbation remains in effect until the perturbed and unperturbed stable manifolds are most nearly colinear.

At this point we point out that the effective use of OGY control requires that the movement of the stable manifold be transverse to the direction of the unstable manifold. It is easily seen that one could
choose a Poincaré section at a phase where the perturbed and unperturbed stable manifolds are colinear. Should this be the case, control by OGY is impossible, as OGY requires that the perturbed and unperturbed manifolds separate so that the perturbed dynamics in the unstable direction force the system state onto the unperturbed stable manifold by cycle's end. It may be that in experiments where it was found necessary to delay the activation of control until sometime between SOS maps that this situation was encountered. Experimentalists sometimes have no choice as to when SOS maps are taken due to the exigencies of the experimental apparatus or type of system under study.

![Diagram showing Poincaré section with control on and off at different phase angles]

*Figure 5.2.7. Successful OGY control. The change in manifold position with perturbation is such that there is a change in the unstable direction. The perturbed unstable dynamics act on the system state in such a way as to place the system state on the stable manifold of the periodic orbit in one system cycle.*
Figure 5.2.8. *Unsuccessful OGY control. The change in manifold position with perturbation is such that there is no change in the unstable direction. The perturbed unstable dynamics act on the system state in such a way as to move the system state away from the stable manifold of the periodic orbit.*

From the schematic of OGY control, we can see that a little larger perturbation could drive the system state onto the stable manifold of the unperturbed system in less than one fundamental period. Figure 5.2.9 illustrates this idea. Control by time proportioned perturbation and control by capture and release will be analyzed more thoroughly and implemented in the next two sections.
5.3. Control by capture and release

Control by capture and release (CR) works only for flip orbits. CR control requires knowledge of the dynamics of the system between Poincaré maps, specifically, the change in fixed point with control perturbation, and the point in time at which the stable manifolds of the perturbed and unperturbed system coincide. Figure 5.3.1 is a plot of the stable and unstable manifolds of a period-one over-the-top orbit of the
pendulum at 16 Poincaré phases and at three different damping levels.

![Diagram of perturbed and unperturbed manifolds of a period one orbit of the vertically driven pendulum.](image)

Figure 5.3.1. The figure above shows the perturbed and unperturbed manifolds of a period one orbit of the vertically driven pendulum. The phase space is periodic at the left and right sides, and only the upper half plane is shown. The orbit moves from left to right.

In each group of three the middle manifolds are the unperturbed ones, with a damping level \( \rho_0 = 0.23 \) in units of actual damping to small angle approximation critical damping. The left members of the trios are at an increased damping level \( \rho = 0.25 \), and the right members have a damping of \( \rho = 0.21 \). In the 6th full group of three from the left, the unstable manifolds are nearest coincidence, and in the 11th group of three the stable manifolds are nearest coincidence. In capture and release control we capture the system state at Poincaré section 6, where the perturbation gives the greatest change in the unstable direction, and release the system at Poincaré section 11, where the stable manifolds are almost colinear. The system then progresses to the center manifold.
along the stable direction.

Figure 5.3.2. The perturbed and unperturbed left and right stable and unstable eigenvectors of the period one orbit are shown.

Let $\frac{\partial x_0}{\partial \rho} = g$ be the change in fixed point $x_0$ with change in parameter $\rho$, and $\delta \rho$ be a small parameter perturbation. Our goal is to find a $\delta \rho$ that will perturb the fixed point $x_0$ at time $t_0$ by $\delta \rho g$ so that the system state $\Delta x$ lies in the stable manifold of $x_0 + \delta \rho g$. The system state $\Delta x$ is now captured by the stable dynamics of the perturbed orbit. We leave the control on for a time $t_r$ until $x$ lies in $e_s$, the stable manifold of the unperturbed system. The control is turned off and the system state is released into the custody of the unperturbed stable manifold $e_s$. The system will now evolve along $e_s$ to the center manifold.

Suppose we have a continuous time chaotic system that we can sample at several different Poincaré phases $\phi_i, i = 1, 2, \ldots, n$ over the course of the fundamental system period. Using standard techniques, we establish the location of the fixed point $x_0$ in the Poincaré map the local linearized dynamics $\Delta x_{n+1} = Ax_n$ near the fixed point, the stable and unstable right and left normalized eigenvectors $e_s, e_u, f_s, f_u$ of $A$, and the change in fixed point with respect to parameter change $g = x_0 \frac{\partial}{\partial \rho}$. We also compute the time $t_c$ at which the unstable manifolds of $g$ and $x_0$ coincide ($e_s = 0$) and the time $t_r$ at which the stable
manifolds of $g$ and $x_0$ coincide ($e_u = 0$). We choose the Poincaré phase $\phi$, at which we take data as close as we can to $t_c$ to maximize the effect of the parameter perturbation. The control-on time is therefore fixed at $t_r$, the time it takes for the manifolds to rotate from unstable coincident to stable coincident.

Let $\delta \rho$ be some small change in $\rho$. As the system parameter $\rho$ varies, the attractor and the periodic orbits move in state space. We will assume that for small parameter change $\delta \rho$ the fixed points move but the local dynamics do not change. The location of the new fixed point with change $\delta \rho$ is $\delta \rho \frac{\partial}{\partial \rho}x_0 = \delta \rho g$. We will choose $\delta \rho$ so that the perturbed stable manifold contains the current system state $\Delta x_n = x_n - x_0$. This condition is satisfied when

$$f_u \cdot (\delta \rho g - \Delta x_n) = 0$$

or

$$\delta \rho = \frac{f_u \cdot \Delta x_n}{f_u \cdot g}.$$

If we apply this correction $\delta \rho$ the phase point will be on the perturbed stable manifold until the correction is turned off at $t_r$. The system will then evolve along the unperturbed stable manifold into the fixed point.

Near a saddle, the rate of departure of an initial condition $\Delta x$ along the unstable direction $e_u$ is a function of the distance from the stable manifold. Thus, perturbing a system so that the new stable manifold contains the current system state $\Delta x$ results in a stable system so long
as the perturbation remains.

Figure 5.2.3. This picture is a time series of the pendulum being controlled by the method of capture and release. The control is alternated between a period one clockwise and a period one counterclockwise rotation.

Figure 5.3.3 shows a time series of a session of control by capture and release. The control was alternately applied to period one clockwise and period one counterclockwise rotation. The $\theta$ scale runs from $-\pi$ to $\pi$. The control parameter is velocity dependent damping, with a control off level of .23 normalized units. The control perturbation is shown in gray, and its scale runs from -.21 to .25. Circles indicate when the control was turned on and off.

5.4. Control by time proportioned perturbations

The method of Ott, Grebogi and Yorke (OGY) may be used to control a low-dimensional continuous time chaotic system when the flow may be sampled at some fundamental period to obtain a surface of section
(SOS) map. Then, small parameter perturbations may suffice to stabilize the system around one of its unstable periodic orbits (UPOs). Using OGY control, we would apply a perturbation, whose size is dependent on the value of the system state and the left eigenvector of the stable manifold, for one iterate of the map. The system state will lie on the stable manifold after one fundamental period.

Frequently the only parameter available in a physical system is capable of only a few discrete states, and sometimes only two; either on or off. Control using this type of parameter is called bang-bang control. An air conditioner or a thermostat controlled gas heater are two types of systems that use bang-bang control. When the temperature in your house gets too low, the mercury switch in the thermostat is tripped and the furnace goes full on until the temperature rises above a preset level, at which time the furnace is turned off. If we can vary the parameter only by a fixed amount, then we must resort to schemes other than standard OGY.

If we can reconstruct some of the dynamics between surface of section maps we can use time proportioned perturbation (TPP), where a fixed perturbation is applied for less than one fundamental period. We show how to implement TPP control when we have access to several SOS maps within one fundamental period. In fact, TPP needs access to several SOS maps only for the learning stage. Then, once the dynamics of the system have been reconstructed linearly in the region near the behavior of interest, we monitor only one surface of section map.

The goal of time proportioned perturbation is to apply a fixed perturbation \( p \) in order to direct the system state onto the stable manifold in less than one fundamental period of the system. Usually, the stable and unstable eigenvectors \( e_s \) and \( e_u \) rotate about the UPO during the course of a system cycle, and the angle between them changes periodically. Furthermore, the distance between the perturbed and unperturbed orbits is periodic, and we must take these factors into account.
when building our control rule.

Suppose we have access to $n$ surface of section maps equally spaced in time during one fundamental period $T$ of a continuous time system for both perturbed and unperturbed states. Then we can determine the perturbed and unperturbed UPOs $\tilde{X}_0(k\Delta t)$ $(k = 0, \ldots, n-1)$, $\Delta t = \frac{T}{n-1}$, and $\tilde{X}_p(k\Delta t)$, respectively, the left unstable eigenvector $f_0(k\Delta t)$ of the unperturbed orbit, and the transition matrix $B_k$ taking a system state $X(k\Delta t)$ near the unperturbed periodic orbit $\tilde{X}_0(k\Delta t)$ at SOS $k$ to its image $X((k+1)\Delta t)$ at SOS $k+1$. From the set of transition matrices $B_k$ we form the set of matrices $A_k$ where $A_0 = B_0$, $A_1 = B_1B_0$, $A_2 = B_2B_1B_0$, etc. In other words, $A_k$ takes an initial state $X(0)$ and evolves it to the system state at Poincaré section $k$.

We now fit smooth periodic functions $f_u(t)$, $\tilde{X}_0(t)$, $\tilde{X}_p(t)$ and $A(t)$ to the data $f_u(k\Delta t)$, $\tilde{X}_0(k\Delta t)$, $\tilde{X}_p(k\Delta t)$ and $A_k$. Note that $A(t)$ is not a periodic matrix in the sense that $A(0)x(0) = x(2\pi)$ (it doesn’t), but the entries of the matrix are periodic.

Suppose we monitor SOS 0 and measure an initial system state $X(0)$ near the UPO $\tilde{X}_0(0)$. We want $X(t)$ to be on the stable manifold $e_s(t)$.
at time $t$. The system evolves as

$$X(t) - \dot{X}_{0}(t) = A(t)[X(0) - \dot{X}_{0}(0)].$$

When we turn on the perturbation $p$ the system evolves as

$$X(t) - \dot{X}_{p}(t) = A(t)[X(0) - \dot{X}_{p}(0)].$$

where we have assumed that $A(t)$ does not differ significantly from $A_{p}(t)$, the dynamics of the perturbed system. Then

$$X(t) = A(t)[X(0) - \dot{X}_{p}(0)] + \dot{X}_{p}(t).$$

We want, at some future time $t$

$$f_{a}(t) \cdot [X(t) - \dot{X}_{0}(t)] = f_{a}(t) \cdot [A(t)[X(0) - \dot{X}_{p}(0)] + \dot{X}_{p}(t) - \dot{X}_{0}(t)] = 0.$$

Rewriting

$$A(t)[X(0) - \dot{X}_{p}(0)] + \dot{X}_{p}(t) - \dot{X}_{0}(t)$$

as

$$A(t)[X(0) - \dot{X}_{p}(0)] - (\dot{X}_{p}(t) - \dot{X}_{0}(t)) = 0,$$

and grouping terms, we have finally

$$f_{a}(t) \cdot (A(t)[\xi_{0}(0) - g_{p}(t)] + g_{p}(t)) = 0,$$

the condition that $X(t)$ is on the stable manifold $e_{s}(t)$.

We are only interested in the first zero, so instead of using a Newton’s method to solve for the zeroes, we step time forward and take products of the current and previous iterate. The first time $t_{c}$ for which this product is negative is the $t$ we want. The fixed control is then applied for this time and turned off. At the next Poincaré section we take the data and determine the on time for the next control cycle.

The vertically driven parametric pendulum

$$\dot{x} = y$$

$$\dot{y} = -p y - \sin(x)(1 - \alpha \cos(w t))$$

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is chaotic for the fixed parameter values $a = 1.2$, $\omega = 1.5$ and for $\rho \in [0, 0.28]$; we choose $c_0 = 0.23$ as our unperturbed parameter value. We choose to operate the system with $\rho \in [0.21, 0.25]$, and as we are implementing control by time proportioned perturbations, we fix two possible perturbations, $\rho_1 = 0.21$ and $\rho_2 = 0.25$.

Data sets were generated by a 4th order Runge-Kutta integrator. Data were extracted from 48 data sets consisting of 4096 values for $x$ and $y$ for each of 16 equally spaced Poincaré phases, and at three values of damping, $\rho_1 = 0.21$, $\rho_0 = 0.23$, and $\rho_2 = 0.25$. A PB computer program extracted the fixed points, the local linear map, its eigenvalues, and its stable and unstable left and right eigenvectors. The local transition matrices, that is, the 16 matrices that evolve a system state $X_k$ from the $kth$ Poincaré section to the $k+1$ Poincaré section were calculated and continuous periodic functions fitted to the sequence of entries in the $i,jth$ position of the local transition matrices for $i, j = 1, 2$. These four functions $a_{1,1}(t)$, $a_{1,2}(t)$, $a_{2,1}(t)$ $a_{2,2}(t)$ form the entries in

$$A(t) = \begin{bmatrix} a_{1,1}(t) & a_{1,2}(t) \\ a_{2,1}(t) & a_{2,2}(t) \end{bmatrix}. \quad (5.4)$$

The functions $a_{1,1}(t)$, $a_{1,2}(t)$, $a_{2,1}(t)$ $a_{2,2}(t)$ appear in the table below.

In this particular case, the Poincaré control plane was sampled at phase $\frac{\pi}{8}$ since at this value the change in parameter moved the fixed point along the unstable manifold. Choosing this phase for control then gave the greatest controllability. Since at some drive phase $\varphi$ the periodic orbit crosses the periodic boundary, the function $A(t)$ will be discontinuous at $t = \varphi$. We avoid the problem by choosing to control in less than $\omega t = \pi$, hence, we actually fit $A(t)$ to the values taken from Poincaré sections 7 to 15 and 0 to 1 for a total of 11 successive Poincaré sections.

$$a_{1,1} = 0.291063 + 0.793685 \times \cos[t]$$
$$-0.0905197 \times \cos[2t] + 0.0296494 \times \cos[3t]$$
$$-0.0559318 \times \sin[t] + 0.00286768 \times \sin[2t] + 0.050346 \times \sin[3t]$$

$$a_{1,2} = 0.408048 - 0.220476 \times \cos[t]$$
\[-0.0905197 \times \cos[2t] + 0.0296494 \times \cos[3t]\]
\[-0.0559318 \times \sin[t] + 0.00286768 \times \sin[2t] + 0.050346 \times \sin[3t]\]

\[a_{2,1} = -1.29693 + 0.430001 \times \cos[t] + 0.776662 \times \cos[2t] + 0.0230557 \times \cos[3t] + 0.786877 \times \sin[t] - 0.240939 \times \sin[2t] - 0.22432 \times \sin[3t]\]

\[a_{2,2} = 0.309087 + 0.400585 \times \cos[t] + 0.136693 \times \cos[2t] + 0.0268144 \times \cos[3t] - 0.13708 \times \sin[t] - 0.0935679 \times \sin[2t] - 0.00711615 \times \sin[3t]\]

Shown in Figures 2a through 2d are plots of the data sets (connected piecewise linear plot) from the four entries of the matrix \(A\) and the functions used to approximate the data.
Figures 5.4.2a, 5.4.2b, 5.4.2c, 5.4.2d show the fits to the data from the entries of A plotted with the piecewise linear fit to the data in the entries. From top to bottom are \( a_{1,1} \), \( a_{1,2} \), \( a_{2,1} \), \( a_{2,2} \).

The unstable periodic orbits for three levels of damping were computed in a similar fashion, by fitting smooth periodic functions to the 7th through the 1st successive state vectors, forming the continuous vector function \( X_0(t) \). The function \( X_0(t) \) is shown below.

\[
\begin{align*}
\bar{x}_0 &= 0.866553 - 1.97963 \cos[\theta] \\
&- 0.494012 \cos[2\theta] + 0.0414162 \cos[3\theta] \\
&- 0.439859 \sin[\theta] + 0.379237 \sin[2\theta] + 0.131035 \sin[3\theta]
\end{align*}
\]

\[
\bar{y}_0 = 1.5033 + 0.371607 \cos[\theta]
\]
\[-0.0176539 \cdot \cos[2t] - 0.0344509 \cdot \cos[3t] + \]
\[0.557474 \cdot \sin[t] + 0.2615 \cdot \sin[2t] + 0.0335449 \cdot \sin[3t] \]

\[x_0^-(t) = 0.707967 - 2.10011 \cdot \cos[x] \]
\[-0.222945 \cdot \cos[2x] + 0.240454 \cdot \cos[3x] \]
\[+ 0.0253629 \cdot \sin[x] + 0.653105 \cdot \sin[2x] + 0.105806 \cdot \sin[3x] \]

\[\bar{y}_0^- = 1.49461 + 0.405109 \cdot \cos[t] \]
\[+ 0.00936983 \cdot \cos[2t] - 0.0279653 \cdot \cos[3t] \]
\[+ 0.553614 \cdot \sin[t] + 0.275949 \cdot \sin[2t] + 0.0387564 \cdot \sin[3t] \]

\[x_0^+(t) = 0.772507 - 2.00497 \cdot \cos[t] \]
\[-0.47772 \cdot \cos[2t] + 0.0564898 \cdot \cos[3t] \]
\[-0.443086 \cdot \sin[t] + 0.390533 \cdot \sin[2t] + 0.130429 \cdot \sin[3t] \]

\[\bar{y}_0^+ = 1.51107 + 0.333787 \cdot \cos[t] \]
\[-0.0459989 \cdot \cos[2t] - 0.0436186 \cdot \cos[3t] \]
\[+ 0.56489 \cdot \sin[t] + 0.248926 \cdot \sin[2t] + 0.0278085 \cdot \sin[3t] \]

\[\text{Figure 5.4.3. This figure shows the fit to the } x \text{ coordinate of the periodic orbit data for the three different damping levels.} \]
Figure 5.4.4. This figure shows the fit to the y coordinate of the periodic orbit data for the three different damping levels.

The function \( f^u(t) \), the left unstable eigenvector is shown below:

\[
\begin{align*}
\frac{f_x^u}{f_y^u} &= 0.351576 + 0.50572 \cos[t] \\
&\quad + 0.0330918 \cos[2t] + 0.0873002 \cos[3t] - 0.640942 \sin[t] + 0.186759 \sin[2t] - 0.051183 \sin[3t] \\
\end{align*}
\]

Figure 5.4.5. This figure shows the data and the fit to the x coordinate of the left unstable eigenvector of the unperturbed orbit.
Figure 5.4.6. This figure shows the data and the fit to the y coordinate of the left unstable eigenvector of the unperturbed orbit.

When control was implemented, we found that it was not as tight as OGY or CR control. We speculate that the accurate modeling of the system is crucial, and that the compounded error of the interpolated functions and inaccuracies in the estimates of other system characteristics conspired to confound our control. The control was, however, robust, and control was never lost over the longest run of one hour. Pictured below is a time series plot of a control session, with time on the horizontal axis and velocity on the vertical. The system was allowed to evolve uncontrolled for a while, then TPP control was turned on. The control was then turned off and the system allowed to evolve.
on its own.

Figure 5.4.7. This figure shows a time series plot of the control of the damped driven pendulum equation by the method of time proportioned perturbations.
6. Appendix A

By definition

\[ Ae = \lambda e, \]

\[ f_u \cdot e_s = f_s \cdot e_u = 0, \]

\[ f_u \cdot e_u = f_s \cdot e_s = 1, \]

so

\[ A[e_u + e_s] = \lambda_u e_u + \lambda_s e_s \]

\[ = \lambda_u e_u (f_u e_u + f_u e_s) + \lambda_s e_s (f_s e_s + f_s e_u). \]

Factoring, we obtain

\[ A[e_u + e_s] = [\lambda_u e_u f_u + \lambda_s e_s f_s] \cdot [e_u + e_s], \]

and finally

\[ A = [\lambda_u e_u f_u + \lambda_s e_s f_s]. \]
7. Appendix B

We can derive the pendulum equation by equating forces. First, ignoring the driving and damping terms and assuming a point mass on the end of a massless rod, equate the forces along the path of the bob:

\[ mr\theta'' = -mg\sin\theta. \]

Assuming velocity dependent damping, we get

\[ mr\theta'' = -mg\sin\theta - \Gamma\theta'. \]

All motion of the pendulum will be relative to the position of the pivot. With this in mind, we note that the acceleration imparted to the bob due to the acceleration of the pivot is proportional to \(\sin\theta\). We differentiate the drive position twice to get drive acceleration, so the force felt by the pivot is

\[ -m\omega^2 \sin\theta \cos\omega t. \]
Now adding the drive term, we have

\[ mr\theta'' = -mg\sin\theta - \Gamma\theta' + am\omega^2 \sin \theta \cos \omega t. \]

Dividing through by \( mr \), we get

\[ \theta'' = -\frac{g}{r}\sin\theta - \frac{\Gamma}{mr}\theta' + \frac{a}{r}\omega^2 \sin \theta \cos \omega t. \]

Finally we introduce a dimensionless time \( \tau \) such that \( \tau = \frac{t}{T} \), and letting the double dot represent differentiation with respect to \( \tau \) we have

\[ \ddot{\theta} = -\frac{g}{rT^2}\sin\theta - \frac{\Gamma}{mrT}\dot{\theta} + \frac{a}{mr}\omega^2 \sin \theta \cos \omega t. \]

Letting \( T = \sqrt{\frac{r}{g}}, A = \frac{a\omega^2}{r}, \) and \( \rho = \frac{\Gamma}{Tmr} \) we have, finally,

\[ \ddot{\theta} = -\rho \dot{\theta} - \sin \theta (1 - A \cos \omega t). \]
References


