$H^\infty$ ROBUST CONTROL SYSTEM DESIGN

by

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ABSTRACT

This thesis presents the application of $H^\infty$ optimal control. Controllers are designed using $H^\infty$ optimal control design techniques described in Robust Control Toolbox [6] and Feedback Control Theory [7]. The mathematical properties as well as the derivation of the $H^\infty$ algorithm are discussed.

Computer simulation case studies using $H^\infty$ robust control design methods are presented to substantiate theory. The emphasis is on formulating the $H^\infty$ control design problem such that the resulting controller provides robustness with reasonable control effort. The simulations show the performance of the algorithms for some practical systems.
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CONTENTS

List of Symbols..........................................................vii

CHAPTER

1. INTRODUCTION...............................................................1

2. H-Infinity Robust Control System Design...........4
   2.1 Infinity-norm Computation.........................4
   2.2 Performance.................................................5
   2.3 Robust Stability...........................................11
   2.4 Robust Performance......................................15
   2.5 Mixed-Sensitivity Formulation......................19
   2.6 H-Infinity Optimal Control Synthesis
       Method.....................................................28

3. Simulation Examples and Results............................32
   3.1 Benchmark Problem........................................32
   3.2 Flexible Body Rocket System.......................48
   3.3 Hydraulic Actuator......................................64
   3.4 Aircraft Lateral Dynamics.............................78

4. Conclusion...............................................................93
APPENDIX

MATLAB Input Files for

A  Benchmark problem.......................... 94
B  Flexible Body Rocket System............... 101
C  Hydraulic Actuator.......................... 108
D  Aircraft Lateral Dynamics.................. 115

REFERENCES..................................... 122
List of Symbols

d plant disturbance

e tracking error

r reference or command input

u control signal, controller output

y plant output

F controller

G nominal plant, system's transfer function

\( \tilde{G} \) perturbed plant

\( \hat{G}(s) \) Laplace transform of transfer function \( G \)

\( \|G\|_\infty \) peak amplitude on the Bode magnitude plot of \( G \)

L loop transfer function \( GF \)

R control transfer function from reference \( r \) to control input \( u \)

S sensitivity function: error transfer function from reference \( r \) to tracking error \( e \)

T complementary sensitivity function: equal to \( 1-S \) (output transfer function from \( r \) to \( y \))

\( P(s) \) \( H^\infty \) synthesis model
$W_1^{-1}$ performance specification weighting function for sensitivity function $S(s)$

$W_2^{-1}$ $R(s)$ weighting transfer function for closed-loop transfer function from $r$ to $u$

$W_3^{-1}$ robustness specification weighting function for complementary sensitivity function $T(s)$

$\text{pow}(e)$ power signal $e$

$\alpha$ robust performance level

$\beta$ uncertainty level

$\infty$-norm least upper bound of its absolute value

$r_\text{pf}$ a pre-filtered input

$\|r\|^2_2$ 2-norm measure of the energy of $r$

$\sup_r\|e\|_2$ supremum of tracking error measure 2-norm of $e$

$\mathcal{P}$ set of plant

$T_{y_1u_1}$ closed-loop transfer function from $u_1$ to $y_1$

$\|T_{y_1u_1}\|_\infty$ infinity norm of $T_{y_1u_1}$

$\sigma$ maximum singular value

$\sigma_i$ singular value: $i=1,2,3,...n-1$

$\Delta_i$ additive plant perturbation
\( \Delta_M \) multiplicative plant perturbation

\( \Delta \) uncertainty to act as a factor on magnitude of the perturbation

\( P_R \) control Riccati equation

\( S_R \) observer Riccati equation

\( \lambda_{\text{max}} \) greatest eigenvalue

\( \gamma \) iteration function to compute the optimal \( H^* \)
CHAPTER 1

INTRODUCTION

In this thesis a robust controller will be looked at using an $H^\infty$ optimality criterion. The systems presented herein are linear and time-invariant.

The $H^\infty$ optimal control theory represents an important advance in control technology. Among the modern control theories developed in 1980's, $H^\infty$ optimal control theory is no doubt the most successful. It offers a direct and efficient methodology for designing highly practical robust multivariable control systems [3]. Generally speaking, the notion of robustness means that some characteristic of the feedback system holds for every plant in the set $\mathcal{P}$. A controller $F$ provides robust stability if it provides internal stability for every plant in $\mathcal{P}$. Given a synthesis model $P(s)$ (Figure 1.1): $H^\infty$ design finds a controller, $F$, with two important properties;

2. Maximum gain, from inputs to the outputs is <1.

Figure 1.1 $H^\infty$ synthesis model control system block diagram

The objective of this thesis is to develop insight into formulating a robust control problem within the $H^\infty$ control design framework and to improve upon the control design. Towards this goal a collection of several problem results of case studies are presented to illustrate $H^\infty$ robust controller design which minimizes the $H^\infty$-norm of a pre-designated closed-loop transfer matrix. The $H^\infty$-norm is the largest singular value of a transfer matrix. Simulations are developed with mixed sensitivity formulation of the robust control design which penalizes both sensitivity $S$ and complementary sensitivity $T$ [6]. The theory is developed in the input-output framework, while computational procedures are presented in the state-space framework. Theory dealing
with performance and robust stability are discussed in detail. This thesis is organized as follows. The $\infty$-norm computations are first discussed. The performance of the $H^\infty$ control design is then presented followed by robust stability and robust performance. Next the $H^\infty$ optimal control synthesis with the emphasis on formulating the problem will be discussed. The resulting controller is robust with reasonable control effort and controller complexity. The first computer simulation presented is a two-mass spring system ACC Benchmark problem with non-colocated sensor and actuator [1]. The second model consists of a flexible body rocket system with rigid body coupled with a bending mode to simulate a launch vehicle problem. The third plant model consists of a hydraulic actuator model control system. The fourth and final control system design model is an aircraft lateral dynamics system [7].
CHAPTER 2

H-Infinity Robust Control System Design

2.1 Infinity-norm Computation

Performance of a control system may be specified in terms of the size of certain signals of interest [7]. For example, the performance of a tracking system could be measured by the size of the error signal. Consider systems that are linear, time-invariant, and finite dimensional. In the time domain an input($u$)-output($y$) model for such a system has the form of a convolution equation,
\[ y = g * u, \]
that is,
\[ y(t) = \int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau. \]

Where $g$ is the system impulse response.

The $\infty$-norm of a signal is the least upper bound of its absolute value. For example, the $\infty$-norm of $(1-e^{-\tau})1(\tau)$ equals 1. Here $1(\tau)$ denotes the unit step function. To
compute $\infty$-norm, this requires a search. Set up a fine grid of frequency points,
\[ \{\omega_1, \ldots, \omega_n\} \]
Then an estimate for \( \| \hat{G} \|_\infty \) is
\[ \max_{\omega_i} |\hat{G}(j\omega_i)|. \]

Here \( \hat{G} \) is the Laplace transform of \( G \).

Alternatively, one could find where \( |G(j\omega)| \) is maximum by solving the equation
\[ \frac{d|\hat{G}|^2}{d\omega}(j\omega) = 0. \]
This derivative can be computed in the closed form because \( \hat{G} \) is rational. It then remains to compute the roots of a polynomial.
The system is analyzed for its ability to track a single reference signal \( r \) - a step or a ramp asymptotically as time increases.

2.2 Performance
Consider a set of reference signals and a bound on the steady-state error. This performance objective will be quantified in terms of a weighted norm bound [7]. Let
\( L \) denote the loop transfer function \( L:=GF \). The transfer function from reference input \( r \) to tracking error \( e \) is

\[
S := \frac{1}{1 + L},
\]
called the sensitivity function. The name sensitivity function comes from the following idea. Let \( T \) denote the transfer function from \( r \) to \( y \) as shown in Figure 2.1:

\[
T = \frac{GF}{1 + GF}.
\]

![Figure 2.1 Block Diagram of the Feedback Control System](image)

One way to quantify how sensitive \( T \) is to variations in \( G \) is to take the limiting ratio of a relative perturbation in \( T \) (i.e., \( \Delta T/T \)) to a relative perturbation in \( G \) (i.e., \( \Delta G/G \)). Thinking of \( G \) as a variable and \( T \) as a function of it, we get
\[
\lim_{\Delta G \to 0} \frac{\Delta T/T}{\Delta G/G} = \frac{dT/G}{dG/T}.
\]

The right hand side is easily evaluated to be \( S \). In this way, \( S \) is the sensitivity of the closed-loop transfer function \( T \) to an infinitesimal perturbation in \( G \).

Now we have to decide on a performance specification, a measure of goodness of tracking. This decision depends on two things: what we know about \( r \) and what measure we choose to assign to the tracking error. Usually, \( r \) is not known in advance. Few control systems are designed for one and only one input [7]. Rather, a set of possible \( r \)'s will be known or at least postulated for the purpose of design.

Let's first consider sinusoidal inputs. Suppose that \( r \) can be any sinusoid of amplitude \( \leq 1 \) and we want \( \varepsilon \) to have amplitude \( < \varepsilon \). Then since \( S \) is the error transfer function, the performance specification can be expressed as

\[ \|S\|_\infty < \varepsilon. \]
The maximum amplitude of $e$ equals the $\infty$-norm of the transfer function. Or if we define the weighting function $W_1(s) = 1/\varepsilon$, then the performance specification is

$$\|W_1S\|_\infty < 1.$$ 

The situation becomes more realistic and more interesting with a frequency-dependent weighting function. Assume that $W_1(s)$ is real-rational, so any poles or zeros in $\text{Re} \ s > 0$ can be reflected into the left half-plane without changing the magnitude. Let us consider four scenarios giving rise to an $\infty$-norm bound on $W_1S$. The first three require $W_1$ to be stable [7].

1. Suppose that the family of reference inputs is defined as all signals of the form $r = Wir_{pf}$, where $r_{pf}$, a pre-filtered input, is any sinusoid of amplitude $\leq 1$. Thus the set of $r$'s consists of sinusoids with frequency-dependent amplitudes. Then the maximum amplitude of $e$ equals $\|W_1S\|_\infty$.

2. The equation

$$\|r\|_2^2 = \frac{1}{2\pi} \int_{-\omega}^{\omega} |r(j\omega)|^2 d\omega$$

8
is a measure of the energy of \( r \). Thus we may think of \( |r(j\omega)|^2 \) as energy spectral density, or energy spectrum.

Suppose that the set of all \( r \)'s is

\[
\{ r: r = W r_f, \|r_f\|_2 \leq 1 \},
\]

that is,

\[
\left\{ r: \frac{1}{2\pi} \int_{-\infty}^{\infty} |r(j\omega)| W_1(j\omega)^2 \, d\omega \leq 1 \right\}.
\]

Thus, \( r \) has an energy constraint and its energy spectrum is weighted by \( 1/|W_1(j\omega)|^2 \). For example, if \( W_1 \) was a bandpass filter, the energy spectrum of \( r \) would be confined to the passband. More generally, \( W_1 \) could be used to shape the energy spectrum of the expected class of reference inputs. Now suppose that the tracking error measure is the 2-norm of \( e \).

Then

\[
\sup_r \|e\|_2 = \sup \{ \| W r_f \|_2: \|r_f\|_2 \leq 1 \} = \| W S \|_\infty,
\]

so \( \| W S \|_\infty < 1 \) means that \( \|e\|_2 < 1 \) for all \( r \)'s in the set above.
3. This scenario is like the preceding one except for signals of finite power. We see that $\|W S\|_\infty$ equals the supremum of $\text{pow}(e)$ (power signal of $e$) over all $r_{pf}$ (pre-filtered input) with $\text{pow}(r_{pf}) \leq 1$ [7]. So $W_1$ could be used to shape the power spectrum of the expected class of $r$'s.

4. In several applications, for example aircraft flight-control design, designers have acquired through experience desired shapes for the Bode magnitude plot of $S$. In particular, suppose that good performance is known to be achieved if the plot of $|S(j\omega)|$ lies under some curve. We could rewrite this as

$$|S(j\omega)| < |W_1(j\omega)|^{-1}, \forall \omega,$$

or in other words, $\|W S\|_\infty < 1$.

There is a nice graphical interpretation of the bound $\|W S\|_\infty < 1$. Note that

$$\|W S\|_\infty < 1 \Leftrightarrow \frac{|W_1(j\omega)|}{|1 + L(j\omega)|} < 1, \forall \omega.$$
The last inequality says that at every frequency, the point $L(j\omega)$ on the Nyquist plot lies outside the disk of center $-1$, with radius $|\mathcal{W}(j\omega)|$ as shown in Figure 2.2.

\[ \text{Figure 2.2 Performance specification} \]

### 2.3 Robust Stability

The notion of robustness can be described as follows. Suppose that the plant transfer function $G$ belongs to a set $\mathcal{P}$. Consider some characteristic of the feedback system, for example, that it is internally stable. A controller $F$ is robust with respect to this characteristic if this characteristic holds for every plant in $\mathcal{P}$. The notion of robustness therefore requires a controller, a set of plants, and some characteristic of system.
A controller $F$ provides robust stability if it provides internal stability for every plant in $\varphi$. We might like to have a test for robust stability, a test involving $F$ and $\varphi$. Or if $\varphi$ has an associated size, the maximum size such that $F$ stabilizes all of $\varphi$ might be a useful notion of stability margin.

The Nyquist plot gives information about stability margin. Note that the distance from the critical point $-1$ to the nearest point on the Nyquist plot of $L$ equals $1/\|S\|_\infty$:

$$\text{distance from } -1 \text{ to Nyquist plot} = \inf_{\omega} |1 - L(j\omega)|$$

$$= \inf_{\omega} |1 + L(j\omega)|$$

$$= \left[ \sup_{\omega} \frac{1}{|1 + L(j\omega)|} \right]^{-1}$$

$$= \|S\|_\infty^{-1}$$

Thus if $\|S\|_\infty >> 1$, the Nyquist plot comes close to the critical point, and the feedback system is nearly unstable. However, as a measure of stability margin this distance is not entirely adequate because it contains no frequency information. More precisely, if the nominal
plant $G$ is perturbed to $\tilde{G}$, having the same number of unstable poles as has $G$ and satisfying the inequality

$$\left| \tilde{G}(j\omega)F(j\omega) - G(j\omega)F(j\omega) \right| < \|S\|_{\infty}^{-1}, \forall \omega,$$

then internal stability is preserved (the number of encirclements of the critical point by the Nyquist plot does not change). But this is usually very conservative; for instance, larger perturbations could be allowed at frequencies where $G(j\omega)F(j\omega)$ is far from the critical point.

Now we look at a typical robust stability test, one for the multiplicative perturbation model. Assume that the nominal feedback system is internally stable for controller $F$. Introduce the complementary sensitivity function

$$T = 1 - S = \frac{L}{1 + L} = \frac{GF}{1 + GF}.$$

Consider perturbed plant transfer functions of the form $\tilde{G} = (1 + \Delta W)G$. Here $W$ is a fixed stable transfer function, the weight, and $\Delta$ is a variable stable transfer function satisfying $\|\Delta\|_{\infty} \leq 1$. The main purpose of $\Delta$ is to account for phase uncertainty and to act as a scaling factor on
the magnitude of the perturbation. For multiplicative uncertainty model, the controller $F$ provides robust stability iff $\|W_T\|_\infty < 1$. The key equation is

$$1+(1+\Delta W)T = (1+L)(1+\Delta W_T).$$

Since

$$\|\Delta W_T\|_\infty \leq \|W_T\|_\infty < 1,$$

the point $1+\Delta W_T$ always lies some closed disk with center $1$, radius $< 1$. The condition $\|W_T\|_\infty < 1$ has a nice graphical interpretation. Note that

$$\|W_T\|_\infty < 1 \iff \left| \frac{W(j\omega)L(j\omega)}{1+L(j\omega)} \right| < 1, \forall \omega$$

$$\iff \left| W(j\omega)L(j\omega) \right| < |1+L(j\omega)|, \forall \omega.$$

The last inequality says that at every frequency, the critical point, $-1$, lies outside the disk of center $L(j\omega)$, with radius $\left| W(j\omega)L(j\omega) \right|$ as shown in Figure 2.3.

```
Figure 2.3 Robust stability
```
2.4 Robust Performance

Now we look into performance of the perturbed plant. Suppose that the plant transfer function belongs to a set $\varphi$. The general notion of robust performance is that internal stability and performance, of a specified type, should hold for all plants in $\varphi$.

When the nominal feedback system is internally stable, the nominal performance condition is $\|W_S\|_\infty<1$ and the robust stability condition is $\|W_S\|_\infty<1$. If $P$ is perturbed to $(1+\Delta W_P)P$, $S$ is perturbed to

$$\frac{1}{1+(1+\Delta W_P)L} = \frac{S}{1+\Delta W_P T}.$$  

Clearly, the robust performance condition should therefore be

$$\|W_P T\|_\infty<1 \text{ and } \left\| \frac{W_S}{1+\Delta W_P T} \right\|_\infty<1, \forall \Delta.$$  

Here $\Delta$ must be allowable. The next equation gives a test for robust performance in terms of the function

$$|W_F(s)S(s)| + |W_F(s)T(s)|,$$

which is denoted $|W_S|+|W_P T|$.  


A necessary and sufficient condition for robust performance is

$$\| W_i S\| + \| W_i T\| < 1. \quad (2.1)$$

Assume (2.1), or equivalently,

$$\| W_i T\| \text{ and } \frac{W_i S}{1-|W_i T|} < 1. \quad (2.2)$$

Fix $\Delta$. In what follows, functions are evaluated at an arbitrary point $j\omega$, but this is suppressed to simplify notation. We have

$$1 = |1 + \Delta W_i T - \Delta W_i S| \leq |1 + \Delta W_i T| + |W_i T|$$

and therefore

$$1 - |W_i T| \leq |1 + \Delta W_i T|.$$ 

This implies that

$$\frac{W_i S}{1-|W_i T|} \leq \frac{W_i S}{1+\Delta W_i T}.$$ 

This and (2.2) yield

$$\frac{W_i S}{1+\Delta W_i T} < 1.$$ 

Assume that

$$\| W_i T\| < 1 \text{ and } \frac{W_i S}{1+\Delta W_i T} < 1, \ \forall \Delta. \quad (2.3)$$

Pick a frequency $\omega$ where
\[ \frac{W_s S}{1 - |W_s T|} \]

is maximum. Now pick \( \Delta \) so that

\[ 1 - |W_s T| = |1 + \Delta W_s T| . \]

The idea here is that \( \Delta(j\omega) \) should rotate \( W_s(j\omega)T(j\omega) \) so that \( \Delta(j\omega)W_s(j\omega)T(j\omega) \) is negative real. Now we have

\[
\left\| \frac{W_s S}{1 - |W_s T|} \right\|_\infty = \frac{|W_s S|}{1 - |W_s T|}
\]

\[
= \frac{W_s S}{|1 + \Delta W_s T|}
\]

\[
\leq \left\| \frac{W_s S}{1 + \Delta W_s T} \right\|_\infty
\]

So from this and (2.3) there follows (2.2).

The robust performance condition says that the robust performance level 1 is achieved. More generally, let's say that robust performance level \( \alpha \) is achieved if

\[
\|W_s T\|_\infty < 1 \text{ and } \left\| \frac{W_s S}{1 + \Delta W_s T} \right\|_\infty < \alpha, \ \forall \Delta .
\]

Noting that at every frequency

\[
\max_{\|\Delta\|_1} \left| \frac{W_s S}{1 + \Delta W_s T} \right| = \frac{|W_s S|}{1 - |W_s T|}
\]

we get that the minimum \( \alpha \) equals
Alternatively, we may wish to know how large the uncertainty can be while the robust performance condition holds. To do this, we scale the uncertainty level; that is, we allow \( \Delta \) to satisfy \( \| \Delta \|_\infty < \beta \). Let's say that the uncertainty level \( \beta \) is permissible if

\[
\left\| \beta W_s T \right\|_\infty < 1 \quad \text{and} \quad \frac{\left\| W_s \right\|_\infty}{1 + \beta W_s T} < 1, \quad \forall \Delta.
\]

Again, noting that

\[
\max_{s \in \text{Re} s \geq 0} \frac{W_s S}{1 + \beta W_s T} = \frac{W_s S}{1 - \beta W_s T} = 1,
\]

we get that the maximum \( \beta \) equals

\[
\left\| \frac{W_s T}{1 - W_s S} \right\|_\infty^{-1}.
\]

Suppose that the loop transfer function \( L \) has a zero \( z \) in \( \text{Re} s \geq 0 \). Then

\[
\| W_s S \|_\infty \geq | W(z) |.
\]

This is a direct consequence of the maximum modulus theorem

\[
| W(z) | = | W(z) S(z) | \leq \sup_{\text{Re} s > 0} | W(s) S(s) | = \| W S \|_\infty.
\]
So a necessary condition that the performance criterion $\|W_S\| < 1$ be achievable is that the weight satisfy $|W_i(z)| < 1$. In words, the magnitude of the weight at a right half-plane zero of $G$ or $F$ must be less than 1.

Similarly, suppose that $L$ has a pole $p$ in $\mathbb{R} s \geq 0$. Then

$$\|W_i T\|_\infty \leq |W_i(p)|,$$

so a necessary condition for the robust stability criterion $\|W_i T\|_\infty < 1$ is that the weight $W_i$ satisfy $|W_i(p)| < 1$.

### 2.5 Mixed-Sensitivity Formulation

Consider the multivariable feedback system shown in Figure 2.4. In order to quantify the multivariable stability margins and performance of such systems, you can use the singular values of the closed-loop transfer function matrices from $r$ to each of the three outputs $e$, $u$ and $y$.

$$S(s) \overset{\text{def}}{=} (I + L(s))^{-1}$$

$$R(s) \overset{\text{def}}{=} F(s)(I + L(s))^{-1}$$

$$T(s) \overset{\text{def}}{=} L(s)(I + L(s))^{-1} = I - S(s)$$
where $L(s) = G(s)F(s)$.

Figure 2.4 Block diagram of the multivariable feedback control system

The two matrices $S(s)$ and $T(s)$ are known as the sensitivity function (closed-loop error transfer function from $r$ to $e$) and complementary sensitivity function (closed-loop output transfer function from $r$ to $y$), respectively. The matrix $R(s)$ (closed-loop control transfer function from $r$ to $u$) has no common name. The singular value Bode plots of each of the loop transfer function matrices $S, R,$ and $T$ play an important role in robust multivariable control system design. The singular values of the loop transfer function matrix $L(s)$ are important because $L$ determines the matrices $S(s)$ and $T(s)$. The singular values of $S$ determine the disturbance attenuation since $S$ is in fact the closed-loop transfer
function from disturbance $d$ to plant output $y$. Thus a disturbance attenuation performance specification may be written as

$$\sigma(S(j\omega)) \leq |W^{-1}_i(j\omega)|$$

(2.4)

where $|W^{-1}_i(j\omega)|$ is the desired disturbance attenuation factor. Allowing $W(j\omega)$ to depend on frequency $\omega$. The singular value Bode plots of $R(s)$ and of $T(s)$ are used to measure the stability margins of multivariable feedback designs in the face of additive plant perturbations ($\Delta_A$) (Figure 2.5) and multiplicative plant perturbations ($\Delta_M$), respectively.

Let us consider how the singular value Bode plot of complementary sensitivity $T(s)$ determines the stability margin for multiplicative perturbations $\Delta_M$. The multiplicative stability margin is, by definition, the "size" of the smallest stable $\Delta_M(s)$ which destabilizes the system in with $\Delta_A = 0$. 
Taking $\sigma(\Delta_M(j\omega))$ to be the definition of the "size" of $\Delta_M(j\omega)$, you have the following stability theorem [6]:

Robustness Theorem 1: Suppose the system in Figure 2.5 is stable with both $\Delta_A$ and $\Delta_M$ being zero. Let $\Delta_A = 0$. Then the size of the smallest stable $\Delta_M(s)$ for which the system becomes unstable is

$$
\sigma(\Delta_M(j\omega)) = \frac{1}{\sigma(T(j\omega))}.
$$

The smaller is $\sigma(T(j\omega))$, the greater will be the size of the smallest destabilizing multiplicative perturbation, and hence the greater will be the stability margin.

A similar result is available for relating the stability margin in the face of additive plant perturbations $\Delta_A(s)$.
to $R(s)$. Let us take $\sigma(\Delta_s(j\omega))$ to be our definition of the "size" of $\Delta_s(j\omega)$ at frequency $\omega$. Then, you have the following stability theorem [6].

Robustness Theorem 2: Suppose the system is stable when $\Delta_s$ and $\Delta_M$ are both zero. Let $\Delta_M = 0$. Then the size of the smallest stable $\Delta_s(s)$ for which the system becomes unstable is

$$\sigma(\Delta_s(j\omega)) = \frac{1}{\sigma(R(j\omega))}.$$ 

As a consequence of Theorems 1 and 2, it is common to specify the stability margins of control systems via singular value inequalities such as

$$\sigma(R(j\omega)) \leq |W_2^{-1}(j\omega)| \quad (2.5)$$

$$\sigma(T(j\omega)) \leq |W_5^{-1}(j\omega)| \quad (2.6)$$

where $|W_2(j\omega)|$ and $|W_5(j\omega)|$ are the respective sizes of the largest anticipated additive and multiplicative plant perturbations.

It is common practice to lump the effects of all plant uncertainty into a single fictitious multiplicative perturbation $\Delta_M$, so that the control design requirements may be written
\[
\frac{1}{\sigma(S(j\omega))} \geq |M(j\omega)|; \quad \sigma(T(j\omega)) \leq |M^{-1}(j\omega)|.
\]

It is not uncommon to see specification on disturbance attenuation and multiplicative stability margin expressed directly in terms of forbidden regions for the Bode plots of \(\sigma(L(j\omega))\) as "singular value loop shaping" requirements.

Here is an important point to note for control system designers. In choosing design specifications \(\mathcal{W}\) and \(\mathcal{W}_3\) the 0 db crossover frequency of the singular value Bode plot of \(W_i^{-1}\) must be sufficiently below 0 db crossover frequency of \(W_3^{-1}\) or the performance requirements (2.4) and (2.6) will not be achievable; more precisely, we require

\[
\sigma(W_i^{-1}(j\omega)) + \sigma(W_3^{-1}(j\omega)) > 1 \quad \forall \omega \quad (2.7).
\]

The Mixed-Sensitivity approach of the robust control system design is a direct and effective way of achieving multivariable loop shaping. In the mixed-sensitivity problem formulation, nominal disturbance attenuation specifications and stability margin specifications equations (2.4) and (2.6) are combined into a single infinity norm specification of the form
where $T_{nu}$ is the closed-loop transfer function from $u_l$ to $y_l$ in Figure 2.6 and

$$T_{nu} \overset{\text{def}}{=} \begin{bmatrix} \mathcal{W} S \\ \mathcal{W} T \end{bmatrix}$$ (2.9)

This is the mixed-sensitivity cost function, so called because it penalizes both sensitivity $S$ and complementary sensitivity $T$.

Note that if you augment the plant $G(s)$ with the weights $\mathcal{W}(s)$ and $\mathcal{W}_3(s)$ as shown in Figure 2.6, then wrap the feedback $F(s)$ from output channel $y_2$ back to input channel $u_l$, the resulting nominal (i.e., $\Delta u = 0$) closed
loop transfer function is precisely the $T_{zm}(s)$ given by (2.9). The Robust Control Toolbox function `augtf` [6] performs this plant augmentation.

The mixed sensitivity cost function has the attractive property that, it provides a much simplified, and nearly equivalent, alternative to the canonical robust control problem for the case of the robust sensitivity problem. It turns out that if (2.8) is strengthened very slightly to

$$\|T_{zm}\|_\infty < 1/\sqrt{2}$$

then robust sensitivity performance can be guaranteed; that is,

$$\frac{1}{\sigma(S(j\omega))} \geq |W(j\omega)|$$

for every multiplicative uncertainty $\Delta_M$ satisfying

$$\bar{\sigma}(\Delta_M(j\omega)) \leq |W_5(j\omega)|$$

This is because in this case the $T_{zm}$ associated with the corresponding canonical robust control problem becomes simply

$$T_{zm} = \begin{bmatrix} W_5S \\ W_5T \end{bmatrix} [I, -I].$$
For any $S(s)$ and $T(s)$, it may be shown that [6]

$$\left\| \begin{bmatrix} W_s S \\ \sqrt{W_s T} \end{bmatrix} \right\| \leq \mu \left( \begin{bmatrix} W_s S \\ \sqrt{W_s T} \end{bmatrix} [I, -I] \right) \leq \sqrt{2} \left\| \begin{bmatrix} W_s S \\ \sqrt{W_s T} \end{bmatrix} \right\|.$$

This relationship simplifies the robust control synthesis problem by achieving a mixed sensitivity design with the transfer function matrix

$$T_{\text{syn}} = \begin{bmatrix} W_s S \\ W_s T \end{bmatrix}.$$
2.6 H-Infinity Optimal Control Synthesis Method

The $H^\infty$ optimal control synthesis method is available in the Robust Control Toolbox [6] to design a robust stabilizing feedback control law such that the robustness inequality $\|T_{yuu}\| < 1$ is satisfied. The method of $H^\infty$ synthesis tool for designing robust design problem can be formulated as follows:

Given a state-space realization of an "augmented plant" $P(s)$ find a stabilizing feedback control law

$$u_2(s) = F(s)y_1(s)$$

such that the norm of the closed-loop transfer function matrix $T_{yuu}$ is small. Such problem addressed by the Robust Control Toolbox is $H^\infty$ Optimal Control [6]:

$$\min \|T_{yuu}\|_{\infty}.$$ 

The standard $H^\infty$ control problem is sometimes also called the $H^\infty$ Small Gain problem.

There are several important properties of $H^\infty$ controllers worth mentioning [6]:

1. The $H^\infty$ optimal control cost function $T_{yuu}$ is all-pass, i.e., $\mathfrak{m}[T_{yuu}] = 1$ for all $\omega \in \mathbb{R}$. 

2. An $H^\infty$ "sub-optimal" controller has order equal to that of the augmented plant ($n$-state). An $H^\infty$ optimal controller can be computed having at most $(n-1)$ states.

3. In any weighted mixed sensitivity problem formulation, the $H^\infty$ controller always cancels the stable poles of the plant with its transmission zeros [3].

4. In the weighted mixed sensitivity problem formulation [2], any unstable pole of the plant inside the specified control bandwidth will be shifted approximately to its $j\omega$-axis mirror image once the feedback loop is closed with $H^\infty$ controller.

Property 1 means that designers can ultimately achieve very precise frequency-domain loop-shaping via suitable weighting strategies [6]. For example, you may augment the plant with frequency dependent weights $W_1$ and $W_2$. Then, if there exists a feasible controller that meets the frequency domain constraints, the software hinf [6] will find one that ultimately "shapes" the signals to the inverse of the weights. This enables the $H^\infty$ to achieve the best results. If you impose demanding design
requirements then the minimal achievable $H^\infty$ norm may be
greater than one, in which case no solution exists to the
standard $H^\infty$ control problem in which the goal is to find
a controller for which $\|T_{ru}\|_\infty < 1$. The $H^\infty$ theory gives four
conditions for the existence of a solution to the
standard $H^\infty$ control problem [5]:

1. There must exist a constant feedback control law
$F(s) =$ "constant matrix" such that the closed-loop $D$
matrix satisfies $\sigma(D) < 1$.

2. Control Riccati $P_\alpha \geq 0$. The $H^\infty$ full-state
feedback control Riccati equation must have a real,
positive semidefinite solution $P_\alpha$. The software
ascertains existence of the Riccati solution by
checking that the associated hamiltonian matrix does
not have any $j\omega$-axis eigenvalues. Positive
semidefiniteness of the solution is verified by
checking, equivalently, that the $H^\infty$ full-state
feedback control-law is asymptotically stabilizing
[5]; this circumvents numerical instabilities
inherent in directly checking positive
semidefiniteness.
3. Observer Riccati $S_R \geq 0$. The Riccati equation associated with the observer dual of the $H^\infty$ full-state feedback control problem must have a real, positive semidefinite solution $S_R$. Again the results of [5] are used to avoid numerical instabilities.

4. $\lambda_{\text{max}}(P_RS_R) < 1$. The greatest eigenvalue of the product of the two Riccati equation solutions must be less than one.

All of these four conditions must hold for there to exist a feedback control law which solves the standard $H^\infty$ control problem.
CHAPTER 3

Simulation Examples and Results

3.1 Benchmark Problem

Plant Description:
Consider a two-mass spring system shown in Figure 3.1 which is a generic model of an uncertain dynamical system with a rigid-body mode and a vibration mode.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{benchmark_problem.png}
\caption{The benchmark problem}
\end{figure}

It is assumed that for the nominal system, \( m_1 = m_2 = 1 \) and \( k = 1 \) with appropriate units and time in units of second. A control force acts on body 1 and the position of body 2 is measured, resulting in a non-colocated control problem. Use Lagrange's equation to derive the dynamic equations of the spring coupled masses.
\[ L = T - V \]
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \]
\[ \dot{x}_1 = x_3 \quad \dot{x}_2 = x_4 \]
\[ T = \frac{1}{2} (m_1 \ddot{z}_1 + m_2 \ddot{z}_2) \]
\[ x_3 = \dot{z}_1 \quad x_4 = \dot{z}_2 \]
\[ V = \frac{1}{2} k(z_2 - z_1)^2 \]
\[ \dot{x}_3 = \ddot{z}_1 \quad \dot{x}_4 = \ddot{z}_2 \]
\[ L = \frac{1}{2} k(m_1 \ddot{z}_1 + m_2 \ddot{z}_2) - \frac{1}{2} k(z_2 - z_1)^2 \]
\[ \frac{\partial L}{\partial \dot{x}_1} = k(z_2 - z_1) \quad \frac{\partial L}{\partial \dot{x}_2} = -k(z_2 - z_1) \]
\[ \frac{\partial L}{\partial x_1} = m_1 \ddot{z}_1 \quad \frac{\partial L}{\partial x_2} = m_2 \ddot{z}_2 \]
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_1} \right) = m_1 \ddot{z}_1 \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_2} \right) = m_2 \ddot{z}_2 \]
\[ m_1 \ddot{z}_1 - k(z_2 - z_1) \quad m_2 \ddot{z}_2 + k(z_2 - z_1) \]
\[ m_1 \dot{x}_3 - k(x_4 - x_1) = 0 \quad m_2 \dot{x}_4 + k(x_4 - x_1) = 0 \]
\[ \ddot{x}_3 = \frac{k(x_2 - x_1)}{m_1} \quad \ddot{x}_4 = -\frac{k(x_2 - x_1)}{m_2} \]
\[ \dot{x}_3 = \left( \frac{k}{m_1} \right) x_2 - \left( \frac{k}{m_1} \right) x_1 \quad \dot{x}_4 = \left( \frac{k}{m_2} \right) x_2 - \left( \frac{k}{m_2} \right) x_1 \]

This system can be represented in state-space form as:
where $x_1$ and $x_2$ are the positions of body 1 and body 2, respectively; $u$ the control input acting on body 1; $y$ the sensor output; $w_1$ and $w_2$ the plant disturbances acting on body 1 and body 2, respectively.

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-k/m_1 & k/m_0 & 0 & 0 \\
k/m_2 & -k/m_0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1/m_1 \\
0
\end{bmatrix}
(u + w_1) +
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
w_2
\]

![Nichols plot](image)

**Figure 3.2** Nichols plot of the open-loop plant
Figure 3.3 Bode plot of the open-loop plant
Nichols and Bode plots of the open-loop plant are shown in Figures 3.2 and 3.3, respectively. Figure 3.4 shows the impulse response of the plant and it is unstable. Now we have to design a control-gain linear feedback controller of the form

\[ \dot{X} = AX + Bu \]
\[ y = CX + Du \]

with the following properties:

1. The closed-loop system is insensitive to high-frequency sensor noise.
2. Reasonable performance/stability robustness and reasonable gain/phase margins are achieved with reasonable bandwidth.

3. Reasonable control effort (e.g., peak control input) is used.

4. Reasonable controller complexity (e.g., controller order) is needed.

The singular value design specifications are:

1. Performance spec.: Minimize the sensitivity function as much as possible. Sensitivity function ($S$) (Figure 3.5) specification;

$$w_1^{-1}(s) = \frac{0.1(s^2 + 2s + 1)}{(1 + (s/10))^2}$$

2. Robustness spec.: -20 db/decade roll-off and at least -20 db at 10 rad/sec. Complementary sensitivity function ($1 - S$) (Figure 3.5) specification

$$w_3^{-1}(s) = \frac{2(1 + (s/30))}{s/10}$$
Figure 3.5 Bode plot of weighting functions

Note that because $W_3(s)$ is an improper transfer function (i.e., has more zeros than poles), it cannot be realized in state-space form; but $W_3(s)G(s)$ is proper. This particular $W_3(s)$ ensures that the matrix $P(s)$ is full rank as required by hinfs [6].
The hinf function solves the small-gain infinity-norm robust control problem; i.e., find a stabilizing controller \( F(s) \) for an augmented system \( P(s) \) such that the closed-loop transfer function satisfies the infinity-norm inequality
\[
\|T\|_\infty = \sup_{\omega} \sigma_{\text{max}}(T(j\omega)) < 1.
\]

Because \( W(s) \) has no state-space realization, the M-file `augtf` [6] can be directly employed here [2].

1. Find a stabilizing controller \( F(s) \) such that the infinity norm of transfer function \( T_{\text{num}} \) is minimized and is less than or equal to one. The \( T_{\text{num}} \) singular value Bode plot associated with each design will indicate how close the design is to the specifications. The Robust Control toolbox function `hinfopt` [6] automates this iteration. `Hinfopt` performs \( H^\infty \) "\( \gamma \)-iteration" to compute the optimal \( H^\infty \) controller using the loop-shifting two-Riccati formulae of `hinfsol` [6]. The output `gamopt` is the optimal "\( \gamma \)" for which the cost function \( T_{\text{num}} \) can achieve under a preset tolerance [6].
As shown in Figure 3.6, the cost function gets pushed to the "all-pass limit" (i.e., to 0 db), the sensitivity function $S$ shown in Figure 3.7 gets pushed down more and more, consequently the complementary sensitivity function $T$ shown in Figure 3.8 approaches to its associated weighting function $W_j^{-1}$.
Figure 3.7 $H^*$ design sensitivity function & $1/W_1$

Figure 3.8 $H^*$ design comp. sensitivity function & $1/W_3$
2. The resulting controller shown in Figure 3.9 is stable, 7th order, strictly proper, and non-minimum phase.

\[ F(s) = \frac{2.36E + 4s^6 + 7.47E + 6s^5 + 1.17E + 7s^4 + 1.15E + 7s^3 + 1.79E + 7s^2 + 9.38E + 6s + 1.147E + 5}{s^7 + 2.40E + 3s^6 + 9.34E + 4s^5 + 7.71E + 5s^4 + 3.45E + 6s^3 + 9.80E + 6s^2 + 1.19E + 7s + 4.85E + 6} \]

with the poles at

\[ s = -2364.0, -30.0, -1.47 \pm 3.64i, -1.0 \pm 1.96E-8i, -4.43. \]
Figure 3.9 Bode plot of controller $F(s)$
3. The loop transfer function $G(s)F(s)$ Nichols and Bode plots are shown in Figures 3.10 and 3.11, respectively. Reasonable gain and phase margins are achieved for the nominal system, G.M. = 5.29 db and P.M. = 5.82 deg.

![Nichols plot of loop transfer func. $G(s)F(s)$](image-url)

*Figure 3.10 Nichols plot of loop transfer func. $G(s)F(s)$*
Figure 3.11 Bode plot of loop transfer function $G(s)F(s)$
4. The closed-loop system step and impulse response plots are presented in Figures 3.12 and 3.13, respectively. A stable controller has been designed for a spring mass coupled system for robust control design. Satisfactory robustness, performance has been achieved with reasonable control effort (e.g., peak control input) and controller complexity (e.g., controller order).

![Figure 3.12 Closed-loop system step response](image)
Figure 3.13 Closed-loop system impulse response
3.2 Robust Control Design of Flexible Body Rocket System

Plant Description:
For a launch vehicle with rigid body mode, a bending mode and a gimbaled engine as shown in Figure 3.14, we need to design a robust autopilot.

![Figure 3.14](https://example.com/f1.png)

Figure 3.14 $H^\infty$ Autopilot design model $G(s)$

![Figure 3.15](https://example.com/f2.png)

Figure 3.15 Block diagram of rocket control system

The airframe model $G(s)$ is shown in Figure 3.14 where:
\( T = \text{Total thrust} \)

\( L_g = \text{Moment arm from center of gravity to effective thrust deflection point} \)

\( I = \text{Moment of inertia} \)

\( M = \text{Total mass of airframe} \)

\( M_a = q S_r d C_{ma} \)

\( q = \text{Dynamic pressure} \)

\( S_r = \text{Aerodynamic reference area} \)

\( \sigma_{\text{imu}} = \text{Relative 1st mode slope at IMU station} \)

\( h_{\text{g}}(x) = \text{Relative 1st mode displacement at gimbal station} \)

\( \omega = \text{1st structural mode frequency} \)

\( \zeta = \text{1st mode damping} \)
Figure 3.16 Nichols plot of the open-loop plant
Figure 3.17 Bode plot of the open-loop plant
Nichols and Bode plot of the open-loop plant are shown in Figures 3.16 and 3.17, respectively. Figure 3.18 shows the impulse response of the plant and it is unstable. Now we have to design a control-gain linear feedback controller of the form

\[ \dot{X} = AX + Bu \]
\[ y = CX + Du \]

with the following properties:

1. The closed-loop system is insensitive to high-frequency sensor noise.
2. Reasonable performance/stability robustness and reasonable gain/phase margins are achieved with reasonable bandwidth.
3. Reasonable control effort (e.g., peak control input) is used.
4. Reasonable controller complexity (e.g., controller order) is needed.

The singular value design specifications are:

1. Performance spec.: Minimize the sensitivity function as much as possible. Sensitivity function ($S$) (Figure 3.19) specification;

   \[ w_1(s) = \frac{s^2 + 20s + 100}{10000} \]

2. Robustness spec.: -40 db/decade roll-off and at least -20 db at 100 rad/sec. Complementary sensitivity function ($I - S$) (Figure 3.19) specification

   \[ w_2(s) = \frac{100000}{s^2} \]
Note that because $W_j(s)$ is an improper transfer function (i.e., has more zeros than poles), it cannot be realized in state-space form; but $W_j(s)G(s)$ is proper. This particular $W_j(s)$ ensures that the matrix of $P(s)$ is full rank as required by hinf [6]. The hinf function solves the small-gain infinity-norm robust control problem; i.e., find a stabilizing controller $F(s)$ for an augmented system $P(s)$ such that the closed-loop transfer function satisfies the infinity-norm inequality.
Because $W(s)$ has no state-space realization, the M-file `augtf` [6] can be directly employed here [2].

1. Find a stabilizing controller $F(s)$ such that the infinity norm of transfer function $T_{su}$ is minimized and is less than or equal to one. The $T_{su}$ singular value Bode plot associated with each design will indicate how close the design is to the specifications. The Robust Control toolbox function `hinf` [6] automates this iteration. `Hinf` does $H^\infty$ "$\gamma$-iteration" to compute the optimal $H^\infty$ controller using the loop-shifting two-Riccati formulae of hinf. The output `gamopt` is the optimal "$\gamma$" for which the cost function $T_{su}$ can achieve under a preset tolerance.
Figure 3.20 $H^\infty$ design cost function $T_{nm}$

As shown in Figure 3.20, the cost function gets pushed to the "all-pass limit" (i.e., to 0 db), the sensitivity function $S$ shown in Figure 3.21 gets pushed down more and more, consequently the complementary sensitivity function $T$ shown in Figure 3.22 approaches to its associated weighting function $W_3^{-1}$. 
Figure 3.21 $H^*$ design sensitivity function & $1/W_1$

Figure 3.22 $H^*$ design comp sensitivity function & $1/W_3$
2. The resulting controller shown in Figure 3.23 is stable, 6th order, strictly proper, and non-minimum phase.

\[ F(s) = \frac{-1.37E+9s^3 - 5.84E+10s^4 - 8.06E+11s^5 - 6.40E+12s^6 - 2.72E+13s - 4.46E+13}{s^6 + 1.40E+4s^5 + 3.77E+6s^4 + 1.87E+9s^3 + 5.11E+10s^2 + 4.74E+11s + 1.47E+12} \]

with the poles at

\[ s = -1.37E+4, -1.18E+2 \pm 3.39E+2i, -10.0, -10.0, -8.31. \]
Figure 3.23 Bode plot of controller $F(s)$
3. The loop transfer function \( G(s)F(s) \) Nichols and Bode plots are shown in Figures 3.24 and 3.25, respectively. Reasonable gain margin is achieved for the nominal system, G.M. = 6.50 db.

\[ \]
Figure 3.25 Bode plot of loop transfer function $G(s)F(s)$
4. The closed-loop system step and impulse response plots are presented in Figures 3.26 and 3.27, respectively. A stable controller has been designed for a flexible body rocket system for robust control design. Satisfactory robustness, performance has been achieved with reasonable control effort (e.g., peak control input) and controller complexity (e.g., controller order).

![Figure 3.26 Closed-loop system step response](image)

Figure 3.26 Closed-loop system step response
Figure 3.27 Closed-loop system impulse response.
3.3 Robust Control Design of Hydraulic Actuator

For a hydraulic actuator control system model shown below we need to design a robust controller.

![Block diagram of actuator control system](image)

Figure 3.28 Block diagram of actuator control system

![Nichols plot of the open-loop plant](image)

Figure 3.29 Nichols plot of the open-loop plant
Figure 3.30 Bode plot of the open-loop plant
Nichols and Bode plot of the open-loop plant are shown in Figures 3.29 and 3.30, respectively. Figure 3.31 shows the impulse response of the plant and it is stable. However, we would like to design a robust controller to get better system margins. We need to design a control-gain linear feedback controller of the form

\[
\dot{X} = AX + Bu \\
y = CX + Du
\]

with the following properties:
1. The closed-loop system is insensitive to high-frequency sensor noise.
2. Reasonable performance/stability robustness and reasonable gain/phase margins are achieved with reasonable bandwidth.
3. Reasonable control effort (e.g., peak control input) is used.
4. Reasonable controller complexity (e.g., controller order) is needed.

The singular value design specifications are:

1. Performance spec.: Minimize the sensitivity function as much as possible. Sensitivity function \((S)\) (Figure 3.32) specification:

\[
\kappa^{-1}(s) = \frac{(s/30)+1}{1.25((s/1000)+1)}
\]

2. Robustness spec.: -40 db/decade roll-off and at least -20 db at 1000 rad/sec. Complementary sensitivity function \((I-S)\) (Figure 3.32) specification

\[
\omega_3^{-1}(s) = \frac{2E-7s^2+1.2E-3s+1}{1E-5s^2+2E-3s+0.1}
\]
Figure 3.32 Bode plot of weighting functions

Note that because $W_5(s)$ is an improper transfer function (i.e., has more zeros than poles), it cannot be realized in state-space form; but $W_5(s)G(s)$ is proper. This particular $W_5(s)$ ensures that the matrix of $P(s)$ is full rank as required by hinf [6]. The hinf function solves the small-gain infinity-norm robust control problem; i.e., find a stabilizing controller $F(s)$ for an augmented system $P(s)$ such that the closed-loop transfer function satisfies the infinity-norm inequality
Because $\mathcal{H}_I(s)$ has no state-space realization, the M-file `augtf` [6] can be directly employed here [2].

2. Find a stabilizing controller $F(s)$ such that the infinity norm of transfer function $T_{pu}$ is minimized and is less than or equal to one. The $T_{pu}$ singular value Bode plot associated with each design will indicate how close the design is to the specifications. The Robust Control toolbox function `hinfopt` [6] automates this iteration. `Hinfopt` does $H^\infty$ "$\gamma$-iteration" to compute the optimal $H^\infty$ controller using the loop-shifting two-Riccati formulae of hinf. The output `gamopt` is the optimal "$\gamma$" for which the cost function $T_{pu}$ can achieve under a preset tolerance.
As shown in Figure 3.33, the cost function gets pushed to the "all-pass limit" (i.e., to 0 db), the sensitivity function $S$ shown in Figure 3.34 gets pushed down more and more, consequently the complementary sensitivity function $T$ shown in Figure 3.35 approaches to its associated weighting function $W_3^{-1}$.
Figure 3.34 $H^\infty$ design sensitivity function & $1/W_1$

Figure 3.35 $H^\infty$ design comp sensitivity function & $1/W_3$
2. The resulting controller shown in Figure 3.36 is stable, 5th order, strictly proper, and non-minimum phase.

\[ F(s) = \frac{8.89E^{-3}s^5 + 1.63E^{-6}s^4 + 1.73E^{-9}s^3 + 9.15E^{-12}s^2 + 1.18E^{-15}s + 1.68E^{-18}}{s^2 + 1.76E^{-4}s + 2.11E^{-7}s^2 + 4.63E^{-9}s^3 + 9.83E^{-11}s^4 + 1.09E^{-13}s^5} \]

with the poles at

\[ s = -1.63E+4, -1.08E+2 \pm 9.93E+1i, -1.03E+3, -30.0. \]
Figure 3.36 Bode plot of controller $F(s)$
3. The loop transfer function $G(s)F(s)$ Nichols and Bode plots are shown in Figures 3.37 and 3.38, respectively. Reasonable gain and phase margins are achieved for the nominal system, G.M. = 17.75 db and P.M. = 59.64 deg.

![Nichols plot](image)

**Figure 3.37** Nichols plot of loop transfer func. $G(s)F(s)$
Figure 3.38 Bode plot of loop transfer function $G(s)F(s)$
4. The closed-loop system step and impulse response plots are presented in Figures 3.39 and 3.40, respectively. A stable controller has been designed for a hydraulic actuator system for robust control design. Satisfactory robustness, performance has been achieved with reasonable control effort (e.g., peak control input) and controller complexity (e.g., controller order).

Figure 3.39 Closed-loop system step response
Figure 3.40 Closed-loop system impulse response
3.4 Robust Control of Aircraft Lateral Dynamics

One of the most important applications of state-space methods is in the design of control system for aircraft. The forces and moments on such vehicles are produced by the motion of the vehicle through the air and are obtained, in principle by integrating the aerodynamic pressure over the entire surface of the aircraft. The aircraft pitch motion is typically controlled by a control surface called the elevator. The roll is controlled by a pair of ailerons, and the yaw is controlled by a rudder. In many cases the system lateral and longitudinal dynamics are only lightly coupled, and the control system can be designed for each channel without regard to the other [7]. Thus in this example controller using the rudder will be designed for the yaw channel.

Plant Description:

\[
\beta = \frac{Y_p}{V} \beta + \frac{Y_r}{V} P + \left(\frac{Y_r}{V} - 1\right) r + \frac{g}{V} \phi + \frac{Y_A}{V} \delta_A + \frac{Y_r}{V} \delta_R
\]

\[
\dot{P} = L_0 \beta + L_r P + L_r r + L_A \delta_A + L_R \delta_R
\]

\[
\dot{r} = N_p \beta + N_r P + N_r r + N_A \delta_A + N_R \delta_R
\]

\[
\dot{\phi} = \phi
\]
Figure 3.41 Nichols plot of the open-loop plant
Figure 3.42 Bode plot of the open-loop plant
Nichols and Bode plots of the open-loop plant are shown in Figures 3.41 and 4.42, respectively. Figure 3.43 shows the impulse response of the plant and it is a very slow system. The system does get to zero after running the simulation for long time (beyond 100 seconds). Now we have to design a control-gain linear feedback controller of the form
\[
\dot{x} = Ax + Bu \\
y = CX + Du
\]

with the following properties:

1. The closed-loop system is insensitive to high-frequency sensor noise.
2. Reasonable performance/stability robustness and reasonable gain/phase margins are achieved with reasonable bandwidth.
3. Reasonable control effort (e.g., peak control input) is used.
4. Reasonable controller complexity (e.g., controller order) is needed.

The singular value design specifications are:

1. Performance spec.: Minimize the sensitivity function as much as possible. Sensitivity function \((S)\) (Figure 3.44) specification:

\[
\gamma^{-1}(s) = \frac{(s+1)(s+1)}{10((s/50)+1)^2}
\]

2. Robustness spec.: -40 db/decade roll-off and at least -20 db at 10 rad/sec. Complimentary sensitivity function \((I-S)\) (Figure 3.44) specification
$$w_3^{-1}(s) = \frac{(s/30+1)^2}{0.03(s+1)^2}$$

Figure 3.44 Bode plot of weighting functions

Note that because $W_3(s)$ is an improper transfer function (i.e., has more zeros than poles), it cannot be realized in state-space form; but $W_3(s)G(s)$ is proper. This particular $W_3(s)$ ensures that the matrix of $P(s)$ is full rank as required by hinf [6].
The hinf function solves the small-gain infinity-norm robust control problem; i.e., find a stabilizing controller $F(s)$ for an augmented system $P(s)$ such that the closed-loop transfer function satisfies the infinity-norm inequality

$$\|T_{rm}\|_\infty = \sup_{s \in \sigma} \sigma_{\max}(T_{rm}(j\omega)) < 1.$$ 

Because $W_5(s)$ has no state-space realization, the M-file `augtf` [6] can be directly employed here [2].

1. Find a stabilizing controller $F(s)$ such that the infinity norm of transfer function $T_{rm}$ is minimized and is less than or equal to one. The $T_{rm}$ singular value Bode plot associated with each design will indicate how close the design is to the specifications. The Robust Control toolbox function `hinfopt` [6] automates this iteration. Hinfopt does $H^\infty$ "$\gamma$-iteration" to compute the optimal $H^\infty$ controller using the loop-shifting two-Riccati formulae of hinf. The output `gamopt` is the optimal "$\gamma$" for which the cost function $T_{rm}$ can achieve under a preset tolerance.
As shown in Figure 3.45, the cost function gets pushed to the "all-pass limit" (i.e., to 0 db), the sensitivity function $S$ shown in Figure 3.46 gets pushed down more and more, consequently the complementary sensitivity function $T$ shown in Figure 3.47 approaches to its associated weighting function $W_i^{-1}$.
Figure 3.46 H* design sensitivity function & 1/W

Figure 3.47 H* design comp sensitivity function & 1/W3
2. The resulting controller shown in Figure 3.48 is stable, 8th order, strictly proper, and non-minimum phase.

\[
F(s) = \frac{-3.6E+5s^7 - 2.4E+7s^6 - 4.4E+8s^5 - 1.9E+9s^4 - 4.1E+9s^3 - 7.3E+9s^2 - 4.0E+9s - 2.4E+7}{s^8 + 8.4E+5s^7 + 1.7E+7s^6 + 1.8E+8s^5 + 4.9E+8s^4 + 5.9E+8s^3 + 3.9E+8s^2 + 1.7E+8s + 4.7E+7}
\]

with the poles at

\[
s = -5.15E+3, -3.20E+3, -7.27, -1.13E-1 \pm 5.52E-1i, -1.23, -10.0E-1 \pm 6.15E-7i.
\]
Figure 3.48 Bode plot of controller $F(s)$
3. The loop transfer function $G(s)F(s)$ Nichols and Bode plots are shown in Figures 3.49 and 3.50, respectively. Reasonable phase margin is achieved for the nominal system, $P.M. = 7.87$ deg.

Figure 3.49 Nichols plot of loop transfer func. $G(s)F(s)$
Figure 3.50 Bode plot of loop transfer function $G(s)F(s)$
4. The closed-loop system step and impulse response plots are presented in Figures 3.51 and 3.52, respectively. A stable controller has been designed for an aircraft lateral dynamics system for robust control design. Satisfactory robustness, performance has been achieved with reasonable control effort (e.g., peak control input) and controller complexity (e.g., controller order).

Figure 3.51 Closed-loop system step response
Figure 3.52 Closed-loop system impulse response
CHAPTER 4

Conclusion

The $H^\infty$ synthesis robust control design presented in this thesis demonstrates the applicability of an $H^\infty$ control synthesis technique to control systems such as two-mass spring, flexible body rocket, hydraulic actuator and aircraft lateral dynamics. The robustness and performance capability of the closed-loop system was demonstrated with satisfactory results. It has also been shown through computer simulation that a satisfactory and reasonable controller was designed incorporating weighting functions. A controller design approach which has the potential to considerably reduce the complexity of control implementation over the system envelope and which exploits the robustness properties of the $H^\infty$ based controller was developed and demonstrated.
APPENDIX A

MATLAB Input File for Benchmark Problem
% File name: cart4.m

!del cart4
!del cart4.met
diary cart4
clear
c1g

format

% Input the nominal system
k = 1
m1 = 1
m2 = 1

% Input the plant in state space form
ag = [0 0 1 0]
   [0 0 0 1]
   [-k/ml k/ml 0 0]
   [k/m2 -k/m2 0 0]

bg = [0 0 1/ml 0]

cg = [0 1 0 0]

dg = 0

% Settling time of about 15 seconds for the nominal system
% with m1=m2, k=1. Tset=4/ZetaW
Tset=15
ZetaW=4/Tset
ag2 = ag + ZetaW*eye(4,4)

format long e
eig(ag2)
format

w = logspace(-2,4);
ni kl_ol(ag,bg, cg, dg, 1, w, 260, [-60, 40]);
grid
title('NICHOLS PLOT OF OPEN LOOP PLANT')
xlabel('Phase (deg)')
ylabel('Gain (db)')
pause
meta cart4
clg
axis

[magp1, phap1] = bode(ag, bg, cg, dg, 1, w);

magp1 = 20*log10(magp1);
semilogx(w, magp1)
title('BODE PLOT OF OPEN LOOP PLANT')
xlabel('Frequency (rad/sec)')
ylabel('Gain (db)')
gtext('Control Force to Position of Body 2')
meta cart4

pause

% Input the weighting functions.
numw1 = conv([1/10 1], [1/10 1]);
denw1 = .1*[1 2 1];

numw3 = [1/10 0];
denw3 = 2*[1/30 1];

[magw1, phaw1] = bode(denw1, numw1, w);
[magw3, phaw3] = bode(denw3, numw3, w);

magw1 = 20*log10(magw1);
magw3 = 20*log10(magw3);
w1 = [numw1; denw1]
w2 = [0.01; 1]
w3 = [numw3;denw3]

semilogx(w,magw1,w,magw3)
grid
title('BODE PLOT OF WEIGHTING FUNCTIONS')
xlabel('Frequency (rad/sec)')
ylabel('Gain (db)')
gtext('Performance Spec (1/W1)')
gtext('Robustness Spec (1/W3)')
pause
meta cart4
clg

subplot(211)
semilogx(w,magw1,w,magw1,w,magw3)
grid
title('BODE PLOT OF PLANT AND WEIGHTING FUNCTIONS')
xlabel('Frequency (rad/sec)')
ylabel('Gain (db)')
gtext('Performance Spec (1/W1)')
gtext('Robustness Spec (1/W3)')
subplot(212)
semilogx(w,phaw1,w,phaw1,w,phaw3)
grid
xlabel('Frequency (rad/sec)')
ylabel('Phase (deg)')
gtext('Performance Spec (1/W1)')
gtext('Robustness Spec (1/W3)')
pause
meta cart4
clg

% Plant augmentation for use in weighted mixed-sensitivity H_inf design.
[a,b1,b2,c1,c2,d11,d12,d21,d22] =
augtf(ag2,bg,cg,dg,w1,w2,w3)

format long e
% H_inf optimal control synthesis via gamma-iteration.
[gamopt,af,bf,cf,df,acl,bcl,ccl,dc1] =
hinfopt(a,b1,b2,c1,c2,d11,d12,d21,d22,1)

97
disp('       ....COMPUTING BODE PLOT OF THE COST
FUNCTION....')
svtt = sigma(acl,bcl,ccl,dcl,l,w);
svtt = 20*log10(svtt);
semilogx(w,svtt);
grid
title('H-INFINITY DESIGN COST FUNCTION Tylul');
xlabel('Frequency (rad/sec)');
ylabel('SV (db)');
meta cart4
pause

disp('       ....COMPUTING BODE PLOTS OF SENS. & COMP. SENS
FUNCTIONS....')
[a,b,c,d] = series(af,bf,cf,df,ag/bg/cg/dg);
[acs,bcs,ccs,dcs] = feedbk(a,b,c,d,2);
svs = sigma(a,b,c,d,3,w);
svs = -20*log10(svs);
svt = sigma(acs,bcs,ccs,dcs,1,w);
svt = 20*log10(svt);

semilogx(w,magw1,w,svs);
grid
title('H-INFINITY DESIGN SENSITIVITY FUNCTION & 1/W1');
xlabel('Frequency (rad/sec)');
ylabel('SV (db)');
gtext('1/W1')
gtext('Sensitivity Function')
meta cart4
pause

semilogx(w,magw3,w,svt);
title('H-INFINITY DESIGN COMP. SENSITIVITY FUNCTION &
1/W3');
xlabel('Frequency (rad/sec)');
ylabel('SV (db)');
grid
gtext('1/W3')
gtext('Complementary Sensitivity Function')
meta cart4
pause

% Find a stabilizing controller F(s).
[numf,denf] = ss2tf(af,bf,cf,df,1)
format
\[
[\text{contmag}, \text{contpha}] = \text{bode}(af, bf, cf, df, 1, w);
\text{contmag} = 20 * \log10(\text{contmag});
\]

\[
[as, bs, cs, ds] = \text{series}(af, bf, cf, df, ag, bg, cg, dg);
[agf, bgf, cgf, dgf] = \text{cloop}(as, bs, cs, ds, 1, 1);
[GFnum, GFden] = \text{ss2tf}(agf, bgf, cgf, dgf, 1);
[GFmag, GFpha] = \text{bode}(agf, bgf, cgf, dgf, 1, w);
\]

\[
\text{nkl} \_\text{ol}(agf, bgf, cgf, dgf, 1, w, 260, [-60, 40]);
\]

\[
\text{grid}
\text{title('NICHOLS PLOT OF LOOP TRANSFER FUNCTION G*F')};
\text{xlabel('Phase (deg)');}
\text{ylabel('Gain (db)');}
\text{pause}
\text{meta cart4}
\text{clg}
\text{axis}
\]

\[
\text{ytstep} = \text{step}(acl, bcl, ccl, dcl, 1, t);
\text{pause}
\text{ytimp} = \text{impulse}(acl, bcl, ccl, dcl, 1, t);
\text{pause}
\%
\text{Find the gain and phase margins}
[gm, pm, wcp, wcg] = \text{margin}(GFmag, GFpha, w);
\]

\[
\text{GFmag} = 20 * \log10(GFmag);
\text{Gainmargin} = 20 * \log10(gm)
\text{Phasemargin} = pm
\]

\[
\text{clf}
\text{plot(t, ytstep(:, 1))};
\text{grid};
\text{title('STEP RESPONSE OF TOTAL CLOSED LOOP')};
\text{gtext('Input=>Control Force  Output=>Position of Body 2')};
\text{xlabel('Time (sec)');}
\text{ylabel('Amplitude');}
\text{meta cart4}
\text{pause}
\text{clf}
\]

\[
\text{plot(t, ytimp(:, 1))};
\text{grid};
\text{title('IMPULSE RESPONSE OF TOTAL CLOSED LOOP')};
\]
gtext('Input=>Control Force, Output=>Position of Body 2')
xlabel('Time (sec)');
ylabel('Amplitude');
meta cart4
pause clg

subplot(211)
semilogx(w,contmag);
grid;
title('BODE PLOT OF CONTROLLER F(s)');
xlabel('Frequency (rad/sec)');
ylabel('Gain (db)');
subplot(212)
semilogx(w,contpha);
grid;
xlabel('Frequency (rad/sec)');
ylabel('Phase (deg)');
meta cart4
pause clg

subplot(211)
semilogx(w,GFmag);
grid;
title('BODE PLOT OF LOOP TRANSFER FUNCTION G*F');
xlabel('Frequency (rad/sec)');
ylabel('Magnitude (db)');
subplot(212)
semilogx(w,GFpha);
grid;
xlabel('Frequency (rad/sec)');
ylabel('Phase (deg)');
meta cart4

diary off
APPENDIX B

MATLAB Input File for Flexible Body Rocket System
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%*** FLEXIBLE BODY ROCKET SYSTEM ***
%*** ROBUST CONTROL DESIGN ***
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

FILE NAME: rocket1.m

!del rocket!
!del rocket1.met
diary rocket!
clear
c1g

format

TLg_Iyy = -3.333333;
MAlpha_Iyy = 1.864e+1;
def_imu = 1.12e-4;
TPhig_M = 3.87e+4;
zeta = 0.5;
omega = 7.87;

num1 = TLg_Iyy;
den1 = [1 0 -MAlpha_Iyy];
num2 = [TPhig_M*def_imu];
den2 = [1 2*zeta*omega omega^2];

[a_rig b_rig c_rig d_rig] = tf2ss(num1,den1);
[a_flex b_flex c_flex d_flex] = tf2ss(num2,den2);
[a_tot b_tot c_tot d_tot] = append(a_rig,b_rig,c_rig,d_rig,...
 a_flex,b_flex,c_flex,d_flex);

m = [1 1]';
f = [0 0;0 0];
n = [1 1];

[ag bg cg dg] = interc(a_tot,b_tot,c_tot,d_tot,m,n,f);

format long e
damp(ag)

format
\( w = \logspace(-2, 4); \)

\[ \text{nikl ol}(ag, bg, cg, dg, l, w, 260, [-60, 40]); \]

\[ \text{grid} \]

\[ \text{title('NICHOLS PLOT OF OPEN LOOP PLANT')} \]

\[ \text{xlabel('Phase (deg)')} \]

\[ \text{ylabel('Gain (db)')} \]

\[ \text{pause} \]

\[ \text{meta rocket!} \]

\[ \text{clg} \]

\[ \text{axis} \]

\[ [\text{magpl}, \text{phapl}] = \text{bode}(ag, bg, cg, dg, l, w); \]

\[ \text{magpl} = 20*\log10(\text{magpl}); \]

\[ \text{semilogx}(w, \text{magpl}) \]

\[ \text{title('BODE PLOT OF OPEN LOOP PLANT')} \]

\[ \text{xlabel('Frequency (rad/sec)')} \]

\[ \text{ylabel('Gain (db)')} \]

\[ \text{grid} \]

\[ \text{pause} \]

\[ \text{meta rocket!} \]

\[ t = 0:0.1:30; \]

\[ \text{pimpulse} = \text{impulse}(ag, bg, cg, dg, l, t); \]

\[ \text{plot}(t, \text{pimpulse}) \]

\[ \text{grid} \]

\[ \text{title('IMPULSE RESPONSE OF THE PLANT')} \]

\[ \text{xlabel('Time (sec)')} \]

\[ \text{ylabel('Amplitude')} \]

\[ \text{gtext('Engine Command to IMU Deflection')} \]

\[ \text{meta rocket!} \]

\[ \text{pause} \]

\% Input the weighting functions.

\[ \text{numwl} = \text{conv}([0 100], [0 100]); \]

\[ \text{denwl} = \text{conv}([1 10], [1 10]); \]

\[ \text{numw3} = [1 0 0]; \]

\[ \text{denw3} = [0 0 100000]; \]

\% numwl = conv([1/300 1], [1/300 1]);

\% denwl = 0.01*conv([1/10 1], [1/10 1]);

\% numw3 = [1/30 1];

\% denw3 = 40*[1/5000 1];
[magw1, phawl] = bode(denw1, numw1, w);
[magw3, phaw3] = bode(denw3, numw3, w);

magw1 = 20*log10(magw1);
magw3 = 20*log10(magw3);

w1 = [numw1; denw1]
w2 = [] & [0.01; 1]
w3 = [numw3; denw3]

semilogx(w, magw1, w, magw3)
grid
title('BODE PLOT OF WEIGHTING FUNCTIONS')
xlabel('Frequency (rad/sec)')
ylabel('Gain (db)')
gtext('Performance Spec (1/W1)')
gtext('Robustness Spec (1/W3)')
pause
meta rocket1
clg

clg

subplot(211)
semilogx(w, magpl, w, magw1, w, magw3)
grid
title('BODE PLOT OF PLANT AND WEIGHTING FUNCTIONS')
xlabel('Frequency (rad/sec)')
ylabel('Gain (db)')
gtext('Open Loop Plant')
gtext('Performance Spec (1/W1)')
gtext('Robustness Spec (1/W3)')

subplot(212)
semilogx(w, phapl, w, phawl, w, phaw3)
grid
xlabel('Frequency (rad/sec)')
ylabel('Phase (deg)')
gtext('Open Loop Plant')
gtext('Performance Spec (1/W1)')
gtext('Robustness Spec (1/W3)')
pause
meta rocket1
clg
pause

% Plant augmentation for use in weighted mixed-sensitivity H_inf design.
format long e

% H_inf optimal control synthesis via gamma-
iteration.
[gamopt,af,bf,cf,df,acl,bcl,ccl,dcl] =
hinfopt(a,b1,b2,c1,c2,d11,d12,d21,d22,1)

disp('....COMPUTING BODE PLOT OF THE COST
FUNCTION.....')
svtt = sigma(acl,bcl,ccl,dcl,1,w);
svtt = 20*log10(svtt);
semilogx(w,svtt);
grid
title('H-INFINITY DESIGN COST FUNCTION Tylul');
xlabel('Frequency (rad/sec)');
ylabel('SV (db)');
meta rocket1
pause

disp('....COMPUTING BODE PLOTS OF SENS. & COMP. SENS
FUNCTIONS....')
[a,b,c,d] = series(af,bf,cf,df,ag,bg,kg,kg);
[acs,bcs,ccs,dcs] = feedbk(a,b,c,d,2);
svs = sigma(a,b,c,d,3,w);
svs = -20*log10(svs);
svt = sigma(acs,bcs,ccs,dcs,1,w);
svt = 20*log10(svt);

semilogx(w,magw1,w,svs);
grid
title('H-INFINITY DESIGN SENSITIVITY FUNCTION & 1/W1');
xlabel('Frequency (rad/sec)');
ylabel('SV (db)');
gtext('1/W1')
gtext('Sensitivity Function')
meta rocket1
pause

semilogx(w,magw3,w,svt);
title('H-INFINITY DESIGN COMP. SENSITIVITY FUNCTION &
1/W3');
xlabel('Frequency (rad/sec)');
ylabel('SV (db)');
grid
gtext('1/W3')
gtext('Complementary Sensitivity Function')
meta rocket1
pause

% Find a stabilizing controller F(s).
[numf,denf] = ss2tf(af, bf, cf, df, l)
format

[contmag, contpha] = bode(af, bf, cf, df, 1, w);
contmag = 20*log10(contmag);

[as, bs, cs, ds] = series(af, bf, cf, df, ag, bg, cg, dg)
[agf, bgf, cgf, dgf] = cloop(as, bs, cs, ds, 1, l);
[GFnum, GFden] = ss2tf(agf, bgf, cgf, dgf, l);
[GFmag, GFpha] = bode(agf, bgf, cgf, dgf, 1, w);

nikl_ol(agf, bgf, cgf, dgf, 1, w, 260, [-60, 40]);
grid
title('NICHOLS PLOT OF LOOP TRANSFER FUNCTION G*F');
xlabel('Phase (deg)');
ylabel('Gain (db)');
pause
meta rocket1
clg
axis

ytstep = step(acl, bcl, -ccl, -dcl, 1, t);
pause
ytimp = impulse(acl, bcl, -ccl, -dcl, 1, t);
pause

% Find the gain and phase margins
[gm, pm, wcp, wcg] = margin(GFmag, GFpha, w);

GFmag = 20*log10(GFmag);
Gainmargin = 20*log10(gm)
Phasemargin = pm

clg
plot(t, ytstep(:, 1));
grid;
title('STEP RESPONSE OF TOTAL CLOSED LOOP');
gtext('Input=>Engine Command     Output=>IMU Deflection')
xlabel('Time (sec)');
ylabel('Amplitude');
meta rocket1
pause
clg

plot(t,ytimp(:,1));
gtext('Input=>Engine Command Output=>IMU Deflection')
grid;
title('IMPULSE RESPONSE OF TOTAL CLOSED LOOP');
xlabel('Time (sec)');
ylabel('Amplitude');
meta rocket1
pause
clg

subplot(211)
semilogx(w,contmag);
gtext('Input=>Engine Command Output=>IMU Deflection')
grid;
title('BODE PLOT OF CONTROLLER F(s)');
xlabel('Frequency (rad/sec)');
ylabel('Gain (db)');
subplot(212)
semilogx(w,contpha);
gtext('Input=>Engine Command Output=>IMU Deflection')
grid;
xlabel('Frequency (rad/sec)');
ylabel('Phase (deg)');
meta rocket1
pause
clg

subplot(211)
semilogx(w,GFmag);
gtext('Input=>Engine Command Output=>IMU Deflection')
grid;
title('BODE PLOT OF LOOP TRANSFER FUNCTION G*F');
xlabel('Frequency (rad/sec)');
ylabel('Magnitude (db)');
subplot(212)
semilogx(w,GFpha);
gtext('Input=>Engine Command Output=>IMU Deflection')
grid;
xlabel('Frequency (rad/sec)');
ylabel('Phase (deg)');
meta rocket1

diary off
APPENDIX C

MATLAB Input File for Hydraulic Actuator
%*******************************************************************************
%*** HYDRAULIC ACTUATOR ***
%*** ROBUST CONTROL DESIGN ***
%*******************************************************************************

%File name: actuator.m

%del actuator
%del actuator.met
diary actuator
clear
clg
axis

format

num = 9000;
den = [1 30 700 1000];

[ag,bg, cg, dg] = tf2ss(num, den);

w = logspace(-2, 4);

nikl_ol(ag, bg, cg, dg, 1, w, 260, [-60, 40]);
grid
title('NICHOLS PLOT OF OPEN LOOP PLANT')
xlabel('Phase (deg)')
ylabel('Gain (db)')
pause
meta actuator
clg
axis

[magpl, phapl] = bode(ag, bg, cg, dg, 1, w);
magpl = 20 * log10(magpl);
semilogx(w, magpl)
title('BODE PLOT OF OPEN LOOP PLANT')
xlabel('Frequency (rad/sec)')
ylabel('Gain (db)')
grid
pause
meta actuator

t = 0:0.1:30;

pimpulse = impulse(ag, bg, cg, dg, 1, t);
plot(t,pimpulse) 
grid 
title('IMPULSE RESPONSE OF THE PLANT') 
xlabel('Time (sec)') 
ylabel('Amplitude') 
gtext('Actuator Input to Actuator Deflection') 
meta actuator 
pause 

% Input the weighting functions. 
numw1 = 1.25*[1/1000 1]; 
denw1 = [1/30 1]; 
numw3 = .1*conv([1/100 1],[1/100 1]); 
denw3 = conv([1/1000 1],[1/5000 1]); 

[magw1,phaw1]=bode(denw1,numw1,w); 
[magw3,phaw3]=bode(denw3,numw3,w); 
magw1=20*log10(magw1); 
magw3=20*log10(magw3); 

w1 = [numw1;denw1] 
w2 = [0.01;1] 
w3 = [numw3;denw3] 

semilogx(w,magw1,w,magw3) 
grid 
title('BODE PLOT OF WEIGHTING FUNCTIONS') 
xlabel('Frequency (rad/sec)') 
ylabel('Gain (db)') 
gtext('Performance Spec (1/W1)') 
gtext('Robustness Spec (1/W3)') 
pause 
meta actuator 
clg 

subplot(211) 
semilogx(w,magp1,w,magw1,w,magw3) 
grid 
title('BODE PLOT OF PLANT AND WEIGHTING FUNCTIONS') 
xlabel('Frequency (rad/sec)') 
ylabel('Gain (db)') 
gtext('Open Loop Plant') 
gtext('Performance Spec (1/W1)') 
gtext('Robustness Spec (1/W3)') 

110
% Plant augmentation for use in weighted mixed-sensitivity H_inf design.
[a,b1,b2,c1,c2,d11,d12,d21,d22] =
augtf(ag,bg,cg,dg,w1,w2,w3)

format long e
% H_inf optimal control synthesis via gamma-iteration.
[gamopt,af,bf,cf,df,ac1,bc1,cc1,dc1] =
hinfopt(a,b1,b2,c1,c2,d11,d12,d21,d22,1)

disp('....COMPUTING BODE PLOT OF THE COST FUNCTION....')
svtt = sigma(ac1,bc1,cc1,dc1,1,w);
svtt = 20*log10(svtt);
semilogx(w,svtt);
grid
title('H-INFINITY DESIGN COST FUNCTION Tylul');
xlabel('Frequency (rad/sec)');
ylabel('SV (db)');
meta actuator
pause

disp('....COMPUTING BODE PLOTS OF SENS. & COMP. SENS FUNCTIONS....')
[a,b,c,d] = series(af,bf,cf,df,ag,bg,cg,dg);
[acs,bcs,ccs,dcss] = feedbk(a,b,c,d,2);
svs = sigma(a,b,c,d,3,w);
svs = -20*log10(svs);
svt = sigma(acs,bcs,ccss,dcss,1,w);
svt = 20*log10(svt);
semilogx(w,magwl,w,svs);
grid
title('H-INFINITY DESIGN SENSITIVITY FUNCTION & 1/W1');
xlabel('Frequency (rad/sec)');
ylabel('SV (db)');
gtext('1/W1')
gtext('Sensitivity Function')
meta actuator
pause

semilogx(w,mag3,w,svt);
title('H-INFINITY DESIGN COMP. SENSITIVITY FUNCTION & 1/W3');
xlabel('Frequency (rad/sec)');
ylabel('SV (db)');
gtext('1/W3')
gtext('Complimentary Sensitivity Function')
meta actuator
pause

% Find a stabilizing controller F(s).
[numf,dnf] = ss2tf(af,bf,cf,df,1)
format

[contmag,contpha] = bode(af,bf,cf,df,1,w);
contmag = 20*log10(contmag);

[as,bs,cs,ds] = series(af,bf,cf,df,ag/bg,cf,df,1);
[agf,bgf,cf,gdf] = cloop(as,bs,cs,ds,1,1);
[GFnum,GFden] = ss2tf(agf,bgf,cf,df,1);
[GFmag,GFpha] = bode(-agf,bgf,cf,df,1,w);

nkl_ol(agf,bgf,cf,df,1,w,260,[-60,40]);
grid
title('NICHOLS PLOT OF LOOP TRANSFER FUNCTION G*F');
xlabel('Phase (deg)');
ylabel('Gain (db)');
pause
meta actuator
clg
taxis

ytstep = step(acl,bcl,ccl,dcl,1,t);
pause
ytmp = impulse(acl,bcl,ccl,dcl,1,t);
pause
% Find the gain and phase margins
[gm, pm, wcp, wcg] = margin(GFmag, GFpha, w);
GFmag = 20*log10(GFmag);
Gainmargin = 20*log10(gm)
Phasemargin = pm

clg
plot(t, ytstep(:, 1));
grid;
title('STEP RESPONSE OF TOTAL CLOSED LOOP');
gtext('Input=>Actuator Command Output=>Actuator Deflection')
xlabel('Time (sec)');
ylabel('Amplitude');
meta actuator
pause

clg
plot(t, ytimp(:, 1));
grid;
title('IMPULSE RESPONSE OF TOTAL CLOSED LOOP');
gtext('Input=>Actuator Command Output=>Actuator Deflection')
xlabel('Time (sec)');
ylabel('Amplitude');
meta actuator
pause

clg
subplot(211)
semilogx(w, contmag);
grid;
title('BODE PLOT OF CONTROLLER F(s)');
xlabel('Frequency (rad/sec)');
ylabel('Gain (db)')
subplot(212)
semilogx(w, contpha);
grid;
xlabel('Frequency (rad/sec)');
ylabel('Phase (deg)')
meta actuator
pause

clg
```matlab
subplot(211)
semilogx(w,GFmag);
grid;
title('BODE PLOT OF LOOP TRANSFER FUNCTION G*F');
xlabel('Frequency (rad/sec)');
ylabel('Magnitude (db)');
subplot(212)
semilogx(w,GFpha);
grid;
xlabel('Frequency (rad/sec)');
ylabel('Phase (deg)');
meta actuator

diary off
APPENDIX D

MATLAB File for Aircraft Lateral Dynamics
% File name: jet3.m

!del jet3
!del jet3.met
diary jet3
clear
clg
axis
format

% Input the nominal system
% Input the plant in state space form
ag = [-.746 .006 -.999 .0369
     -12.9 -.746 .387 0
     4.31 .024 -.174 0
     0 1 0 0]  
bg = [.0012 6.05 -.416 0; .0092 .952 -1.76 0]';

cg = [0 1 0 0; 0 0 1 0]
dg = [0 0; 0 0]

[ag,bg,cg,dg] = ssselect(ag,bg,cg,dg,2,2);

format long e
damp(ag)

format

w = logspace(-2,4);
nikl_ol(ag,bg,cg,dg,1,w,260,[-60,40]);
grid
title('NICHOLS PLOT OF OPEN LOOP PLANT')
xlabel('Phase (deg)')
ylabel('Gain (db)')
pause
meta jet3
clg
axis

[magp1,phap1]=bode(ag,bg, cg, dg, 1, w);
magp1=20*log10(magp1);
semilogx(w,magp1)
title('BODE PLOT OF OPEN LOOP PLANT')
xlabel('Frequency (rad/sec)')
ylabel('Gain (db)')
grid
pause
meta jet3

% Input the weighting functions.
numw1 = 10*conv([1/50 1],[1/50 1]);
denw1 = conv([1/1 1],[1/1 1]);

numw3 = .03*conv([1/1 1],[1/1 1]);
denw3 = conv([1/30 1],[1/30 1]);

[magw1,phaw1]=bode(denw1,numw1,w);
[magw3,phaw3]=bode(denw3,numw3,w);

magw1=20*log10(magw1);
magw3=20*log10(magw3);

w1 = [numw1;denw1]
w2 = [0.01; 1]
w3 = [numw3;denw3]

semilogx(w,magw1,w,magw3)
grid
title('BODE PLOT OF WEIGHTING FUNCTIONS')
xlabel('Frequency (rad/sec)')
ylabel('Gain (db)')
gtext('Performance Spec (1/W1)')
gtext('Robustness Spec (1/W3)')
pause
meta jet3
clg

subplot(211)
semilogx(w,magp1,w,magw1,w,magw3)
grid
title('BODE PLOT OF PLANT AND WEIGHTING FUNCTIONS')
xlabel('Frequency (rad/sec)')
ylabel('Gain (db)')
gtext('Open Loop Plant')
gtext('Performance Spec (1/W1)')
gtext('Robustness Spec (1/W3)')
subplot(212)
semilogx(w,phap1,w,phaw1,w,phaw3)
grid
xlabel('Frequency (rad/sec)')
ylabel('Phase (deg)')
gtext('Open Loop Plant')
gtext('Performance Spec (1/W1)')
gtext('Robustness Spec (1/W3)')
pause
meta jet3
clg

% Plant augmentation for use in weighted mixed-sensitivity H_inf design.
[a,b1,b2,c1,c2,d11,d12,d21,d22] =
augtf(ag,bg, cg,dg,w1,w2,w3)
format long e
% H_inf optimal control synthesis via gamma-iteration.
[gamopt,af,bf,cf,df, acl,bcl,ccl,dcl] =
hinfpot(a,b1,b2,c1,c2,d11,d12,d21,d22,1)
disp(' ..........COMPUTING BODE PLOT OF THE COST FUNCTION......')
svtt = sigma(acl,bcl,ccl,dcl,1,w);
svtt = 20*log10(svtt);
semilogx(w,svtt);
grid
title('H-INFINITY DESIGN COST FUNCTION Tyu1.');
xlabel('Frequency (rad/sec)');
ylabel('SV (db)');
meta jet3
pause

disp('...COMPUTING BODE PLOTS OF SENS. & COMP. SENS FUNCTIONS...')
[a,b,c,d] = series(af,bf,cf,df,ag,bg,cg,dg);
[acs,bcs,ccs,dcs] = feedbk(a,b,c,d,2);
svs = sigma(a,b,c,d,3,w);
svs = -20*log10(svs);
svt = sigma(acs,bcs,ccs,dcs,1,w);
svt = 20*log10(svt);

semilogx(w,magw1,w,svs);
grid
title('H-INFINITY DESIGN SENSITIVITY FUNCTION & 1/W1');
xlabel('Frequency (rad/sec)');
ylabel('SV (db)');
gtext('1/W1');
gtext('Sensitivity Function')
meta jet3
pause

semilogx(w,magw3,w,svt);
title('H-INFINITY DESIGN COMP. SENSITIVITY FUNCTION & 1/W3');
xlabel('Frequency (rad/sec)');
ylabel('SV (db)');
grid
gtext('1/W3');
gtext('Complementary Sensitivity Function')
meta jet3
pause

% Find a stabilizing controller F(s).
[numf,denf] = ss2tf(af,bf,cf,df,1)
format

[contmag,contpha] = bode(af,bf,cf,df,1,w);
contmag = 20*log10(contmag);

[as,bs,cs,ds] = series(af,bf,cf,df,ag,bg,cg,dg)
[agf,bgf,cgf,dgf] = cloop(as,bs,cs,ds,1,1);
[GFnum,GFden] = ss2tf(agf,bgf,cgf,dgf,1);
[GFmag,GFpha] = bode(agf,bgf,cgf,dgf,1,w);
grid

title('NICHOLS PLOT OF LOOP TRANSFER FUNCTION G*F');
xlabel('Phase (deg)');
ylabel('Gain (db)');
pause
meta jet3
clg
axis

ytstep = step(acl,bc1,ccl,dcl,l,t);
pause
ytimp = impulse(acl,bc1,ccl,dcl,l,t);
pause

% Find the gain and phase margins
[gm,pm,wcp,wcg] = margin(GFmag,GFpha,w);

GFmag = 20*log10(GFmag);
Gainmargin = 20*log10(gm)
Phasemargin = pm

clg
plot(t,ytstep(:,1));
grid;
title('STEP RESPONSE OF TOTAL CLOSED LOOP');
gtext('Input=>Rudder Deflection  Output=>Yaw Rate')
xlabel('Time (sec)');
ylabel('Amplitude');
meta jet3
pause

clg

plot(t,ytimp(:,1));
grid;
title('IMPULSE RESPONSE OF TOTAL CLOSED LOOP');
gtext('Input=>Rudder Deflection  Output=>Yaw Rate')
xlabel('Time (sec)');
ylabel('Amplitude');
meta jet3
pause

clg

subplot(211)
semilogx(w,contmag);

120
grid;
title('BODE PLOT OF CONTROLLER F(s)');
xlabel('Frequency (rad/sec)');
ylabel('Gain (db)')
subplot(212)
semilogx(w,contpha);
grid;
xlabel('Frequency (rad/sec)');
ylabel('Phase (deg)')
meta jet3
pause

clf

subplot(211)
semilogx(w,GFmag);
grid;
title('BODE PLOT OF LOOP TRANSFER FUNCTION G*F');
xlabel('Frequency (rad/sec)');
ylabel('Magnitude (db)');
subplot(212)
semilogx(w,GFpha);
grid;
xlabel('Frequency (rad/sec)');
ylabel('Phase (deg)')
meta jet3

diary off
REFERENCES


