SHIP MOTION MODELS

by

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Ship Motion Models

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ABSTRACT

Ship motion is a dynamic system with six degrees of freedom. The principle movements studied are roll and heave. Two approaches help in forming a comprehensive view of ship motion. One approach examines actual ship motion data with numerical tools, primarily the tools in Chaos Data Analyzer: Professional Version. The most significant result of the analysis is finding the correlation time. The second approach is to construct a model of ship motion from basic naval-architecture principles. With a model of two coupled differential equations, one can learn about the important components of the data. Mathieu equations are similar in structure to the modeled equations, have been used to model ship motion, and help model a coupled non-autonomous four-dimensional system. This thesis helps form a basis for future projects.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed

Randall P. Tagg
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1. Introduction

Naval crane ships need to transport cargo from one ship to another ship without injury to people or property. Problems arise from different sea states or conditions causing oscillation in the crane’s cable, thus creating a giant pendulum. From the work of John Starrett, chaotic behavior can occur from pendulum movements. People have approached this problem through mechanical engineering solutions, which help most of the time, but do not solve the underlying problem. By understanding the motion of the ship and predicting its movements, one is more likely to transport cargo safely over a greater range of oceanic conditions.

One method for understanding ship motion is to study actual ship data to tease out the important characteristics that allow for predicting the ship’s movements. However, to obtain ship motion data some noise is introduced into the system, which impedes analysis. Noise can be somewhat stochastic and originate from human activity and instrumentation. Humans may move cargo, which changes the distribution of mass on the ship. Some noise may be intrinsic to the system such as waves, currents, and fish swimming by the ship. To grasp the fundamental characteristics of the system, a ship model needs to be created from fundamental principles of ship behavior and shape. With these mathematical models, noise is eliminated; thus, the basic principles of the system can be studied through numerical integration, and the creation of sample time series for analysis.

A number of researchers have done work on ship motion. Some experts chose to work strictly numerically, without experimentation. Other researchers chose to justify their models with ship model data.

Chen, Shaw, and Troesch (1999) created a system of equations that involve roll, sway, and heave (see Figure 2.1). They examined the nonlinear, large-amplitude motion of the ship in response to beam seas, where waves strike the vessel broadside. The authors found that heave was on a fast manifold, while roll and sway were on a slower manifold.

Ohtsu (1990) created an optimum autopilot system with a single-input/single-output, control-type autoregressive model. The autopilot system was fixed-gain and noise-adaptive. This system controlled the rudder and thus, roll. Ohtsu created ship components for a model ship and recorded actual ship motion data. He found that the yaw (see Figure 2.1) of a small rudder induced roll motion.
Sanchez and Nayfeh (1997) used numerical means to analyze ship motion. They examined parametric excitation (time varying), and external excitation (inhomogeneity). The authors identified instabilities that appeared when one of the excitations is slowly varied. They fixed the level of parametric excitation for a model boat. They studied the stability and bifurcation of an equation with some heave and roll coupling.

Falzarano, Esparza, and Taz Ul Mulk (1995) studied roll motion in isolation. They produced an analogy between pendulums and ship motion. They performed steady-state bifurcation analysis through their numerical studies. They observed the changes to the restoring moment of roll by changing the height of the center of gravity and damping. The authors altered damping by examining the presence and size of bilge keels. They found that bilge keels add nonlinear damping and are influential in resistance to capsizing.

Iseki (1990) focused on the cross spectrums of heave, pitch, and roll (see Figure 2.1). The basic idea is that the ship is a giant wave probe and by understanding the motion of the ship, one can estimate the directional wave spectra. The directional wave spectrum describes the wave energy in terms of frequency and direction. He created a model ship that was rigidly fixed to restrict surge motion and used springs to loosely restrict sway and yaw motion. The ship was excited by long crested irregular waves. Iseki found that the ship's pitch frequency was influenced by the waves, while roll frequency was nearly independent of the waves.

Allievi and Soudack (1990) modeled roll motion with damping using basic naval architectural principles. The authors used a Mathieu-like equation for their analysis as they calculated the stable and unstable regions based on their parameters. They examined undamped, linearly damped, and nonlinearly damped Mathieu systems and the phase portraits of those systems.

Despite these hard-working researchers, a conclusive understanding of ship motion has not been found. The thesis chapters are written as follows:

Chapter two examines the work of the Digital Sealegs Group. The Digital Sealegs Group was a group of students and faculty of the University of Colorado at Denver involved in the study of cranes and crane ships. This chapter focuses on the numerical tools used to study the rolling of an actual crane ship. The most significant result is the evidence of measurable correlation times for several windows of 1,024 seconds of data for each window.
Chapter three investigates the formation of a ship model for the coupled heave and roll of a moored ship. Three different ship geometries are examined with different parameters (e.g. "metacentic height"). An argument is made to restrict the model to a coupling of roll (rotation about the ship's axis from bow to stern) and heave (vertical translation), with forcing only in the heave direction.

Chapter four presents coupled Mathieu equations, describing the parametric forcing of two pendula whose pivots are moved sinusoidally up and down. These equations have been used to model theories and, when coupled, produced a non-autonomous four-dimensional system. This system helps in visualizing the behavior of such larger-dimensional systems for dynamics that, in the uncoupled case, is understood and rich in behavior (i.e., periodic motion, period doubling, and chaos).

Chapter five presents the conclusion. In addition, this chapter assesses the model as a tool for evaluating approaches to predicting real data. Further work is suggested, including adding noise to the model-generated data and constructing an experiment with a physical ship model.

Included in the appendix are MatLab programs, a glossary, and additional plots of data. Some of the MatLab programs used to generate the plots in chapters two, four, and five. Also, in the appendix are some additional graphs for the roll and heave of the other data files.
2. Analyzing Actual Ship Motion Data

The Digital Sealegs Group analyzed the ship motion data for the NOAA ship, Discoverer, and a naval crane ship. Preliminary investigations by Margo Martinez and John Slavich involved finding coupling between degrees of freedom spectral analysis. John Slavich analyzed the data from the Discoverer ship. Margo Martinez analyzed crane ship data provided through the Carderock Division of the Naval Surface Warfare Center. There were data from July 15, 17, 18, and 19, 1993. Most of the analysis was performed on the July 19 roll data, which was a long contiguous set of recorded data during sea state 3.

The ship data recorded from both ships had six degrees of freedom (Figure 2.1) (Gillmer & Johnson, 1982). There are three translational motions: surge, sway, and heave. There are three rotational motions: roll, pitch, and yaw. Of the six degrees of freedom, pitch, heave, and roll are predominately used in ship motion. This section focuses on roll motion.

The software, Chaos Data Analyzer: The Professional Version (CDA), was used to extract the Hurst exponent, the power spectrum, and the correlation function. A MatLab program calculated the mean, kurtosis, variance, and skewness for windows of 1,024 seconds (2,048 samples) throughout the entire data file. Two time scales are examined: a single window of approximately 17 minutes of data, and the entire day of approximately 19 hours of data. The purpose of these tests is to identify the characteristic features of the data, such as dominant frequency, and to investigate the data's degree of "stationarity," i.e., how the features and statistics behave over longer time scales.

![Figure 2.1 Degrees of Freedom](image-url)
2.1 Each Window

Each file or window was sampled for 17 minutes and 4 seconds. Each window had its time series, power spectrum, and correlation function analyzed as plots. The sampling rate was two samples per second, well within observed frequencies of motion.

2.1.1 Time Series

Events such as ship motion change over time creating a progression of data points known as a time series (Williams, 1997). The time series plots of July 19, 1993 involved some files that seemed too erratic to have structure (Figure 2.2). Other time series appeared organized and had a low frequency envelope (Figure 2.3). Under a smaller period, the signal looks smooth and thus there is not a lot of instrumental noise in the data.

![Figure 2.2 Time Series of Roll 38 of July 19](image-url)
2.1.2 Power Spectrum

Data may have some periodic components. Differing amplitudes and phases of sines and cosines form the periodic components of the data. Fourier analysis allows for the extraction of the different waves. The power spectrum involves taking a fast Fourier transform of the time series. The power is the mean square amplitude and is plotted against the frequency. A broad spectrum often suggests random and/or chaotic data. A spectrum of a few dominant peaks usually means periodic and quasi-periodic data. Fourier analysis examines superimposed simultaneous multiple waves with various heights (Williams, 1997). Within the signal is a standard or reference wave, which is often the longest wave available, or the length of the record. This wave is called the fundamental wave. The primary characteristics of the fundamental wave are its wavelength (fundamental wavelength) and its frequency (fundamental frequency). Fourier analysis uses waves whose frequencies are integer multiples of the fundamental frequency. All waves are based on the wavelength of the composite wave. Fourier analysis indicates which frequencies are in the signal and their relative importance. The equation for the Fourier analysis is:

$$\sum_{h=0}^{N/2} \left( \alpha_h \cos[h\theta] + \beta_h \sin[h\theta] \right)$$

(2.1)
The discrete Fourier coefficients are:

\[ \alpha_h = \frac{2}{N} \sum_{n=0}^{N-1} y_n \cos \frac{2\pi nh}{N} \]  \hspace{1cm} (2.2) \\

and

\[ \beta_h = \frac{2}{N} \sum_{n=0}^{N-1} y_n \sin \frac{2\pi nh}{N} \]  \hspace{1cm} (2.3) \\

The variables in these equations are:

- \( N \) = number of observations
- \( h \) = harmonic number (1 for first harmonic)
- \( t_n \) = time
- \( y_n \) = data value at time \( t_n \)

Least-squares estimates involve minimizing the average squared difference between the value of the composite wave and the sum of the components.

\[ s_h^2 = \left( \alpha_h^2 + \beta_h^2 \right)/2 \]  \hspace{1cm} (2.4) \\

CDA uses 128 frequency intervals. A larger number of intervals would improve resolution, but "exacerbates the spurious responses" (Sprott & Rowlands, 1995). Since CDA truncates the data to the largest power of two, the window chosen for the data was a power of two. CDA uses non-overlapping segments. The maximum frequency used by CDA is the Nyquist critical frequency, which is the reciprocal of twice the interval between data points.

For most of the CDA windows, a dominant peak was found (Figure 2.4 and 2.5). Random and chaotic data often have broad spectrum, because of all the contributions from different frequencies. Since in most windows in this system have a pronounced peak, this system is to some degree periodic.

\( N \) data points \((N = 2048)\) with sample time \((\Delta t = 0.5 \text{ seconds})\) has a maximum number \( \frac{N}{2} \) frequency intervals with frequency interval \( \Delta f = \frac{1}{N\Delta t} \). The maximum (Nyquist) frequency is:
\[ f_{nyq} = \frac{N}{2} \times \Delta f = \frac{1}{2 \Delta t} = \frac{1}{2 \times 0.5s} = 1 \text{ Hz} \] (2.5)

If \( m \) intervals are averaged to end up with 128 averaged frequency intervals, then:

\[ m = \left( \frac{N}{2} \right) \frac{1}{128} = \frac{1024}{128} = 8 \] (2.6)

Therefore, the frequency interval of the averaged spectrum is:

\[ \Delta f_{av} = m \Delta f = \frac{m}{N \Delta t} = \frac{8}{2048 \times 0.5s} = \frac{1}{12.8} \text{ Hz} \] (2.7)

The peaks appear to lie at approximately corresponding to a period of 12.8 seconds. This is probably the ship’s natural roll period. The bottom axis for each plot needs to be multiplied by 1/128.

Figure 2.4 Power Spectrum of Roll 38 of July 19
2.1.3 Correlation Function

In a time series, there may be repeated data points separated by some time lag (Williams, 1997). Autocorrelation shows to what extent two time segments with certain time lags differ from each other. For a zero value, the two time segments are not correlated or the sum of the products is close to zero. The correlation time states the time it takes until two segments match each other in value. The CDA correlation function was obtained by multiplying each $x(t)$ with $x(t-tau)$ and summing the result over all of the data points (equation 2.8) (Fenny & Moon, 1989). The correlation function is the sum plotted as a function of $\tau$ or $n$ (Sprott & Rowlands, 1995). CDA calculates the correlation time as the $tau$ when the correlation function first falls to $1/e$ in value. $N$ is the sample size minus one.

$$r(n) = \frac{1}{N} \sum_{k=1}^{N} u(k+n)u(k)$$  \hspace{1cm} (2.8)

$$n = 0,1,2,\ldots,N$$  \hspace{1cm} (2.9)

There were correlation functions that seemed reasonable (Figure 2.6) and other correlation functions that did not make sense (Figure 2.7). The reasonable functions appeared to have slower oscillating envelopes. Roll 47 has the erratic time
series (Figure 2.2) as well as the erratic correlation function (Figure 2.7). Roll 38 has a more structured time series (Figure 2.3) and also has a more structured correlation function (Figure 2.6).

![Figure 2.6 Correlation Function of Roll 38 of July 19](image1)

![Figure 2.7 Correlation Function of Roll 47 of July 19](image2)
A problem with how CDA calculates the correlation time is that the time is taken without consideration of dampened oscillating systems. The system may not return to the same value, but the system may return to close in value. This semi-memory in values is not considered in CDA's code and may be a more accurate correlation time for this system.

2.2 The Entire Day

For July 19, the data were collected for 19 hours, 20 minutes, and 32 seconds. The time series is shown in Figure 2.8.

![Figure 2.8 Time Series of Roll of July 19](image)

For mean, variance, skewness, and kurtosis, time is based on the middle point of the time for each window. After the statistics are discussed, dominant frequencies, correlation times, and Hurst exponents will be discussed.

2.2.1 Mean

The sampling rate and the total time of observation limit a time series (Williams, 1997). Increasing sampling rates allows for greater resolution; while averaging over nearby sampling rates helps decrease noise, and aids in illustrating
general trends. In addition, some help in finding the general trend in the data is the use of the arithmetic mean:

\[
\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i
\]  

(2.10)

The mean of the roll data was approximately one until the 25\textsuperscript{th} file at which point the mean became very erratic (Figure 2.9). This change in the mean can be observed through the time series (Figure 2.8) as the general trend is fairly constant for the first 20 kiloseconds and then loses some stationarity. Figure 2.9 shows the mean for each individual window.

![Figure 2.9 Mean of Roll of July 19](image)

2.2.2 Variance

It is useful to know how much the data deviates from the mean (Williams, 1997). Means and variances are tests of stationarity. The variance gives the magnitude of the average deviation from the mean:

\[
s^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2
\]  

(2.11)
The standard deviation is the square root of the variance. For a small total number of data points, N-1 should be used as a divisor so as not to underestimate the values. The variances of the roll windows are spaced fairly close together until the 20th window, then the variance of the windows are farther apart (Figure 2.10).

These changes in variance can be observed in the time series as the relative "thickness" of the data. Between 30 kiloseconds and 40 kiloseconds, the elevated data peaks in the time series that correlate elevated peaks on the variance plot (Figure 2.10). The variance for each window is calculated. The peaks in the plot correspond to places of large variation in the time series.

\[ \text{Figure 2.10 Variance of Roll of July 19} \]

2.2.3 Skewness

Skewness is defined by (Sprott & Rowlands, 1995):

\[
\frac{\sum (x_i - \bar{x})^3}{n \cdot s^3}
\]  

(2.12)
The skewness of the roll data deviated only slightly from zero (Figure 2.11). There are no apparent trends. Some windows are skewed in one direction and other windows are skewed in the other direction. Skewness describes the lack of symmetry about the mean (James & James, 1992). When comparing the skewness to the time series, it is not easy to verify the skewness plot since the data is so concentrated around the mean.

![Figure 2.11 Skewness of Roll of July 19](image)

### 2.2.4 Kurtosis

Kurtosis is defined by (Sprott & Rowlands, 1995):

\[
\frac{\sum (x_i - \bar{x})^4}{n \cdot s^4}
\]

(2.13)

Figure 2.12 shows the kurtosis for July 19 each point represents one window of data. Kurtosis describes the concentration about the mean (James & James, 1992). There are no apparent patterns in the plot. The 63rd window has the highest kurtosis value and is the most skewed in the positive direction.
2.2.5 Dominant Frequencies

In the dominant frequencies, correlation times, and the Hurst exponent, some smoothing was performed. The process involved a nine-point low pass filter was constructed by convolving the data with nine points of 0.1111 in value and interpolating the result. Smoothing was performed on the data in order to reduce the amount of noise in the signal. Noise will have more erratic peaks and valleys, which with smoothing will be decreased in steepness to allow the underlying structure of the data to be seen and analyzed.

The average dominant frequency for the raw roll data of July 19 was 0.10 ± 0.04 Hz (Figure 2.13). With smoothing, the average dominant frequency for the roll data of July 19 was 0.10 ± 0.04 Hz (Figure 2.14).
2.2.6 Correlation Time

The correlation time is the amount of time it takes for a time segment to repeat itself. The correlation time was $1.8 \pm 0.7$ seconds (Figure 2.15) and with smoothing it was $1.9 \pm 0.7$ seconds (Figure 2.16). One approach to correlation time was to take correlation functions from the CDA and construct an exponential fit to the sine wave.

Figure 2.13 Dominant Frequency of Raw Roll Data of July 19

Figure 2.14 Dominant Frequency of Smoothed Roll Data of July 19
with a declining amplitude. The reciprocal of the exponential power is the correlation time (Figure 2.17).

![Figure 2.15 Correlation Time of Raw Roll Data of July 19](image1)

![Figure 2.16 Correlation Time of Smoothed Roll Data of July 19](image2)
2.2.7 Hurst Exponent

The Hurst exponent represents how random or uncorrelated the data points are to each other. White or uncorrelated noise has a Hurst exponent of 0.5. The Hurst exponent is from "the slope of the root-mean-square displacement of various initial conditions" (i.e., each point) versus time (Sprott & Rowlands, 1995). For data, the line starts at the first data point and ends at the square root of the total duration of the data record. Hurst exponents greater than 0.5 show that trends will continue into the future. Hurst exponents less than 0.5 show that trends will reverse in the future. Hurst exponents show how values move away from the initial value using each point in time series as an initial condition.

The average Hurst exponent of raw roll data of July 19 was $0.32 \pm 0.07$ (Figure 2.18). With smoothing, the average Hurst exponent was $0.34 \pm 0.07$ (Figure 2.19). Smoothing hardly affected the Hurst exponent. The roll data had Hurst exponents less than 0.5, so the data is correlated and trends are to be reversed in the future.
2.3 Summary

Margo Martinez and John Slavich were able to obtain ship motion data and perform numerical tests on the data. Further tests (i.e., Hurst exponent, power spectrum, autocorrelation) were performed on a set of roll data of July 19. There was
no definitive answer to knowing the underlying ship dynamics, perhaps due to too much noise in the system. In the short term there is a dynamical system behavior "quasi-stationarity." For the entire day, there is non-stationary behavior, which can be explained by tidal changes.
3. Modeling Moored Ships

In order to gain insight into the dynamics represented by the July 19 roll data, a model was developed. By simplifying the system to a mathematical expression, one can obtain the fundamental attributes of the system without having to deal with noise. With the construction of a mathematical model, there is always a question of how complex to make the model. A complex model may be more realistic, but it often results in large amounts of computing time, more debugging, and may impede the conceptualizing of the basic components of the system. Therefore, a compromise must exist between a simple and complex model.

The ship moves with six degrees of freedom (see Figure 2.1). All of the literature describes models with roll, but there is some discrepancy as to what the next important degree of freedom is after roll. To load and unload cargo on a ship, the crane ship and the transport vessel must be parallel. Thus, the movement of the transverse cross section is most important. Therefore, heave is an important component of ship motion.

This chapter describes the basic knowledge needed to understand the movement of a rigid body and then states a coupled system of equations describing roll and heave. Three basic hull shapes will be examined. For these shapes, the parameters of the coupled equations that can be derived from ship geometry include mass, volume, center of gravity, and moment of inertia. Additional parameters require analysis of the ship's position and orientation relative to the sea surface: equilibrium waterline, center of buoyancy, and the metacenter.

3.1 Rigid Body Motion

The following equations form the basic mathematical theory behind rigid body motion, which is the fundamental basis for the coupled equations. Harrison and Nettleton (1997) and Ginsberg (1995) describe the dynamics of rigid body motion, which are demonstrated below.
Imagine a solid body divided into a set of small masses, \( m_i \). The total mass, \( m \), is then:

\[
m = \sum_i m_i
\]  

(3.1)

The center of mass, \( \vec{r}_{cm} \), is defined as:

\[
\vec{r}_{cm} = \frac{\sum_i m_i \vec{r}_i}{m}
\]  

(3.2)

Let \( \vec{r}_i' \) be the position of the \( i^{th} \) mass relative to the center of mass:

\[
\vec{r}_i' = \vec{r}_i - \vec{r}_{cm}
\]  

(3.3)

Note that:

\[
\sum_i m_i (\vec{r}_i - \vec{r}_{cm}) = 0
\]  

(3.4)

Substituting equation 3.3 into equation 3.4:

\[
\sum_i m_i \vec{r}_i' = 0
\]  

(3.5)

Let the force be distributed over the body with force, \( \vec{F}_i \), acting at each small mass. The total force is:
From Newton's 2nd Law:

\[ \vec{F} = \sum_i m_i \vec{r}_i \]  

(3.7)

\[ \vec{F} = \sum m_i (\vec{r}_{cm} + \vec{r}'_i) \]  

(3.8)

\[ = \sum m_i \vec{r}_{cm} + \sum m_i \vec{r}'_i \]  

(3.9)

\[ = m \vec{r}_{cm} + \sum m_i \vec{r}'_i \]  

(3.10)

If the distribution of masses is fixed in time:

\[ \sum m_i \vec{r}'_i = 0 \]  

(3.11)

Then since

\[ \sum m_i \vec{r}'_i = 0 \]  

(3.12)

The second term of equation 3.11 vanishes and the equation for the center of mass motion is:

\[ m \vec{r}_{cm} = \vec{F} \]  

(3.13)

Now examine the torques on the solid body about the center of mass. Let

\[ \vec{\tau}_i = \vec{r}'_i \times \vec{F}_i \]  

(3.14)

The total torque about the center of mass is:
\[ \vec{\tau} = \sum_i \vec{\tau}_i = \sum_i \vec{r}'_i \times \vec{F}'_i \]  \hspace{1cm} (3.15)

The second term vanishes because of the property that for \( \vec{a} \times \vec{a} = 0 \) any vector \( \vec{a} \) (again assuming mass distribution remains fixed).

\[ \begin{align*}
\vec{\tau} &= \sum_i \vec{r}'_i \times m_i (\vec{v} + \vec{r}'_i) \\
&= \left( \sum_i m_i \vec{r}'_i \right) \times \vec{v} + \sum_i m_i \vec{r}'_i \times \vec{r}'_i \\
&= \left( \sum_i m_i \vec{r}'_i \right) \times \vec{v} + \sum_i m_i \vec{r}'_i \times \vec{r}'_i \\
\end{align*} \]  \hspace{1cm} (3.16)

\[ \begin{align*}
\vec{\tau} &= \left( \sum_i m_i \vec{r}'_i \right) \times \vec{v} + \sum_i m_i \vec{r}'_i \times \vec{r}'_i \\
\end{align*} \]  \hspace{1cm} (3.17)

The first term vanishes (equation 3.5) and the second term can be rearranges to give:

\[ \vec{\tau} = \sum_i m_i \left[ \frac{d}{dt} \left( \vec{r}'_i \times \frac{d\vec{r}'_i}{dt} \right) - \left( \frac{d}{dt} \vec{r}'_i \times \frac{d}{dt} \vec{r}'_i \right) \right] \]  \hspace{1cm} (3.18)

\[ \vec{\tau} = \frac{d}{dt} \sum_i m_i \left( \vec{r}'_i \times \frac{d\vec{r}'_i}{dt} \right) \]  \hspace{1cm} (3.19)

Again assuming the mass distribution remains fixed.

For rigid body motion:

\[ \frac{d\vec{r}'_i}{dt} = -\vec{\omega} \times \vec{r}'_i \]  \hspace{1cm} (3.20)

for some angular velocity vector \( \vec{\omega} \).

Substituting 3.20 into equation 3.19:

\[ \vec{\tau} = \frac{d}{dt} \sum_i m \vec{r}'_i \times (\vec{\omega} \times \vec{r}'_i) \]  \hspace{1cm} (3.21)
\[
\frac{d}{dt} \sum m_i \left[ \omega (\vec{r}_i \cdot \vec{r}_i) - \vec{r}_i (\vec{r}_i \cdot \vec{w}) \right]
\]  

The result is:

\[
\vec{\tau} = \frac{d}{dt} (I \vec{\omega})
\]  

where the moment of inertia tensor is:

\[
I = \begin{pmatrix}
\sum m_i (y_i^2 + z_i^2) & -\sum m_i x_i y_i & -\sum m_i x_i z_i \\
-\sum m_i x_i y_i & \sum m_i (x_i^2 + z_i^2) & -\sum m_i y_i z_i \\
-\sum m_i x_i z_i & -\sum m_i y_i z_i & \sum m_i (x_i^2 + y_i^2)
\end{pmatrix}
\]  

and

\[
\vec{\omega} = \begin{pmatrix}
\omega_x \\
\omega_y \\
\omega_z
\end{pmatrix}
\]  

3.2 Simplified Class of Hull Shapes

A trapezoidal hull (Figure 3.2) is a good approximation to an actual ship. The top left picture in Figure 3.2 shows the side view. The top right in Figure 3.2 shows the transverse section of the ship. The bottom right picture shows the top view.

Through limits on the bottom of the ship, rectangular (Figure 3.3) and triangular hulls (Figure 3.4) can be created from the trapezoidal hull. The width of the deck for all the vessels is \(d\). The height of the vessel is \(h\). The length of the vessel is \(l\). The width of the bottom of the vessel is \(k\).
Figure 3.2 Views of the Trapezoidal Vessel

The rectangles indicate a simplification of the ship geometry that ignores the shape of the bow and stern. The equilibrium waterline depth $\omega_0$ is the value of the neutral waterline depth for $\theta = 0$ (zero roll). Later, all dimensions will be scaled by the width of the deck.

$$h' \equiv \frac{h}{d}$$ \hfill (3.26)

$$k' \equiv \frac{k}{d}$$ \hfill (3.27)

$$l' \equiv \frac{l}{d}$$ \hfill (3.28)

$$\omega_0' \equiv \frac{\omega_0}{d}$$ \hfill (3.29)
Figure 3.3 Rectangular Transverse Hull

For the rectangular hull (Figure 3.3),

\[
\frac{k}{d} = 1 \quad (3.30)
\]

or

\[
k' = 1 \quad (3.31)
\]

Figure 3.4 Triangular Transverse Hull

For the triangular hull (Figure 3.4),

\[
\frac{k}{d} = 0 \quad (3.32)
\]
or

\[ k' = 0 \] (3.33)

### 3.3 Parameters Derived from Ship Hull Shape and Mass Distribution

There are certain characteristics of the vessel that are independent of the ship's position relative to the sea, such as ship mass, volume, center of gravity, and moment of inertia. These factors will be described in order. To discuss these factors, certain variables and scaled values must be defined.

Some important variables are:

- \( \bar{c} \) = center of gravity
- \( \rho_0 \) = density of water
- \( V \) = volume
- \( M \) = mass
- \( I \) = moment of inertia about the x-axis

For a vessel symmetric about the midplane fore to aft, the center of gravity lies a (scalar) distance \( c \) above the keel. Some scaled values are:

\[ c' = \frac{c}{d} \] (3.34)

\[ I' = \frac{I}{Md^2} \] (3.35)

\[ V' = \frac{V}{d^3} \] (3.36)

#### 3.3.1 Ship Mass

Let the vessel have a total mass, \( \hat{M} \):

\[ \hat{M} = M_1 + M_* \] (3.37)
$M_i$ is the mass of the interior contents of the ship, which will be assumed to be uniformly distributed with density $\rho_i$. $M_e$ is the mass of the exterior surface of the ship and is described:

$$M_e = M_d + M_k + 2M_h$$

(3.38)

$M_d$ is the mass of the deck, $M_k$ is the mass of the keel, and $M_h$ is the mass of each side of the hull. Let $\rho_0$ be the density of the water in which the vessel floats. Masses will be scaled by the mass of water whose volume equals that of the total volume of the vessel.

Thus, the scaled mass is:

$$M' = \frac{M}{\rho_0 V}$$

(3.39)

The scaled interior mass is:

$$M'_i = \frac{M_i}{\rho_0 V} = \frac{\rho_i V}{\rho_0 V} = \frac{\rho_i}{\rho_o}$$

(3.40)

The scaled exterior mass is:

$$M'_e = \frac{M_e}{\rho_0 V}$$

(3.41)

In order for a vessel to float, $M' < 1$. A neutrally buoyant vessel (e.g., a submarine stationary at constant depth) has $M' = 1$. If $M' > 1$, the vessel sinks.

3.3.2 Volume

The volume is the cross-sectional area of the ship multiplied by the length of the ship. The trapezoidal hull transverse area can be viewed in Figure 3.5.
Figure 3.5 Volume of Trapezoidal Hull

The volume of the trapezoidal vessel is:

\[ V = (A_1 + A_2 + A_3) \cdot l \]  

\[ = \left( k h + \frac{1}{2} \frac{d-k}{2} h + \frac{1}{2} \frac{d-k}{2} h \right) l \]  

\[ = \left( k + \frac{d-k}{2} \right) hl \]  

\[ = \frac{d+k}{2} hl \]  

The resulting equation for the trapezoidal hull volume is:

\[ V = \frac{1}{2} lhd \left( 1 + \frac{k}{d} \right) \]  

(3.46)

By substituting equation 3.46 into equation 3.36, the scaled volume of a trapezoidal hull is:

\[ = \frac{1}{2} \left( 1 + \frac{k}{d} \right) h \frac{l}{d} \]  

(3.47)
\[ \frac{1}{2} (1 + k') h' l' \]  \hspace{1cm} (3.48)

By letting \( \frac{k}{d} = 0 \), the volume of the triangular hull is:

\[ V_{tri} = \frac{1}{2} ldh \]  \hspace{1cm} (3.49)

or, in scaled form:

\[ V'_{tri} = \frac{1}{2} l'h' \]  \hspace{1cm} (3.50)

By letting \( \frac{k}{d} = 1 \), the rectangular hull's volume is:

\[ V_{rec} = ldh \]  \hspace{1cm} (3.51)

or, in scaled form:

\[ V'_{rec} = h'l' \]  \hspace{1cm} (3.52)

Table 3.1 summarizes the results of the volume equations.

<table>
<thead>
<tr>
<th>Hull Shape</th>
<th>( V )</th>
<th>( V' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trapezoidal</td>
<td>( \frac{1}{2} \left( 1 + \frac{k}{d} \right) dhl )</td>
<td>( \frac{1}{2} \left( 1 + k' \right) h'l' )</td>
</tr>
<tr>
<td>Rectangular</td>
<td>( dhl )</td>
<td>( h'l' )</td>
</tr>
<tr>
<td>Triangular</td>
<td>( \frac{1}{2} dhl )</td>
<td>( \frac{1}{2} h'l' )</td>
</tr>
</tbody>
</table>
3.3.3 Center of Gravity

With an understanding of the volumes, the centers of gravity of these vessels can be calculated. The centers of gravity will remain constant for the vessels as long as their distributions of mass remain the same (i.e., no shifting of their masses).

In this section, there is a progression of calculations for different shapes. First, the center of gravity for a right triangle is calculated. Then the center of gravity of a more complex triangle is calculated. Finally, the center of gravity for the different hull shapes is calculated for distributed and non-distributed masses.

To calculate the center of gravity, integrate over the volume of the vessel:

\[
\iiint \rho_0 g \begin{pmatrix} x \\ y \\ z \end{pmatrix} \, dx \, dy \, dz \\
\frac{1}{\rho_0 g V}
\]

(3.53)

To analyze shapes, it is easier to have the centroid of a right triangle (Figure 3.6).

\[z = p \left(1 - \frac{y}{p}\right)\]

Figure 3.6 Center of Gravity for a Right Triangle

The center of gravity, \(\bar{c}\), that has a uniform mass per unit area, \(\sigma\), for a right triangle is:

\[
\begin{pmatrix} y_g \\ z_g \end{pmatrix} = \frac{1}{\iint_{\text{triangle}} \sigma \, dy \, dz} \begin{pmatrix} \iint_{\text{triangle}} y \, dy \, dz \\ \iint_{\text{triangle}} z \, dy \, dz \end{pmatrix}
\]

(3.54)
The mass per unit area divides out, because it is assumed constant.

\[
\frac{1}{\int_0^p \int_0^q (1 - \frac{y}{p}) \ dy \ dz} = \frac{1}{\int_0^p \int_0^q (1 - \frac{y}{p}) \ dy \ dz}
\]

Integrating equation 3.55:

\[
\begin{bmatrix}
\frac{y_g}{z_g}
\end{bmatrix} = \frac{1}{2 \ pq} \left[ \int_0^p \left( \frac{y_1}{p} \right) dy \right] (3.56)
\]

\[
\begin{bmatrix}
\frac{y_g}{z_g}
\end{bmatrix} = \frac{2}{pq} \left[ \int_0^{\frac{p}{2}} \left( \frac{y - \frac{y^2}{p}}{2} \right) dy \right] (3.57)
\]

Integrating equation 3.57:

\[
\begin{bmatrix}
\frac{y_g}{z_g}
\end{bmatrix} = \frac{2}{pq} \left[ \frac{y^2}{p} - \frac{y^3}{3p} \right] (3.58)
\]

\[
\begin{bmatrix}
\frac{y_g}{z_g}
\end{bmatrix} = \frac{2}{pq} \left( \frac{1}{6} \frac{pp^2}{q^2} \right) (3.59)
\]

The resulting center of gravity for a right triangle is:
This result will be useful below and in section 3.3.3.1 for the center of gravity of the trapezoidal hull with distributed mass.

Many triangles involve the side of the trapezoid and require special attention to find the center of gravity (Figure 3.7).

From trigonometry:

\[ q = s \sin(\theta) \quad (3.61) \]

From the Pythagorean Theorem:

\[ (q \tan(\phi) + p)^2 + q^2 = s^2 \quad (3.62) \]

Substituting equation 3.61 into equation 3.62:

\[ q^2 \tan^2(\phi) + 2q \cdot p \tan(\phi) + p^2 + q^2 = \frac{q^2}{\sin^2(\theta)} \quad (3.63) \]

\[ q^2 \left(1 - \frac{1}{\sin^2(\theta)} + \tan^2(\phi)\right) + 2q \cdot p \tan(\theta) + p^2 = 0 \quad (3.64) \]
As a side note:

\[
1 - \frac{1}{\sin^2(\theta)} = \frac{\sin^2(\theta) - 1}{\sin^2(\theta)}
\]  
(3.65)

\[
- \frac{\cos^2(\theta)}{\sin^2(\theta)}
\]  
(3.66)

\[
- \frac{1}{\tan^2(\theta)}
\]  
(3.67)

Substituting the result of equation 3.67 into equation 3.64:

\[
q^2 \left( \tan^2(\phi) - \frac{1}{\tan^2(\theta)} \right) + 2p \cdot q \tan(\phi) + p^2 = 0
\]  
(3.68)

Using the quadratic formula:

\[
q = \frac{-2p \tan(\phi) \pm \sqrt{4p^2 \tan^2(\phi) - 4 \left( \tan^2(\phi) - \frac{1}{\tan^2(\theta)} \right) p^2}}{2 \left( \tan^2(\phi) - \frac{1}{\tan^2(\theta)} \right)}
\]  
(3.69)

\[
= \frac{-p \tan(\phi) \pm \sqrt{p^2 \tan^2(\phi) - p^2 \tan^2(\phi) + p^2 \frac{1}{\tan^2(\theta)}}}{\tan^2(\phi) - \frac{1}{\tan^2(\theta)}}
\]  
(3.70)

\[
= \frac{-p \tan(\phi) \pm p \frac{1}{\tan(\theta)}}{\tan^2(\phi) - \frac{1}{\tan^2(\theta)}}
\]  
(3.71)
\[ q = \begin{cases} 
\frac{-\tan(\phi) + \frac{1}{\tan(\theta)}}{(\tan(\phi) + \frac{1}{\tan(\theta)})(\tan(\phi) - \frac{1}{\tan(\theta)})}^P \\
\frac{-\tan(\phi) - \frac{1}{\tan(\theta)}}{(\tan(\phi) + \frac{1}{\tan(\theta)})(\tan(\phi) - \frac{1}{\tan(\theta)})}^P 
\end{cases} \quad (3.72) 
\]

\[ q = \begin{cases} 
-\frac{1}{\tan(\phi) + \frac{1}{\tan(\theta)}}^P \\
-\frac{1}{\tan(\phi) - \frac{1}{\tan(\theta)}}^P 
\end{cases} \quad (3.73) 
\]

Exclude the top root to get positive \( q \) when \( \phi \to 0 \).

\[ q = \frac{\tan(\theta)}{1 - \tan(\theta)\tan(\phi)}^P \quad (3.74) \]

The area \( A_1 \) of the triangle \( T_1 \) is the area \((A_1 + A_2)\) of the larger triangle \((T_1 + T_2)\) minus the area \( A_2 \) of the smaller triangle.

\[ A_1 = \frac{1}{2} q(q\tan(\phi) + p) - \frac{1}{2} q^2 \tan(\phi) \quad (3.75) \]

\[ = \frac{1}{2} qp \quad (3.76) \]

Substituting equation 3.76 into equation 3.74:

\[ A_1 = \frac{p^2}{2} \cdot \frac{\tan(\theta)}{1 - \tan(\theta)\tan(\phi)} \quad (3.77) \]
Let \( \bar{c}_1 \) be the center of gravity of triangle \( T_1 \) with respect to origin \( \mathcal{O}' \). Let \( \mathcal{O}^* \mathcal{O}' \) be the vector from origin \( \mathcal{O}^* \) to \( \mathcal{O}' \) origin. Let \( \bar{c}_2 \) be the center of gravity of triangle \( T_2 \) with respect to origin \( \mathcal{O}^* \). Let \( \bar{c}_{12}^* \) be the center of gravity of the \( T_1 + T_2 \) combined triangle with respect to origin \( \mathcal{O}^* \). Then:

\[
A_1 (\bar{c}_1 + \mathcal{O}^* \mathcal{O}') + A_2 \bar{c}_2 = (A_1 + A_2) \bar{c}_{12}^* \tag{3.78}
\]

\[
\bar{c}_1^* = \mathcal{O}^* \mathcal{O}' + \frac{(A_1 + A_2) \bar{c}_{12}^* - A_2 \bar{c}_2^*}{A_1} \tag{3.79}
\]

\[
= \mathcal{O}^* \mathcal{O}' \frac{A_1 + A_2 \bar{c}_{12}^* - A_2 \bar{c}_2^*}{A_1} \tag{3.80}
\]

The vector between origins is:

\[
\mathcal{O}^* \mathcal{O}' = \begin{pmatrix} q \tan(\phi) \\ 0 \end{pmatrix} \tag{3.81}
\]

Using the result 3.60 for right triangles, the center of gravity for the combined triangle \( T_1 + T_2 \) relative to \( \mathcal{O}^* \) is:

\[
\bar{c}_{12}^* = \frac{1}{3} \begin{pmatrix} q \tan(\phi) + p \\ \frac{1}{3} \end{pmatrix} \tag{3.82}
\]

The center of gravity for the triangle \( T_2 \) relative to \( \mathcal{O}^* \) is:

\[
\bar{c}_2^* = \frac{1}{3} \begin{pmatrix} q \tan(\phi) \\ \frac{1}{3} \end{pmatrix} \tag{3.83}
\]

Substituting equations 3.76, 3.81, 3.82, and 3.83 into equation 3.80:
\[-\frac{1}{2} qp \left( q \tan(\phi) + \frac{1}{2} qp + \frac{1}{2} q^2 \tan(\phi) \right) \left( \frac{1}{3} q \tan(\phi) + p \right) - \frac{1}{2} q^2 \tan(\phi) \left( \frac{1}{3} q \tan(\phi) \right) \]

\[ \overline{c}_1' = \frac{1}{2} qp \]  

\[
\overline{c}_1' = \left( -q^2 p \tan(\phi) + \left( qp + q^2 \tan(\phi) \right) \frac{1}{3} \left( q \tan(\phi) + p \right) - \frac{1}{3} q^3 \tan^2(\phi) \right) \left( \frac{1}{3} q - \frac{1}{3} q^3 \tan(\phi) \right) \left( \frac{1}{qp} \right) \]  

\[
\overline{c}_1' = \left( -q^2 p \tan(\phi) + \frac{1}{3} q^2 p \tan(\phi) + \frac{1}{3} qp^2 + \frac{1}{3} q^2 p \tan(\phi) \right) \left( \frac{1}{3} q^2 p \right) \left( \frac{1}{qp} \right) \]  

\[
\overline{c}_1' = \left( -\frac{1}{3} q^2 p \tan(\phi) + \frac{1}{3} qp^2 \right) \left( \frac{1}{3} q^2 p \right) \left( \frac{1}{qp} \right) \]  

\[
\overline{c}_1' = \left( p - q \tan(\phi) \right) \left( \frac{1}{3} q \right) \left( \frac{1}{q} \right) \]  

Substituting equation 3.74 into equation 3.88:

\[
\overline{c}_1' = \left( p - \frac{\tan(\theta)}{1 - \tan(\phi) \tan(\theta)} \tan(\phi) \right) p \left( \frac{1}{3} \right) \left( \frac{1}{1 - \tan(\phi) \tan(\theta)} \right) \]  

\[
\overline{c}_1' = \left( 1 - \tan(\phi) \tan(\theta) - \tan(\theta) \tan(\phi) \right) p \left( \frac{1}{3} \right) \left( \frac{1}{1 - \tan(\phi) \tan(\theta)} \right) \]  

38
\[
\vec{c}'_1 = \left(1 - 2 \tan(\phi)\tan(\theta)\right) \frac{p}{\tan(\theta)} \frac{1}{3 - \tan(\phi)\tan(\theta)}
\] (3.91)

This result will be useful in finding the center of buoyancy in section 3.4.2.

3.3.3.1 Center of Gravity with Distributed Mass

Figure 3.8 provides the parameters used to calculate the center of gravity for a uniformly massed vessel.

\[h = \frac{d + k}{2}
\] (3.92)

\[= kh + \frac{h}{2} \left(\frac{1}{2} d - k\right) + \frac{h}{2} \left(\frac{1}{2} d - k\right)
\] (3.93)

\[= \frac{h}{2} (d + k)
\] (3.94)

\[\bar{c} = \frac{1}{A} \left( A_1 \bar{c}_1 + A_2 \bar{c}_2 + A_3 \bar{c}_3 \right)
\] (3.95)
\[
\bar{c} = \frac{1}{2(d+k)h} \left[ k \left( \frac{0}{h} \right) + \frac{1}{2} (d-k)h \left( \frac{k}{2} + \frac{1}{3} \left( \frac{1}{2} \right) (d-k)h \right) \right] + \frac{1}{2} (d-k)h \left( \frac{k}{2} + \frac{1}{3} \left( \frac{1}{2} \right) (d-k)h \right) \] (3.96)

\[
= \frac{1}{h(d+k)} \left[ k \left( \frac{0}{h} \right) + (d-k)h \left( \frac{0}{h} \right) \right] \] (3.97)

\[
= \frac{h}{d+k} \left( \frac{0}{d+k} \right) + \frac{2}{3} (d-k) \] (3.98)

where the result from equation 3.60 has been used for finding \( \bar{c}_2 \) and \( \bar{c}_3 \).

The height \( c \) of the center of gravity above the keel in the trapezoid case is:

\[
c = \frac{1}{3} \frac{h}{d+k} (2d+k) \] (3.99)

\[
= \frac{h}{3} \left( \frac{2d+k}{d+k} \right) \] (3.100)

\[
= \frac{h}{3} \left( \frac{2+k}{1+k} \right) \] (3.101)

In scaled form:

\[
c' = \frac{h'}{3} \left( \frac{2+k'}{1+k'} \right) \] (3.102)

For the triangle, use \( \frac{k}{d} = 0 \):
\[ c_{\text{tri}} = \frac{2}{3} h \]  
\[ (3.103) \]

or, in scaled form:

\[ c'_{\text{tri}} = \frac{2}{3} h' \]  
\[ (3.104) \]

For the rectangle, \( \frac{k}{d} = 1 \):

\[ c_{\text{rec}} = \frac{1}{2} h \]  
\[ (3.105) \]

or, in scaled form:

\[ c'_{\text{rec}} = \frac{1}{2} h' \]  
\[ (3.106) \]

Table 3.2 summarizes the center of gravity as a length measured from the keel. Both scaled and unscaled values are presented in the table.

**Table 3.2 Centers of Gravity (Distribute Mass) for Different Transverse Hull Shapes**

<table>
<thead>
<tr>
<th>Hull Shape</th>
<th>( c )</th>
<th>( c' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle</td>
<td>( \frac{1}{2}h )</td>
<td>( \frac{1}{2}h' )</td>
</tr>
<tr>
<td>Trapezoid</td>
<td>( \frac{2 + k}{d} ) ( \frac{h}{3} ) (1 + ( \frac{k}{d} ))</td>
<td>( \frac{2 + k'}{1 + k'} ) ( \frac{h'}{3} )</td>
</tr>
<tr>
<td>Triangle</td>
<td>( \frac{2}{3}h )</td>
<td>( \frac{2}{3}h' )</td>
</tr>
</tbody>
</table>
3.3.3.2 Center of Gravity with Non-Distributed Mass

Some important variables in this section are:

- $m =$ total mass of vessel
- $m_d =$ mass of deck
- $m_k =$ mass of the bottom
- $m_l =$ mass of left hull
- $m_r =$ mass of right hull
- $c =$ center of gravity of the vessel
- $c_d =$ center of gravity of deck
- $c_k =$ center of gravity of bottom
- $c_l =$ center of gravity of left side of hull
- $c_r =$ center of gravity of right side of hull
- $\mu_d =$ mass per unit length of deck
- $\mu_h =$ mass per unit length of sides
- $\mu_k =$ mass per unit length of bottom

\[
\begin{align*}
    m_l &= \mu_h \left[ h^2 + \frac{1}{2} (d-k)^2 \right] \\
    \tilde{c}_l &= \left( \frac{k}{2} + \frac{1}{2} \left( \frac{1}{2} (d-k) \right) \frac{h}{2} \right) \\
    \tilde{c}_k &= (0,0) \\
    \tilde{c}_r &= \left( \frac{k}{2} + \frac{1}{2} \left( \frac{1}{2} (d-k) \right) \frac{h}{2} \right) \\
    m_k &= \mu_k k \\
    m_d &= \mu_d d
\end{align*}
\]

**Figure 3.9 Center of Gravity of Trapezoidal Hull of Nonuniformly Distributed Mass**

\[
\tilde{c} = \frac{m_d \tilde{c}_d + m_k \tilde{c}_k + m_l \tilde{c}_l + m_r \tilde{c}_r}{m} \quad (3.107)
\]

42
\[
\tilde{c} = \frac{1}{\mu_k k + \mu_d d + 2\mu_h \sqrt{h^2 + \frac{1}{4}(d-k)^2}} \left[ \mu_k \left(\frac{0}{0}\right) + \mu_d \left(\frac{0}{h}\right) + \mu_h \sqrt{h^2 + \frac{1}{4}(d-k)^2} \left( \frac{k}{2} \right) \frac{1}{4} \left( \frac{d-k}{h} \right) \mu_h \sqrt{h^2 + \frac{1}{4}(d-k)^2} \left( \frac{k}{2} \right) \frac{1}{4} \left( \frac{d-k}{h} \right) \right]
\]

(3.108)

\[
\tilde{c} = \frac{1}{\mu_k k + \mu_d d + 2\mu_h \sqrt{h^2 + \frac{1}{4}(d-k)^2}} \left( \mu_d d + \mu_h \sqrt{h^2 + \frac{1}{4}(d-k)^2} \right)
\]

(3.109)

\[
\tilde{c} = \frac{h}{\mu_d + \mu_h \sqrt{\left( \frac{h}{d} \right)^2 + \frac{1}{4} \left( 1 - \frac{k}{d} \right)^2}}
\]

(3.110)

For the rectangle, let \( \frac{k}{d} = 1 \):

\[
\tilde{c}_{rec} = h \left( \frac{0}{\mu_d + \mu_h \sqrt{\left( \frac{h}{d} \right)^2}}} + 0 \right)
\]

(3.111)

\[
\tilde{c}_{rec} = h \left( \frac{0}{\mu_d + \mu_k + 2\mu_h \left( \frac{h}{d} \right)}} \right)
\]

(3.112)
For the triangle, let \( \frac{k}{d} = 0 \):

\[
\dot{c}_m = h \left( \frac{0}{\mu_d + \mu_h \sqrt{\left( \frac{h}{d} \right)^2 + \frac{1}{4}}} \right)
\]

\[
(\mu_d + \mu_h \sqrt{\left( \frac{h}{d} \right)^2 + \frac{1}{4}})
\]

(3.113)

3.3.4 Moment of Inertia

The moment of inertia of a uniform distribution of mass for all three hull shapes is given below. The moment usually restores the ship to an upright position (Gillmer & Johnson, 1982). Figure 3.10 presents the important parameters of the moment of inertia of a trapezoidal hull.

![Figure 3.10 Moment of Inertia of Trapezoidal Hull](image)

The ship is assumed to be symmetric part to starboard (a good assumption) and fore and aft about its midplane (ignores change in cross-section at bow and stern). With this degree of symmetry of this ship model, roll motions only deal with the upper left element of the matrix in equation 3.24 (i.e., \( \sum m_i (y_i^2 + z_i^2) \)).
Integrating equation 3.114:

\[
I = \rho l \int_0^h \left[ \frac{k d - k z}{2} \right]^3 \left( y^2 + z^2 \right) dy
\]  

(3.114)

Using a mathematical software program such as MathCad results in:

\[
I = \frac{1}{48} \rho l h \left( k^3 + d^3 + k^2 d + k d^2 + 12 h^2 d + 4 k h^2 \right)
\]  

(3.118)

\[
= \frac{1}{24} M \left( \frac{k^3 + d^3 + k^2 d + k d^2 + 12 h^2 d + 4 k h^2}{k + d} \right)
\]  

(3.119)

where

\[
M = \rho l h \frac{d + k}{2}
\]  

(3.120)

The result for the moment of inertia for the trapezoid is:

\[
I = \frac{1}{24} M \left[ d^2 + k^2 + 4 \left( 1 + 2 \frac{d}{k + d} \right) h^2 \right]
\]  

(3.121)
\[ I = \frac{1}{24} M \left[ 1 + \left( \frac{k}{d} \right)^2 + 4 \left( 1 + \frac{2}{1 + \frac{k}{d}} \right) \left( \frac{h}{d} \right)^2 \right] \]  

\[ (3.122) \]

The scaled moment of inertia is:

\[ I' = \frac{1}{24} \left[ 1 + k' \right] + 4 \left( 1 + 2 \frac{d}{k + d} \right) h'^2 \]  

\[ (3.123) \]

For the triangle, let \( \frac{k}{d} = 0 \):

\[ I_{\text{tri}} = \frac{1}{24} Md^2 \left[ 1 + 12 \left( \frac{h}{d} \right)^2 \right] \]  

\[ (3.124) \]

For the scaled triangle:

\[ I'_{\text{tri}} = \frac{1}{24} \left[ 1 + 12h'^2 \right] \]  

\[ (3.125) \]

For the rectangle, let \( \frac{k}{d} = 1 \):

\[ I_{\text{rec}} = \frac{1}{24} Md^2 \left[ 2 + 8 \left( \frac{h}{d} \right)^2 \right] \]  

\[ (3.126) \]

For the scaled rectangle:

\[ I'_{\text{rec}} = \frac{1}{24} \left[ 2 + 8h'^2 \right] \]  

\[ (3.127) \]

Table 3.5 summarizes the moments of inertia for uniformly massed hull shapes.
Table 3.5 Moments of Inertia of Uniformly Massed Hull Shapes

<table>
<thead>
<tr>
<th>Hull Shape</th>
<th>Moment of Inertia</th>
<th>Scaled Moment of Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>( \frac{1}{24} M d^2 \left[ 2 + 8 \left( \frac{h}{d} \right)^2 \right] )</td>
<td>( \frac{1}{24} \left[ 2 + 8 h'^2 \right] )</td>
</tr>
<tr>
<td>Triangular</td>
<td>( \frac{1}{24} M d^2 \left[ 1 + 12 \left( \frac{h}{d} \right)^2 \right] )</td>
<td>( \frac{1}{24} \left[ 1 + 12 h'^2 \right] )</td>
</tr>
<tr>
<td>Trapezoidal</td>
<td>( \frac{1}{24} M \left[ 1 + \left( \frac{k}{d} \right)^2 + \left( 1 + \frac{2}{k} \left( \frac{h}{d} \right)^2 \right) \right] )</td>
<td>( \frac{1}{24} \left[ 1 + k'^2 + 4 \left( 1 + 2 \frac{d}{k + d} \right) h'^2 \right] )</td>
</tr>
</tbody>
</table>

3.4 Parameters Derived from Ship Position Relative to Sea Surface

In this section, equilibrium waterline and center of buoyancy are described.

3.4.1 Equilibrium Waterline Depth

Figure 3.11 shows how the equilibrium water line can be determined.

![Figure 3.11 Equilibrium Waterline Depth](image-url)
The volume of the water is needed. Using the result from equation 3.46, replacing \( h \) with \( \omega_0 \) and replacing \( d \) with \( k + \frac{\omega_0}{h}(d - k) \):

\[
V_0 = \frac{\left[k + \frac{\omega_0}{h}(d - k)\right] + k}{2} \omega_0 l
\]  
(3.128)

Archimedes’ principle requires that the mass of displaced water equal the mass of vessel.

\[
\rho_0 V_0 = M
\]  
(3.129)

Substituting equation 3.128 into equation 3.129:

\[
\rho_0 l \omega_0 \left(k + \frac{\omega_0}{h} \frac{d - k}{2}\right) = M
\]  
(3.130)

Dividing both sides of the equation by \( l \rho_0 \):

\[
\omega_0 \left(k + \frac{\omega_0}{h} \frac{d - k}{2}\right) = \frac{M}{l \rho_0}
\]  
(3.131)

\[
\frac{d - k}{2h} \omega_0^2 + k \omega_0 - \frac{M}{l \rho_0} = 0
\]  
(3.132)

Dividing each term of equation 3.132 by \( (d - k) \):

\[
\omega_0^2 + \frac{2hk}{d - k} \omega_0 - \frac{2Mh}{\rho_0 l (d - k)} = 0
\]  
(3.133)

Using the quadratic formula:

\[
\omega_0 = \frac{-2hk + \sqrt{4\left(\frac{hk}{d - k}\right)^2 + 4\frac{2Mh}{\rho_0 l (d - k)}}}{2}
\]  
(3.134)
Factoring \((d - k)\) from the terms of equation 3.135, gives the final result for the trapezoidal equilibrium water level:

\[
\omega_0 = -\frac{hk}{d-k} + \sqrt{\left(\frac{hk}{d-k}\right)^2 + \frac{2Mh}{\rho_0 l(d-k)}}
\]  

(3.136)

Dividing both sides by \(d\) to scale the equations:

\[
\frac{\omega_0}{d} = -\frac{hk}{d-k} + \sqrt{\left(\frac{hk}{d-k}\right)^2 + \frac{2Mh}{\rho_0 l(d-k)}}
\]  

(3.137)

Scaling the denominator:

\[
\omega_0 = -\frac{hk}{d} + \sqrt{\left(\frac{hk}{d}\right)^2 + \frac{2Mhd}{l\rho_0 d \left(1 - \frac{k}{d}\right)}}
\]  

(3.138)

Scaling the other variables in the equation:

\[
\omega'_0 = -\frac{h k}{d d} + \sqrt{\left(\frac{h k}{d d}\right)^2 + \frac{2M h}{\rho_0 l d^3} \left(1 - \frac{k}{d}\right)}
\]  

(3.139)

\[
-\frac{h'k'}{1-k'} + \sqrt{\left(\frac{h'k'}{1-k'}\right)^2 + \frac{M h'}{\rho_0 d^3 l' \left(1-k'\right)}}
\]  

(3.140)
As a side note:

\[ 2 \frac{M}{\rho_0 d^3} = 2 \frac{MV \rho_0}{\rho_0 d^3} \]  
\[ = \frac{M' \rho_0}{\rho_0 d^3} \frac{1}{2} (d + k) h l \]  
\[ = 2 \frac{d^3 \rho_0}{d^3 \rho_0} \]  
\[ = M' (1 + k') h' l' \]  
(3.141)

Substituting equation 3.143 into equation 3.140:

\[ \omega_0' = -h'k' + \frac{\sqrt{(h'k')^2 + M'(1 + k')h'^2(1 - k')}}{1 - k'} \]  
(3.144)

For the scaled triangular hull, let \( k' = 0 \):

\[ \omega_0' = h' \sqrt{M} \]  
(3.145)

The unscaled expression for the triangular hull is obtained by letting \( k = 0 \) in equation 3.138:

\[ \omega_0 = \frac{2kh}{\sqrt{\rho_0 ld}} \]  
(3.146)

For the rectangular hull, let \( k = d \) in equation 3.133 and solve for \( \omega_0 \):

\[ \omega_0 = \frac{M}{\rho_0 ld} \]  
(3.147)

For the scaled rectangle, use \( M = M' \rho_0 ldh \) in equation 3.147:

\[ \omega_0' = \frac{M' \rho_0 ldh}{\rho_0 ld} \]  
(3.148)
Dividing through by $d$:
\[
\omega_0' = M' h' \tag{3.149}
\]

Table 3.4 summarizes the equilibrium waterline depth for a variety of hull shapes.

<table>
<thead>
<tr>
<th>Hull Shape</th>
<th>$\omega_0$</th>
<th>$\omega_0'$</th>
<th>$h' M'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle</td>
<td>$h \frac{M}{\rho_0 V}$</td>
<td>$h' M'$</td>
<td></td>
</tr>
<tr>
<td>Trapezoid</td>
<td>$h \left[ k \frac{k}{d} + \frac{M}{\rho_0 V} \left( 1 - \frac{k}{d}^2 \right) \right] - k' \sqrt{\frac{k'}{d} + M' \left( 1 - \frac{k'}{d}^2 \right)}$</td>
<td>$h' \sqrt{\frac{k'}{d} + M' \left( 1 - \frac{k'}{d}^2 \right)}$</td>
<td></td>
</tr>
<tr>
<td>Triangle</td>
<td>$h \frac{M}{\rho_0 V}$</td>
<td>$h' \sqrt{M'}$</td>
<td></td>
</tr>
</tbody>
</table>

3.4.2 Center of Buoyancy

Consider now a vessel that has rolled to angle $\theta$ relative to the sea surface. Figure 3.12 shows the situation in the ship's frame of reference with the origin $\mathcal{O}$ at the keel. The shaded region $A$ is the submerged portion. The center of buoyancy $\bar{b}$ is identified as the centroid of this region. The location of $\bar{b}$ in ship coordinates is found by constructing $A$ out of a trapezoid $A_1$ (Figure 3.13) plus a triangle $A_2$ and minus a triangle $A_3$ (Figure 3.14).
Figure 3.12 The Actual Area of the Water

Figure 3.13 The Area of the Vessel's Waterline
Figure 3.14 The Areas of the Two Triangles Formed by the Waterline and Tilted Vessel

Figure 3.15 presents all relevant parameters used in computing the center of buoyancy of the vessel from this construction. The vertical bisector of the ship cross section is $\mathcal{D}$ and $W, W_r$ is the waterline, which intersects $\mathcal{D}$ at point $W_e$. Line $\mathcal{D}_1\mathcal{D}_2$ is the horizontal (in ship coordinates) through $W_e$. Note that $\phi$ represents the angle of the sides of the hull away from the vertical and is a constant of the ship geometry.
The important parameters of Figure 3.15 are as follows.

The center of gravity of the ship is:

\[ \bar{c} = (c_x, c_y) \]  
(3.150)

The center of buoyancy (the center of gravity of the displaced volume of water) is:

\[ \bar{b} = (b_y, b_z) \]  
(3.151)

\[ \lim_{\theta \to 0} b_z = b_0 \]  
(3.152)

The centroid of \( A_1 \) is \( \bar{b}_1 \), \( \bar{b}_2 \) is the centroid of \( A_2 \), and \( \bar{b}_3 \) is the centroid of \( A_3 \).
According to the construction in Figure 3.15:

\[ p_2 = p_3 = p \]  \hspace{1cm} (3.153)

and

\[ p = \frac{1}{2} k + \omega \tan(\phi) \]  \hspace{1cm} (3.154)

where

\[ \tan(\phi) = \frac{d - k}{h} \]  \hspace{1cm} (3.155)

The center of buoyancy in ship's coordinates relative to the keel involves:

\[ \bar{b} = \frac{A_1 \bar{b}_1 + A_2 \bar{b}_2 - A_3 \bar{b}_3}{A} \]  \hspace{1cm} (3.156)

where

\[ A = A_1 + A_2 - A_3 \]  \hspace{1cm} (3.157)

The submerged area \( A \) is found by evaluating the areas \( A_1, A_2, \) and \( A_3 \) by replacing \( d \) with \( 2p \) and \( h \) with \( \omega \) in equation 3.45 for the area of trapezoid \( A_1 \) and by using equation 3.77 with angle \( \phi \) for \( A_2 \) and \( -\phi \) for \( A_3 \).

\[ A_1 = \left( p + \frac{1}{2} k \right) \omega \]  \hspace{1cm} (3.158)

\[ A_2 = \frac{\tan(\theta)}{1 - \tan(\theta)\tan(\phi)} \frac{p^2}{2} \]  \hspace{1cm} (3.159)

\[ A_3 = \frac{\tan(\theta)}{1 + \tan(\theta)\tan(\phi)} \frac{p^2}{2} \]  \hspace{1cm} (3.160)

Substitute equations 3.158, 3.159, and 3.160 into equation 3.157:
\[ A = \left( p + \frac{1}{2} k \right) \omega + \frac{1}{2} \frac{\tan(\theta) p^2}{1 - \tan(\theta) \tan(\phi)} - \frac{1}{2} \frac{\tan(\theta) p^2}{1 + \tan(\theta) \tan(\phi)} \] (3.161)

Substitute \( p \) from equation 3.157 into equation 3.161 results in:

\[ A = -\frac{1}{4} 4\omega k + 4\omega^2 \tan(\phi) + k^2 \tan(\theta)^2 \tan(\phi) \]
\[ = \frac{4}{4 \left(-1 + \tan(\theta) \tan(\phi)\right)(1 + \tan(\theta) \tan(\phi))} \]
(3.162)

The centroid of \( A_1 \) is the centroid of a trapezoid (use equation 3.101 replacing \( d \) with \( 2p \) and \( h \) with \( \omega \)):

\[ \tilde{b} = \left( \frac{1}{3} \frac{\omega(4p + k)}{2p + k} \right) \] (3.163)

The vertex \( \mathcal{G}_2 \) of triangle \( A_2 \) (Figure 3.15):

\[ \left( -p \right) \left( \omega \right) \] (3.164)

The centroid of \( A_2 \) relative to \( \mathcal{G}_2 \):

\[ \frac{p}{3(1 - \tan(\theta) \tan(\phi))} \left( 1 - 2 \tan(\theta) \tan(\phi) \right) \left( \frac{\tan(\theta)}{\tan(\phi)} \right) \] (3.165)

The centroid of \( A_2 \) relative to the keel thus:

\[ \left( -p \right) + \frac{p}{3(1 - \tan(\theta) \tan(\phi))} \left( 1 - 2 \tan(\theta) \tan(\phi) \right) \left( \frac{\tan(\theta)}{\tan(\phi)} \right) \] (3.166)

The vertex \( \mathcal{G}_3 \) of triangle \( A_3 \) (Figure 3.15) is:

\[ \left( p \right) \left( \omega \right) \] (3.167)

The centroid of triangle \( A_3 \) relative to \( \mathcal{G}_3 \):
\[
\frac{p}{3(1 + \tan(\theta)\tan(\phi))} \left( \frac{1 + 2 \tan(\theta)\tan(\phi)}{\tan(\theta)} \right)
\]

(3.168)

The centroid of \(A_3\) relative to the keel is thus:

\[
\left( \frac{p}{\omega} \right) \frac{1 + 2 \tan(\theta)\tan(\phi)}{3(1 - \tan(\theta)\tan(\phi))} \frac{1}{\tan(\theta)}
\]

(3.169)

Substituting the centroids of the various areas (equations 3.158, 3.159, and 3.160) and their corresponding areas (equations 3.161, 3.163, 3.166, and 3.169) into equation 3.156 enables one to find the centroid of the entire underwater region:

\[
\bar{b} = \left[ \frac{-2\tan(\theta)^3}{3} \left( -1 + \tan(\theta)\tan(\phi) \right)^2 \frac{p^3}{(1 + \tan(\theta)\tan(\phi))^2} \right] \left( \frac{1}{3\tan(\theta)^2} \frac{p^2}{2\tan(\theta)^2} \frac{3\omega\tan(\phi) - 3\omega\tan(\theta)^2\tan(\phi)^3 + p + p\tan(\theta)^2\tan(\phi)^2}{(1 + \tan(\theta)\tan(\phi))^2(1 + \tan(\theta)\tan(\phi))^2} \right)
\]

(3.170)

Substitute for \(p\) from equation 3.154 into equation 3.170:

\[
\bar{b} = \left[ \frac{-1}{12} \left( -1 + \tan(\theta)\tan(\phi) \right)^2 \frac{\tan(\theta)(k + 2\omega\tan(\phi))^3}{(1 + \tan(\theta)\tan(\phi))^2} \right] \left( \frac{1}{24} \frac{\tan(\theta)^2(k + 2\omega\tan(\phi))^3}{(1 + \tan(\theta)\tan(\phi))^2} \frac{(8\omega\tan(\phi) - 4\omega\tan(\theta)^2\tan(\phi)^3 + k + k\tan(\theta)^2\tan(\phi)^2)}{(1 + \tan(\theta)\tan(\phi))^2(1 + \tan(\theta)\tan(\phi))^2} \right)
\]

(3.171)

In the above expression, recall that \(\tan(\phi) = \frac{d - k}{\omega}\) (3.158). The angle \(\theta\) is, however, one of the dynamical variables, i.e., it is a function of time \(\theta(t)\).

The other dynamical variable is \(\bar{z}\), the position of the center of gravity of ship relative to earth coordinates. In this model, the sea surface lies a distance \(s\) above the ocean floor (Figure 3.16).
From the construction in Figure 3.16 and recalling that the center of gravity of the ship lies a distance \( c \) above the keel.

\[
\begin{align*}
s &= \tilde{s} + (\omega - c)\cos(\theta) \\
\end{align*}
\]  

(3.172)

Solving for \( \omega \) gives:

\[
\omega = c + \frac{s - \tilde{z}}{\cos(\theta)}
\]  

(3.173)

Thus the dynamical variable \( \tilde{z} \) enters into the model by substituting equation 3.174 for \( \omega \) in the expression for submerged area \( A \) (equation 3.162) and for center of buoyancy \( \tilde{b} \) (equation 3.171).
3.5 **Modeling Sea Surface Variations**

The ship will be forced into motion by sea surface variations. Normally, waves are thought of in this context, so that the ship does not sit in a level sea as assumed in Figure 3.16. However, the waves that are significant in driving ship motion will have periods comparable to the ship roll period. In Chapter 2, the roll period of the crane ship was calculated to be about 12 seconds. The period $T$ of deep-water ocean waves is related to the wavelength $\lambda$ by (Aubrecht II, 1996):

$$ T_{\text{wave}} = \sqrt{\frac{2\pi \lambda}{g}} \quad (3.174) $$

where $g$ is the acceleration due to gravity. Thus,

$$ \lambda = \frac{g T_{\text{wave}}^2}{2\pi} \quad (3.175) $$

Substituting 9.80 m/s\(^2\) for $g$ and 12 sec for $T$ gives $\lambda = 225$ meters

The ships of interest have deck dimensions (approximating the "beam of the ship"): $d = 30$ meters, which is about 13% of the wavelength. Thus, to a first approximation, the sea surface is a level surface that is periodically rising up and down.

$$ s = s_0 + s_1 \sin \left( \frac{2\pi t}{T_{\text{wave}}} \right) \quad (3.176) $$

where $s_0$ is a sufficiently large average depth and $s_1$ is a variation small relative to the wavelength:

$$ s_1 \ll 2 \text{ meters} \quad (3.177) $$

or
\[
\frac{s_1}{d} \approx 0.07 
\] (3.178)

Note that the maximum variation in surface height will be on the order of:

\[
\Delta s = \frac{2\pi s_1}{\lambda} d
\] (3.179)

\[\approx 2\pi \times \frac{2}{225} d
\] (3.180)

\[\approx 6\% \text{ of } d
\] (3.181)

This variation is ignored in the dynamical equations discussed below.

3.6 Equations of Motion

In this model, ship motion is governed by Newton's Laws as they describe:

(1) the acceleration \( \ddot{z} \) in the true vertical \( \ddot{z} \) direction as a result of imbalance between the ship's weight \( Mg \) and the buoyancy force \( \bar{F}_B \).

(2) the angular acceleration \( \ddot{\theta} \) due to the moment of buoyancy around the center of gravity.

The magnitude of the buoyancy force is given by:

\[ F_B = \rho_0 A \lg 
\] (3.182)

where \( A \) is the submerged cross-sectional area given in equation

\[ A = -\frac{1}{4} \frac{4\omega k + 4\omega^2 \tan(\phi) + k^2 \tan(\theta)^2 \tan(\phi)}{4 (-1 + \tan(\theta) \tan(\phi)) (1 + \tan(\theta) \tan(\phi))} 
\] (3.162)

where
\[ \tan(\phi) = \frac{d - k}{2h} \]  
\[ \omega = c + \frac{s - z}{\cos(\theta)} \]

The dynamics is then given by:

\[ m\ddot{z} = \rho_0 A l g - Mg \]

The moment arm \( h \) of the buoyancy force \( F_B \) about the center of gravity is found using Figures 3.17 and 3.18.

Figure 3.17 Construction to Find Buoyancy Moment
It is useful to identify the intersection of the true vertical through the center of buoyancy with the ship's vertical through the center of gravity as a point called the "metacenter."

If the distance from the metacenter to the center of gravity is $m$ then:

$$n = m \sin(\theta)$$

(3.185)

Thus, using equation 3.184:

$$m = \frac{b_y}{\tan(\theta)} - (c - b_z)$$

(3.186)

This is called the metacentric height and must be positive for the buoyancy force to exert a righting moment rather than cause the ship to capsize. That is, $m$ determines the static stability of the ship.

The roll dynamics involves:

$$I \ddot{\theta} = F_b n$$

(3.187)
where $I$ is the moment of inertia of the ship. Substituting for $F_B$ from equation 3.182 and $n$ from equation 3.184:

$$I \ddot{\theta} = F_B \left[ b_y \cos(\theta) - (c_0 - b_z) \sin(\theta) \right] \quad \text{(3.188)}$$

Introducing phenomenological damping terms $\gamma$ and $\beta$, our model then arrives at the coupled system of two second-order equations:

$$\ddot{z} = \frac{\rho g l}{M} A(z, 0) - g - \beta (z, 0) \dot{z} \quad \text{(3.189)}$$

$$\ddot{\theta} = \frac{\rho g l}{I} A(z, \theta) \left[ b_y (z, \theta) \cos(\theta) - (c_0 - b_z (z, \theta)) \sin(\theta) \right] - \gamma (z, \theta) \dot{\theta} \quad \text{(3.190)}$$

Further elaboration of the model will require specifying dependence of the damping coefficients on $z$ and $\theta$. Additionally, the effect of inertia of the water surrounding the ship (the added-mass effect) would need to be included.

3.7 Summary

Modeling of ship motion involves ship geometries and the interaction between the water and the ship. To simplify the calculations a simple trapezoidal transverse hull is used with a rectangular side view. By taking limits of the bottom of the vessel, rectangular and triangular hulls can be examined. Using the geometries, the volumes of the ships, the centers of gravity, and the moments of inertia can be calculated. With the addition of water, the centers of buoyancy can be calculated. Using all of these parameters, one can understand the coupled dynamical equations for the roll and heave motion of the ship.
4. Coupled Mathieu Equations

The Mathieu equation has been used for a variety of applications from springs and pendulums, to ships. A coupled spring and pendulum system is similar to ship motion where the pendulum represents the rolling motion of the ship and the spring represents the heave motion of the ship.

Allievi and Soudack (1990) actually modeled ship roll motion using the Mathieu equation. They analyzed naval architectural principles, such as the metacentric height, to create a differential equation. They also examined the presence of damping and examined Poincaré sections to understand the stability and instability regions based on changing parameters.

To understand a non-autonomous four-dimensional coupled system, a progressively more complex series of Mathieu equations was used. For each step of complexity, the solutions to the differential equations were plotted often with Poincaré sections (the x's or points on the graphs).

First, the equation was undamped with two degrees of freedom. By being undamped, the parametric instability regions or "tongues" could easily be identified. Second, the equation was damped and had two degrees of freedom. The introduction of damping added realism to the system and aided in the understanding of the influence of damping on instability. Third, nonlinearity was added to the Mathieu equation, with two degrees of freedom, to explore full nonlinear behavior such as limit cycles and chaos. Fourth, two nonlinear equations each with two degrees of freedom were coupled in order to see what happens to the geometrical description of dynamics as the coupling between these two equations was increased. MatLab program B.2 was used to create all of the graphs used in this section.

4.1 Undamped Linear Mathieu Equation –
Two Degrees of Freedom

First, a linear equation with no damping was examined (equation 4.1) (Anićin, Davidović, and Babović, 1993).

\[
\frac{d^2 x}{d\tau^2} + [a + 16g \cos(2\tau)]x = 0
\]  

(4.1)
Values for $a$ were chosen using the Ince-Strutt stability chart (Anićin, Davidović, and Babović, 1993). Figure 4.1 shows a parameter space, which shows the stable (outside the marked regions) and unstable (the $\mu$ regions) regions.

![Figure 4.1 Ince-Strutt Stability Chart](image)

There is a series of alternating unstable and stable regions. The unstable regions alternate between those regions with resonances drive frequencies that are multiples of twice the natural frequencies (subharmonic response) (Figure 4.2) and those regions with resonances at the natural frequency of the system (Figure 4.3). The instability is noticed by the fact that the trajectory spirals outward from the initial conditions of $x(t) = 0.05$ and $x'(t) = 0$. 
Figure 4.2 Solutions of Mathieu Linear Equation with $a = 1$ and $q = 0.005$

Figure 4.3 Solutions of Mathieu Linear Equation with $a = 4$ and $q = 0.05$
The stable region is a circle (Figure 4.4), thus showing quasi-periodicity in the system.

Figure 4.4 Solutions of Mathieu Linear Equation with $a = 2$ and $q = 0.005$

4.2 Damped Linear Mathieu Equation – Two Degrees of Freedom

Second, damping was added (equation 4.2) to the linear equation (equation 4.1). The damping coefficient $\beta$ was chosen to be 0.5 for the different $a$ values.

\[
\frac{d^2 \bar{x}}{d\tau^2} + \left[ a + 16q \cos(2\tau) \right] \bar{x} + \beta \frac{d\bar{x}}{d\tau} = 0
\] (4.2)

A damped Mathieu equation can be converted to an equivalent undamped Mathieu equation:

\[
\tilde{x} = x e^{\frac{\beta}{2}}
\] (4.3)

or

\[
x = \tilde{x} e^{-\frac{\beta}{2}}
\] (4.4)
\[
\frac{dx}{dt} = \frac{d\tilde{x}}{dt} e^{-\frac{\beta t}{2}} - \frac{\beta}{2} \tilde{x} e^{-\frac{\beta t}{2}}
\]  
(4.5)

\[
\frac{d^2\tilde{x}}{dt^2} = \frac{d^2\tilde{x}}{dt^2} e^{-\frac{\beta t}{3}} - \frac{\beta}{2} \frac{d\tilde{x}}{dt} e^{-\frac{\beta t}{2}} + \frac{\beta^2}{4} \tilde{x} e^{-\frac{\beta t}{2}}
\]  
(4.6)

So,

\[
\frac{d^2\tilde{x}}{dt^2} e^{-\beta t} - \beta \frac{d\tilde{x}}{dt} e^{-\beta t} + \frac{\beta^2}{4} \tilde{x} e^{-\beta t} + (\alpha + 16q\cos(2\tau))\tilde{x} e^{-\beta t} + \beta \frac{d\tilde{x}}{dt} e^{-\beta t} - \frac{\beta^2}{2} \tilde{x} e^{-\beta t} = 0
\]  
(4.7)

Dividing through by \(e^{-\beta t}\):

\[
\frac{d^2\tilde{x}}{dt^2} + \left(\alpha - \frac{\beta^2}{4} + 16q\cos(2\tau)\right)\tilde{x} = 0
\]  
(4.8)

The damping raised the tapered ends of the unstable regions from the \(\alpha\) axis of Figure 4.1. Therefore, it takes a higher \(q\) value to remain in the unstable region for the same \(\alpha\) value. The stability can be seen by the fact that the trajectory is spiraling inward from the initial conditions of \(x(t) = 0.05\) and \(x'(t) = 0\) (Figure 4.4.).
Figure 4.5 Solutions of Damped Linear Mathieu Equation with $a = 1$ and $q = 0.0025$

4.3 Nonlinear Damped Mathieu Equation – Two Degrees of Freedom

Third, nonlinearity was added (equation 4.9) to the damped equation (equation 4.2).

$$\frac{d^2x}{d\tau^2} + \left[a + 16q \cos(2\tau)\right] \sin(x) + \beta \frac{dx}{d\tau} = 0$$  (4.9)

With certain parameters, a strange attractor can be observed (Figure 4.6). The existence of a strange attractor implies chaotic activity in the system.
4.4 Nonlinear Damped Coupled Mathieu Equations – Two Degrees of Freedom Each

Last, two coupled equations were used (equation 4.10 and equation 4.11). The coupling coefficient is $\chi$. The initial value for $x$ is 0.05, while the initial value for $y$ is 0.5.

\[
\frac{d^2 x}{d\tau^2} + [a + 16q \cos(2\tau)] \sin(x) + \beta \frac{dx}{d\tau} + \chi(x - y) = 0
\]

(4.10)

\[
\frac{d^2 y}{d\tau^2} + [a + 16q \cos(2\tau)] \sin(y) + \beta \frac{dy}{d\tau} + \chi(y - x) = 0
\]

(4.11)

For figures 4.7 and 4.8, $a = 1.469$, $q = 0.1102$, and $\beta = 0.42$. For figure 4.8, $a = 1.5$, $q = 0.11$, and $\beta = 0.42$. Figure 4.7 shows low coupling of 0.05. The two graphs look slightly different. Figure 4.8 shows moderate coupling 0.07, where there appears to be some higher dimensional activity based on the fuzziness of the strange attractor. The s-attractor can no longer be seen. High coupling Figure 4.9 has twice as many points as Figures 4.8 and 4.7 to show the detail. With more points, another s-curve can be seen.
Figure 4.7 Weakly Coupled Mathieu Equations
Figure 4.8 Moderately Coupled Mathieu Equations
Figure 4.9 Highly Coupled Mathieu Equations

4.5 Mathieu Equation Conclusion

Mathieu equations are versatile. They model various dynamic systems including ship motion. These equations enable people to handle non-autonomous four-dimensional systems in a relatively simple fashion. As a linear undamped system, unstable spirals and circles are possible. By adding damping, regions of instability are altered allowing for stable spirals. Adding a nonlinear term into the equation allows for chaotic activity, such as the s-curve attractor. The coupling of Mathieu equations shows that for a set of $a$, $q$, and $\beta$, there is a coupling coefficient that shows some higher dimensional activity. Future research will involve placing the ship equations into this program and seeing how changing parameters demonstrates the dynamics of the system.
5. Conclusion

This thesis strives to help understand the motion of ships and how to predict ship motion. Analysis of the Carderock ship roll data, recorded on July 19\textsuperscript{th} 1993, began the whole progression of thought. The most important finding is the extraction of a correlation time from the data. With a lack of insight into the ship motion system, a different approach was taken. This new approach began with using the basic principles of naval architecture to form equations. Next, I examined coupled equations, from the dynamic system of Mathieu. By analyzing the well-known Mathieu equation and coupling it with itself, one can understand the dynamics that the ship model equations might exhibit. Despite all of this work, there are still some problems.

Trying to tease apart the essence of a system is very difficult. Actual data is limited by the technology used to collect it, as well as the researchers and the experiment design. The placement of the sensors on the ships is important so that only the degree of freedom is recorded. The designers need to ensure that the instruments work properly and the data is taken at regular intervals. The experiment design needs to ensure that the proper instruments are used and that the instruments are used in such a way to get the proper data. To understand the results of the output, one needs to look critically at how one analyzed the data.

As with any scientific endeavor, assumptions were made which directly and/or indirectly tainted the data. CDA Professional Version was heavily used without a strict validation of its assumptions. CDA's many tools, along with other nonlinear tools, were not used (such as Liaponov exponent). The data was biased by calculations based on strict 2,048 data point files in a contiguous fashion. Although discussed, the idea of a moving window and/or using a different window size was not implemented. CDA is limited to 32,000 or fewer data points; MatLab: Student Version 5.0 does not accept arrays with more than 16,000 elements. Biases come from both the analysis of actual ship data and the formation of models.

Models need validation through experimentation and must mimic real-life behavior. Without experimental evidence validating the model, modeled equations are theoretical or just mathematical exercises. A suggested experiment is as follows: Construct a physical ship model with a scale of roughly 1:24. Place the ship in a tank. Raise and lower the fluid in the tank to a frequency equal to the model's natural roll frequency. This tests the idea that heave and roll are coupled as in the
demonstrated Mathieu's equations. Experimentation and noise help to test mathematical equations.

Mathematical models simplify the complex interactions observed by the instruments used in experiments. The instruments are influenced by human actions, such as sailors moving cargo around, or uncontrollable events, such as a change in the climate. The circuitry in the instruments often adds noise to the data collected. These different components come together as the time series, full of noise that complicates the analysis. The mathematical equations have the principle elements of the system without the perturbations of instrumentation and events. For future study, it would be helpful to add noise to the system and run through the same tests as in chapter two (i.e., mean, variance, kurtosis, power spectrum, etc.).

Another future study would be to understand the ship model equations with the same MatLab program that was used for the Mathieu equations in chapter four. Another useful project from chapter four is to investigate the "fuzzy" Poincaré section results from the particular range of parameters for the coupled, nonlinear Mathieu equations. There is some higher dimensional activity illustrated by the plots that should be analyzed. Different parameters will need to be used to understand the system of ship motions. Overall, this research is a significant beginning to understand the complex nature of ship motion.
Appendix A
Plots of Different Days and of Heave

Here are some plots of time series, mean, variance, skewness, kurtosis, dominant frequency, correlation time, and Hurst exponent of roll from the July 18, July 17, and July 15 respectively. For July 18, the sampling was taken over 11 hours, 22 minutes, and 40 seconds. For July 17, the sampling was taken over 3 hours, 24 minutes, and 48 seconds. For July 15, the sampling was taken over 7 hours, 57 minutes, and 52 seconds. Next are the plots of time series, mean, variance, skewness, kurtosis, dominant frequency, correlation time, and Hurst exponent of heave from July 19, July 18, July 17, and July 15 respectively. Dominant frequencies and correlation times were left out if CDA gave out a value of zero or a value that was extremely high.

Figure A.1 Time Series of Roll of July 18
Figure A.2 Mean of Roll of July 18

Figure A.3 Variance of Roll of July 18
Figure A.4 Skewness of Roll of July 18

Figure A.5 Kurtosis of Roll of July 18
The dominant frequency of roll of July 18 is $0.078 \pm 0.003$ Hz.

The correlation time of roll of July 18 is $2.55 \pm 0.075$ seconds.
The Hurst exponent of roll of July 18 is $0.38 \pm 0.03$.
Figure A.10 Mean of Roll of July 17

Figure A.11 Variance of Roll of July 17
Figure A.12 Skewness of Roll of July 17

Figure A.13 Kurtosis of Roll of July 17
Figure A.14 Dominant Frequency of Roll of July 17

The dominant frequency of roll of July 17 is $0.078 \pm 0.000$ Hz.

Figure A.15 Correlation Time of Roll of July 17

The correlation time of roll of July 17 is $2.1 \pm 0.9$ seconds.
For roll of July 17, the Hurst exponent value for the raw data was $0.35 \pm 0.05$ and $0.37 \pm 0.05$ for the smoothed data.
Figure A.18 Mean of Roll of July 15

Figure A.19 Variance of Roll of July 15
Figure A.20 Skewness of Roll of July 15

Figure A.21 Kurtosis of Roll of July 15
The dominant frequency of roll of July 15 was 0.1 ± 0.2 Hz for both raw and smooth data.

The correlation time for roll of July 15 was 1.05 ± 0.15 seconds for the raw data and 1.9 ± 0.1 seconds for the smoothed data.
For the Hurst exponent of roll July 15, the raw data value was 0.30 ± 0.05 and 0.33 ± 0.05 for the smoothed data.

Figure A.25 Time Series of Heave of July 19
Figure A.26 Mean of Heave of July 19

Figure A.27 Variance of Heave of July 19
Figure A.28 Skewness of Heave of July 19

Figure A.29 Kurtosis of Heave of July 19
The dominant frequency of heave of July 19 is $0.15 \pm 0.05$ Hz.

The correlation time of heave of July 19 is $1.2 \pm 0.2$ seconds.
The Hurst exponent of heave of July 19 is 0.25 \pm 0.02.

Figure A.32 Hurst Exponent of Heave of July 19

Figure A.33 Time Series of Heave of July 18
Figure A.34 Mean of Heave of July 18
Figure A.35 Variance of Heave of July 18

Figure A.36 Skewness of Heave of July 18
The dominant frequency of heave of July 18 is $0.011 \pm 0.03$ Hz.
The correlation time of heave of July 18 is $1.4 \pm 0.2$ seconds.

The Hurst exponent of heave of July 18 is $0.25 \pm 0.05$. 
Figure A.41 Time Series of Heave of July 17

Figure A.42 Mean of Heave of July 17
Figure A.43 Variance of Heave of July 17

Figure A.44 Skewness of Heave of July 17
The dominant frequency of heave of July 17 was $0.14 \pm 0.02$ Hz for the raw data and $0.12 \pm 0.06$ Hz for the smoothed data.
The correlation time for heave of July 17 was 1.3 ± 0.2 seconds for the raw data and 1.4 ± 0.2 seconds for the smoothed data.

For the Hurst exponent of heave of July 17, the value was for the raw data 0.27 ± 0.03 and 0.31 ± 0.03 for the smooth data.
Figure A.49 Time Series of Heave of July 15

Figure A.50 Mean of Heave of July 15
Figure A.51 Variance of Heave of July 15

Figure A.52 Skewness of Heave of July 15
The dominant frequency of heave of July 15 is $0.14 \pm 0.04$ Hz.
The correlation time of heave of July 15 is $1.44 \pm 0.06$ seconds.

The Hurst exponent of heave of July 15 is $0.23 \pm 0.04$. 
Figure A.57 CDA File with Six Point Gaussian Smoothing
Figure A.58 Skew Derivative for Raw Roll of July 19

Figure A.59 Smoothed Skew Derivative for Roll of July 19
Figure A.60 Correlation Dimension of Raw Roll Data of July 19

Figure A.61 Correlation Dimension of Smoothed Roll Data of July 19
Appendix B
MatLab Programs

In this section, MatLab programs used in chapters two, four, and five are discussed.

Appendix B.1 – Chapter 2

% This program calculates the mean, skewness, variance, and kurtosis for the user % chosen time series. The time series provided by the naval crane ship % have different numbers of data points depending on the day. So, this % change is taken into account in both the graphs and the calculations.

clear all

% Prompt the user for the appropriate time series

% Change the end point of the calculation and the titles of the plots
% for the appropriate day

% July 19 time series
if file == 1 | file == 2
    num_end = 68;
    file_num = 19;

% July 18 time series
elseif file == 3 | file == 4
    num_end = 40;
    file_num = 18;

% July 17 time series
elseif file == 5 | file == 6
    num_end = 12;
    file_num = 17;

% July 15 time series
else
    num_end = 28;
    file_num = 15;
end

% Change the plot titles and the time series loading to the appropriate type of time
series
% (i.e., roll or heave).
if mod(file,2) == 0
    title_nam = 'Heave';
    file_nam =['hv' int2str(file_num) ' '];
    mean_unit = '(ft / s^2)';
    var_unit = '(ft^2 / s^4)';
else
    title_nam = 'Roll';
    file_nam =['r' int2str(file_num) ' '];
    mean_unit = '(degrees)';
    var_unit = '(degrees^2)';
end

% Initialize the matrices used for the entire program
matrix_kurt = zeros(num_end,1);
matrix_skew = zeros(num_end,1);
matrix_vari = zeros(num_end,1);
matrix_mean = zeros(num_end,1);

% Increment for all of the data files.
for i = 1 : num_end;

    %Initialize the variables that are used each loop.
    kurtsum = 0;
    skewsum = 0;
    varsum = 0;

    % Load each file
    load([file_nam int2str(i) '.dat']);

    % Initialize a variable with the data information
    eval(["cda_file = ' file_nam int2str(i) '']);

    % The variable for the mean of the file
xbar = mean(cda_file);

% Increment for each data point
for x = 1 : 2048

% Calculate the kurtosis.
kurt = (cda_file(x) - xbar)^4;
kurtsum = kurtsum + kurt;

% Calculate the skewness.
skew = (cda_file(x) - xbar)^3;
skewsum = skewsum + skew;

% Calculate the variance.
var = (cda_file(x) - xbar)^2;
varsum = varsum + var;
end

% Calculate the standard deviation of each file
deviation = std(cda_file);

% Record the current file's kurtosis information
matrix_kurt(i,1) = (kurtsum / 2048) / deviation^4;

% Record the current file's mean information
matrix_mean(i,1) = xbar;

% Record the current file's skewness information
matrix_skew(i,1) = (skewsum / 2048) / deviation^3;

% Record the current file's variance information
matrix_vari(i,1) = (varsum) / (2047);

% Clear the data variable
clear cda_file;
end

% Index for plotting time
time_end = 512:1024:num_end*1024;
time_end = time_end/10^3;
Appendix B.2 – Chapter 4

The functions used in the program follow the code of the program.

% This program plots a user inputed amount of Matheiu equation graphs. The user
% chooses the a value for the whole set of
% equations. For each individual graph, the user inputs a q value, and the type of
% graph. The user may choose a linear
% with no damping graph, a damped linear graph, a nonlinear damped graph, and a coupled, nonlinear, damped graph.
% For the differential equations the initial values were 0.05 for non-prime variables and 0 prime variables.
% ode45 was the solver used for the equations. The user may choose the damping coefficients for the damped equations
% and the coupling coefficient for the coupled equations.
clear all
quit = 'n';
fintime=10000;
strtim = 0;
endtim = fintime;
yd1 = 0.05;
yd2 = 0;

tm=0:pi:fintime;
timestp=[0,fintime];

while quit == 'n'

clc

% initializing parameters
a = input('Enter the value for a: ');

% Prompt for the number of plots. To allow for long titles on the plots,
% users are not allowed to have more than one column of graphs.
numans = input('Enter the number of plots including coupled: ');

% initialize plot counter
numbr = 1;

% create plots until the total number of graphs is reached
while numbr <= numans
    clear ym
    clear y
    clear t
    % prompt for type of graph
mansw = menu('Choose an equation', 'Linear', 'Damped', 'Damped Nonlinear', 'Coupled');

% clear the screen
clc

% Prompt for parameter value
q = input('Please enter the q value for the plot: ');

% User chose the linear Mathieu case
switch mansw
  case 1
    % Calculate the linear case with no damping
    [t, y] = ode45('Am', timestp, [0.05 0], [], a, q);
    i = 1;
    while t(i) < fintime
      if (y(i, 1) < 0)
        y(i, 1) = rem(y(i, 1), -pi);
      else
        y(i, 1) = rem(y(i, 1), pi);
      end
      i = i + 1;
    end
    ym = interp1(t, y, tm);

    % Plot the linear case with no damping and label the graph
    subplot(numans, 1, numbr), plot(y(:, 1), y(:, 2), 'm', ym(:, 1), ym(:, 2), 'b*');
    set(gca, 'FontName', 'Times New Roman', 'FontSize', 10);
    title(['Free Section with a = ', num2str(a), ', and q = ', num2str(q)]);
    xlabel('x', 'FontName', 'Times New Roman', 'FontSize', 10);
    ylabel('x''', 'FontName', 'Times New Roman', 'FontSize', 10);
  end

% User chose the damped Mathieu case
  case 2
    % Prompt for the damping coefficient
    beta = input('Please enter damping coefficient: ');

  end
% Calculate the differential equation for given parameters
[t,y] = ode45('Am1', timestp, [0.05 0], [], a, q, beta);
i=1;

while t(i) < fintime
    if (y(i,1) < 0)
        y(i,1)=rem(y(i,1),-pi);
    else
        y(i,1)=rem(y(i,1),pi);
    end
    i=i+1;
end

ym = interp1(t,y,tm);

% Plot the damped linear case and label the graph
subplot(numans, 1, numbr), plot(y(:,1), y(:,2), 'm', ym(:,1), ym(:,2), 'bx');
title(['Damped Section with a =', num2str(a), ', q = ', num2str(q), ', and \beta = ', num2str(beta)]);
xlabel ('x');
ylabel ('x''');

% User chose Nonlinear Mathieu case with dampening
case 3

% Prompt for the damping coefficient
beta = input('Please enter a damping coefficient: ');

% Calculate the nonlinear differential equation with the selected parameters
[t,y] = ode45('Am2', timestp, [yd1 yd2], [], a, q, beta);
i=1;

while t(i) < fintime
    if (y(i,1) < 0)
        y(i,1)=rem(y(i,1),-pi);
    else
        y(i,1)=rem(y(i,1),pi);
    end
i = i + 1;
end

ym = interp1(t, y, tm);

% Plot the Nonlinear equation and label the graph
subplot(numans, 1, numbr), plot(ym(:,1), ym(:,2), 'b.');</p>

```
title(['Damped and Nonlinear with a = ', num2str(a), ', q = ', num2str(q), ', and 
\beta = ', num2str(beta)]);
xlabel ('x');
ylabel ('x''');
```

% User chose the coupled equations
otherwise

% Prompt for the damping coefficient
beta = input('Please enter a damping coefficient: ');

% Prompt for the coupling coefficient
gamma = input('Please enter a coupling coefficient: ');

% Calculate the coupled differential equations with selected parameters
[t, y] = ode45('Am3', timestp, [0.05 0 0.05 0], [], a, q, beta, gamma);

i = 1;
while t(i) < fintime
    if (y(i,1) < 0)
        y(i,1) = rem(y(i,1), -pi);
    else
        y(i,1) = rem(y(i,1), pi);
    end
    i = i + 1;
end

i = 1;
while t(i) < fintime
    if (y(i,3) < 0)
        y(i,3) = rem(y(i,3), -pi);
    else
        y(i,3) = rem(y(i,3), pi);
    end
```
end
   i=i+1;
end

ym = interp1(t,y,tm);

% Plot the first coupled plot and label the graph
subplot(numans,1,numbr), plot(ym(:,1), ym(:,2), 'bx');
title ([\'Coupled System with a = ', num2str(a), ', q = ', num2str(q), ', and \beta= ', num2str(beta)]);
xlabel ('x');
ylabel ('x\''');

% Increment for the second coupled plot
numbr = numbr + 1;

% Plot the second coupled plot and label the graph
subplot(numans,1,numbr), plot(ym(:,3), ym(:,4), 'bx');
title ([\'Dampened and Nonlinear and coupling = ', num2str(gamma)]);
xlabel ('y');
ylabel ('y\''');

end   % end of switch statement

% Increment the number of plots
numbr = numbr + 1;

end   % end of while statement

% present the graphs
figure(1)

% let the figure window fill the whole screen and clear the screen
set(gcf,'Position',[-2, 2, 644, 442]);
clc
quit = input ('Do you want to quit ((y)es or (n)o)? ', 's');
end

Here are the functions used in the program above:
% Am.m

function ode = Fm(t, y, unused, a, q)

% Original Mathieu's equation
ode = [y(2);-y(1)*(a+16*q*cos(2*t))];

% Am1.m
function ode = Fm(t, y, unused, a, q, beta)

% adding damping beta with Am
ode = [y(2);-y(1)*(a+16*q*cos(2*t))-beta*y(2)];

% Am2.m
function ode = Fm(t, y, unused, a, q, beta)

% adding damping and nonlinearity beta with Am
ode = [y(2);-sin(y(1))*(a+16*q*cos(2*t))-beta*y(2)];

% Am3.m
function ode = Fm(t, y, unused, a, q, beta, gamma)

% adding damping of beta, nonlinearity, and coupling, gamma, with Am1
ode = [y(2);-sin(y(1))*(a+16*q*cos(2*t))-beta*y(2)-gamma*(y(1)-y(3)); y(4); -sin(y(3))*(a+16*q*cos(2*t))-beta*y(4)-gamma*(y(3)-y(1))];
Appendix C
Glossary of Naval Architecture

All terms are from Tupper (1996), except for keel.

Amidships – This is the vertical transverse section of the ship that has the largest section of area of the ship. This is located in the midpoint of the ship.

Beam – A term for the ship’s greatest width in a transverse horizontal direction, often quoted at amidships.

Center of Buoyancy – Where the centroid of the displaced volume of water is located through which all buoyancy forces act.

Center of Flotation – Where the centroid of the waterplane is located.

Center of Gravity – Where the centroid of the ship is located, through which all gravitational forces act.

Displacement – The weight of water displaced by a floating ship, which is equal to the ship’s weight.

Draught – This term is the distance between the keel and the waterline.

Half Breadth Plan – This view of the ship shows how the waterlines are grouped. This view is a series of z-x slices of the ship.

Heave – This degree of freedom is the vertical translation. Heave is also the movement along the z-axis.

Keel – This part of the ship is the lowest and most central part of the bottom of the vessel that runs from the fore to the aft of the vessel (Watson & Watson, 1991).

Metacenter – The point where the direction of buoyancy from the center of buoyancy intersects with the original noninclined vertical line of the vessel.
Metacentric Height – This is the distance from the center of gravity to the metacenter. For small inclinations, for a given center of gravity, the metacenter can be considered fixed in position. Metacentric height is a good indicator of the stability of the ship.

Midships – See Amidships.

Pitch – This degree of freedom is the rotation about a transverse axis. Pitch is also the rotation about the y-axis.

Profile Plan – This view of the ship is a side elevation of a ship's form.

Righting Lever – This term is the distance between the lines of action of the gravity and buoyancy forces. This is numerically equal to \( m \times \sin(\theta) \), where \( m \) is the metacentric height and \( \theta \) is the angle of roll.

Roll – This degree of freedom is the rotation about a fore and aft axis. Roll is also the rotation about the x-axis.

Sheer Plan – This is a vertical longitudinal center line section of a vessel, which involves the intersections of vertical fore and aft planes.

Surge – This degree of freedom is the fore and aft translation. Surge can also be thought of as the movement along the x-axis.

Sway – This degree of freedom is the transverse translation. Sway is also the movement along the y-axis.

Transverse section – This term describes a z-y slice of the ship.

Trim – The difference between the draught of the fore and aft of the vessel.

Waterplanes – These are the horizontal planes parallel to the surface of the water that mark the depth of the water along the vessel.

Waterlines – These are the lines of intersection of the body of water and the hull.

Yaw – This degree of freedom is the rotation about a vertical axis. Yaw is also the rotation about the z-axis.
References


