CASCADED LATTICE BOLTZMANN MODEL FOR ISOTROPIC AND ANISOTROPIC CONVEKTIVE THERMAL FLOWS WITH LOCAL HEAT SOURCES

by

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Cascaded Lattice Boltzmann Model for Isotropic and Anisotropic Convective Thermal Flows with Local Heat Sources
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ABSTRACT

A new cascaded central moment based lattice Boltzmann (LB) method for solving low Mach number convective thermal flows with source terms in two-dimensions in a double distribution function framework is presented. For the passive temperature field, which satisfies a convection diffusion equation (CDE) along with a source term to represent internal/external local heat source, a new cascaded collision kernel is presented. Due to the use of a single conserved variable in the thermal energy equation, the cascaded structure in its collision operator begins from the first order moments and evolves to higher order moments. This is markedly different from the collision operator for the fluid flow equations, constructed in previous work, where the cascaded formulation starts at the second order moments in its collision kernel. A consistent implementation of the spatially and temporally varying source terms in the thermal cascaded LB method representing the heat sources in the CDE that maintains second order accuracy via a variable transformation is discussed. In addition, the first order equilibrium moments in this model are augmented with spatial temperature gradient terms obtained locally and involving a tunable coefficient to maintain additional flexibility in the representation of the transport coefficient for the temperature field. The consistency of the thermal cascaded LB method including a source term with the macroscopic convection-diffusion equation is demonstrated by means of a Chapman-Enskog analysis. The emergent tunable diffusivity is shown to be dependent on the relaxation times of the first order moments as well as the tunable parameter in the additional gradient terms in our cascaded multiple-relaxation-time formulation. The
new model is tested on a set of benchmark problems such as the thermal Poiseuille flow, thermal Couette flow with either wall injection or including viscous dissipation and natural convection in a square cavity. The validation studies show that the thermal cascaded LB method with source term is in very good agreement with analytical solutions or numerical results reported for benchmark problems. In addition, the numerical results show that our new thermal cascaded LB model maintains second order spatial accuracy.

The new LBE model is modified to simulate anisotropic fluids that are characterized by different diffusion coefficients along different directions. The applicability of the LBE model is validated by numerical simulations including the convection and diffusion of a Gaussian Hill, solving anisotropic convection diffusion equation with variable diffusion tensor and variable source term, and anisotropic natural convection in a Square Cavity. The validation study shows that the anisotropic thermal cascaded LB model with source term is in very good agreement with the analytical solutions or numerical results reported for the benchmark problems. In addition, the numerical results show that our new anisotropic thermal cascaded LB model maintains second order accuracy as does the isotropic model.

A stability test of the present cascaded LBE model is conducted to compare our model by single relaxation time (SRT) LB model and multiple relaxation times (MRT) LB model using the diffusion in a Gaussian Hill as a test problem by varying the fluid diffusivity to compare the stability characteristics of the cascaded central moment. This stability study concurs with other work that indicates superior stability characteristics of the cascaded LB method.

Finally, the central moment cascaded LB model was adapted to simulate of fluid flow with temperature-dependent viscosity, where the fluid viscosity is exponentially varying with temperature. The simulation results of couette flow with shear heating confirmed the validity of the present central moment LB model to incorporate correctly fluid flow with variable viscosity at different Brinkman numbers.
The form and content of this abstract are approved. I recommend its publication.

Approved: Samuel Welch
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# CONTENTS

## CHAPTER

### I THERMAL LATTICE BOLTZMANN MODEL

1.1 Introduction ................................. 1

1.2 Collision Models .............................. 2

1.2.1 Single Relaxation Time (SRT) LBM ............ 3

1.2.2 Multiple Relaxation Times (MRT) LBM .......... 5

1.2.3 Cascaded LBE ............................... 7

1.3 Thermal flow approaches ....................... 8

1.3.1 Multispeed approach (MSLBE) ................. 9

1.3.2 Hybrid approach (HybLBE) .................... 9

1.3.3 Double distribution functions (DDfLBE) ........ 10

1.4 Construction of the Cascaded Collision Operator for the Temperature Field .......... 11

1.5 Anisotropic Cascaded Model for the Temperature Field .................. 28

1.6 Boundary Conditions ............................ 29

1.6.1 Halfway bounce-back boundaries ............... 30

1.6.1.1 Velocity Distribution Function ............... 32

1.6.1.2 Temperature Distribution Function ........... 32

1.6.2 Periodic Boundary Conditions ................. 33

### II ISOTROPIC THERMAL FLOW

2.1 Numerical Results ............................. 35
LIST OF TABLES

TABLE

2.1 Values of $g\beta$ corresponding to each Rayleigh number. .......................... 46
2.2 Comparison between numerical results obtained using the cascaded LBM and the published results (de Vahl Davis(1983) and Hortmann et al(1990)) at different Rayleigh numbers ($Ra = 10^3 - 10^6$). ......................... 57
3.1 Global relative error and stability characteristics after 1000 time increments for SRT, MRT, and Cascaded LBM for different relaxation times for the convection-diffusion of a Gaussian hill problem at Mach number 0.25. ................................................................. 65
3.2 Values of $g\beta$ corresponding to each Rayleigh number. ............................ 73
3.3 Numerical results obtained using the cascaded LBM for anisotropic natural convection in a square cavity at different Rayleigh numbers ($Ra = 10^3, 10^4, 10^5, 10^6, 10^7$ and $10^8$). ................................................................. 78
3.4 Comparison between numerical results obtained using the cascaded LBM and the published results Dubois et al(2016) [64] for anisotropic natural convection in a square cavity at different Rayleigh numbers ($Ra = 10^3, 10^4, 10^5$ and $10^6$) ......................................................... 79
3.6 Comparison between numerical results obtained using the cascaded LBM and the published results (de Vahl Davis(1983) and Hortmann et al(1990)) for isotropic natural convection in a square cavity at different Rayleigh numbers ($Ra = 10^3 - 10^6$). ......................................................... 81
LIST OF FIGURES

FIGURE

1.1 D2Q9 lattice, Collision (right) and Streaming (left). .................. 3
1.2 D2Q9 lattice diagram, showing the 9 velocity directions. .......... 4
1.3 D2Q13 lattice diagram, showing the 13 velocity directions. ...... 10
1.4 Location of boundary nodes. ........................................ 30
1.5 Boundary Conditions with unknown and known populations. ....... 31
1.6 Location of boundary nodes. ........................................ 31
1.7 Schematic illustrating the periodic boundary conditions. .......... 33
1.8 Periodic boundary condition on the inflow and outflow boundaries. . 33

2.1 Temperature profile for the unsteady reaction-diffusion problem with a variable source term at $y = 0.5$ and diffusion coefficient $\alpha = 10^{-3}$ at different times. Markers represent the Cascaded LBE results and lines represent the analytical solution. ................................. 36

2.2 Temperature profile for the unsteady reaction-diffusion problem with a variable source term at $y = 0.5$ and diffusion coefficient $\alpha = 10^{-4}$ at different times. Markers represent the Cascaded LBE results and lines represent the analytical solutions. ................................. 37

2.3 Temperature global relative error at different values of the diffusion coefficient $\alpha = 10^{-3}$, $D = 0.397$, and $\alpha = 10^{-4}$, $D = 0.3997$ for the unsteady reaction-diffusion problem with variable source term. ...... 38
2.4 Velocity profiles for thermal Couette flow in a channel with wall injection at Reynolds numbers: \( Re = 5,10,15 \). Markers represent the Cascaded LBE results and lines represent the analytical solutions.

2.5 Temperature profiles for thermal Couette flow in a channel with wall injection at Reynolds numbers: \( Re = 5,10,15 \). Markers represent the Cascaded LBE results and lines represent the analytical solutions.

2.6 Velocity relative global error of Couette flow with wall injection at \( Re = 10 \).

2.7 Temperature relative global error of Couette flow with wall injection at Reynold numbers: \( Re = 5,10,15 \).

2.8 Velocity and temperature profiles of Poiseuille flow with thermal diffusion at different values of \( Re \) and \( Pe \). Markers represent the Cascaded LBE results and lines represent the analytical solution.

2.9 Schematic illustrating the cavity boundary conditions.

2.10 Temperature profiles along horizontal centerline of the cavity at various Rayleigh numbers: \( Ra = 10^3,10^4,10^5 \), and \( 10^6 \) computed using the cascaded LBM.

2.11 Temperature profiles along the vertical centerline of the cavity flow at various Rayleigh numbers \( Ra = 10^3,10^4,10^5 \), and \( 10^6 \) computed using the cascaded LBM.

2.12 Isotherms at different values of Rayleigh numbers \( Ra = 10^3,10^4,10^5 \) and \( 10^6 \) for natural convection in a square cavity computed using the cascaded LBM.

2.13 Streamlines at different values of Rayleigh numbers \( Ra = 10^3,10^4,10^5 \) and \( 10^6 \) for natural convection in a square cavity computed using the cascaded LBM.

2.14 Vorticity contours at various Rayleigh numbers \( Ra = 10^3,10^4,10^5 \) and \( 10^6 \) for natural convection in a square cavity computed using the LBM.
2.15 Temperature profiles in Couette flow at various values of Eckert number. Markers represent the cascaded LBE simulations and lines represent the analytical solutions. .......................................................... 52

2.16 Temperature profiles in Couette flow with different values of Prandtl number. Markers represent the cascaded LBE simulations and lines represent the analytical solutions .......................................................... 52

2.17 Temperature global relative error at different Eckert numbers 7, 14, and 28 for thermal Couette flow with viscous heat dissipation. .......................................................... 53

2.18 Temperature global relative error at different Prandtl numbers 0.25, 1.25, and 2.5 for thermal Couette flow with viscous heat dissipation. .......................................................... 53

2.19 Temperature profiles in thermal Couette flow at various values of Peclet numbers: \( Pe = 10, 10^2, 10^3, 10^4, 10^5 \) and \( 10^6 \). Markers represent the Cascaded LBE results and lines represent the analytical solution. .......................................................... 55

2.20 Temperature global relative error at different Peclet numbers: \( Pe = 10, 10^2, \) and \( 10^3 \) for thermal Couette flow with viscous heat dissipation. .......................................................... 55

3.1 Distribution of the scalar variable \( \phi \) at the time \( t = t_m \) and \( u = v = 0.0 \) .......................................................... 60

3.2 Distribution of the scalar variable \( \phi \) for diffusion \( u = (0,0) \) of a Gaussian hill computed using the LBM. .......................................................... 61

3.3 Concentration contours for diffusion of a Gaussian hill \( u = (0,0) \) computed using the LBM. .......................................................... 62

3.4 Contours of the scaler variable \( \phi \) at \( u = (0.05,0.05) \) computed using LBE .......................................................... 63

3.5 Analytical Contours of the scaler variable \( \phi \) at \( u = (0.05,0.05) \) .......................................................... 64

3.6 Distribution of the scalar variable \( \phi \) at the time \( t = 3 \) and \( Pe = 100 \) .......................................................... 68

3.7 Contours of the scalar variable \( \phi \) at the time \( t = 3 \) and \( Pe = 100 \) .......................................................... 69

3.8 Temperature global relative errors at different grid sizes for diffusion of a Gaussian hill. .......................................................... 70

3.9 Temperature global relative error with variable diffusion tensor and source term. .......................................................... 70
3.10 Streamlines at different values of Rayleigh numbers $Ra = 10^{3}, 10^{4}, 10^{5}, 10^{6}, 10^{7}$ and $10^{8}$ for natural convection in a square cavity computed using the cascaded LBM. (Left) $\alpha_{x} = \alpha_{y}/2$, (center) $\alpha_{x} = \alpha_{y}$, (right) $\alpha_{x} = 2\alpha_{y}$ 

3.11 Vorticity at different values of Rayleigh numbers $Ra = 10^{3}, 10^{4}, 10^{5}, 10^{6}, 10^{7}$ and $10^{8}$ for natural convection in a square cavity computed using the cascaded LBM. (Left) $\alpha_{x} = \alpha_{y}/2$, (center) $\alpha_{x} = \alpha_{y}$, (right) $\alpha_{x} = 2\alpha_{y}$

3.12 Isothermals at different values of Rayleigh numbers $Ra = 10^{3}, 10^{4}, 10^{5}$ and $10^{6}$ for natural convection in a square cavity computed using the cascaded LBM. (Left) $\alpha_{x} = \alpha_{y}/2$, (center) $\alpha_{x} = \alpha_{y}$, (right) $\alpha_{x} = 2\alpha_{y}$

3.13 First component of velocity($u$) at different values of Rayleigh numbers $Ra = 10^{3}, 10^{4}, 10^{5}, 10^{6}, 10^{7}$ and $10^{8}$ for natural convection in a square cavity computed using the cascaded LBM. (Left) $\alpha_{x} = \alpha_{y}/2$, (center) $\alpha_{x} = \alpha_{y}$, (right) $\alpha_{x} = 2\alpha_{y}$

3.14 Second component of velocity($v$) at different values of Rayleigh numbers $Ra = 10^{3}, 10^{4}, 10^{5}, 10^{6}, 10^{7}$ and $10^{8}$ for natural convection in a square cavity computed using the cascaded LBM. (Left) $\alpha_{x} = \alpha_{y}/2$, (center) $\alpha_{x} = \alpha_{y}$, (right) $\alpha_{x} = 2\alpha_{y}$

3.15 Pressure at different values of Rayleigh numbers $Ra = 10^{3}, 10^{4}, 10^{5}, 10^{6}, 10^{7}$ and $10^{8}$ for natural convection in a square cavity computed using the cascaded LBM. (Left) $\alpha_{x} = \alpha_{y}/2$, (center) $\alpha_{x} = \alpha_{y}$, (right) $\alpha_{x} = 2\alpha_{y}$

3.16 Temperature profiles along horizontal centerline of the cavity at various Rayleigh numbers: $Ra = 10^{3}, 10^{4}, 10^{5}, 10^{6}, 10^{7}$, and $10^{8}$ computed using the cascaded LBM.

3.17 Temperature profiles along the vertical centerline of the cavity flow at various Rayleigh numbers $Ra = 10^{3}, 10^{4}, 10^{5}, 10^{6}, 10^{7}$, and $10^{8}$ computed using the cascaded LBM.

4.1 Schematic illustrating the problem configuration.
4.2 Temperature profiles for velocity driven flow at various Brinkman numbers: $Br = 3.43, 5, 10, \text{ and } 20$. Markers represent the Cascaded LBE results and lines represent the analytical solution. . . . . . . . . . . . . . . 94

4.3 Velocity profiles for velocity driven flow at various Brinkman numbers: $Br = 3.43, 10, \text{ and } 20$. Markers represent the Cascaded LBE results and lines represent the analytical solution. . . . . . . . . . . . . . . 94
**LIST OF ABBREVIATIONS**

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>LBE</td>
<td>Lattice Boltzmann Equation</td>
</tr>
<tr>
<td>LBM</td>
<td>Lattice Boltzmann Method</td>
</tr>
<tr>
<td>LBM</td>
<td>Lattice Boltzmann Model</td>
</tr>
<tr>
<td>CDE</td>
<td>Convection Diffusion Equation</td>
</tr>
<tr>
<td>NSE</td>
<td>Navier Stokes Equation</td>
</tr>
<tr>
<td>SRT</td>
<td>Single Relaxation time</td>
</tr>
<tr>
<td>MRT</td>
<td>Multiple Relaxation time</td>
</tr>
<tr>
<td>BGK</td>
<td>Bhatnagar- Gross- Krook</td>
</tr>
<tr>
<td>DDF</td>
<td>Double Distribution Function</td>
</tr>
<tr>
<td>MSLB</td>
<td>Multi Speed Lattice Boltzmann</td>
</tr>
<tr>
<td>HybLB</td>
<td>Hybrid Lattice Boltzmann</td>
</tr>
<tr>
<td>D2Q9</td>
<td>Discretization of 2 dimention and 9 velocity</td>
</tr>
</tbody>
</table>
### LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>α</td>
<td>Discrete diffusion coefficient</td>
</tr>
<tr>
<td>α</td>
<td>Particle velocity direction</td>
</tr>
<tr>
<td>τ</td>
<td>Non dimensional relaxation time for density</td>
</tr>
<tr>
<td>τ_\text{g}</td>
<td>Non dimensional relaxation time for energy</td>
</tr>
<tr>
<td>τ_\text{f}</td>
<td>Non dimensional relaxation time for fluid</td>
</tr>
<tr>
<td>t</td>
<td>time</td>
</tr>
<tr>
<td>T</td>
<td>Non dimensional time</td>
</tr>
<tr>
<td>ρ</td>
<td>Density</td>
</tr>
<tr>
<td>P</td>
<td>Pressure</td>
</tr>
<tr>
<td>δ_\text{t}</td>
<td>Time step</td>
</tr>
<tr>
<td>δ_\text{x}</td>
<td>Spacing step</td>
</tr>
<tr>
<td>λ</td>
<td>Relaxation time parameter for temperature field</td>
</tr>
<tr>
<td>ω</td>
<td>Relaxation time parameter for velocity field</td>
</tr>
<tr>
<td>f_\alpha</td>
<td>discrete particle density distribution function for velocity field</td>
</tr>
<tr>
<td>g_\alpha</td>
<td>discrete particle density distribution function for temperature field</td>
</tr>
<tr>
<td>f_\alpha^{eq}</td>
<td>Equilibrium particle distribution function for velocity field</td>
</tr>
<tr>
<td>g_\alpha^{eq}</td>
<td>Equilibrium particle distribution function for temperature field</td>
</tr>
<tr>
<td>S_\alpha</td>
<td>source term contribution from internal and external forces</td>
</tr>
<tr>
<td>e_\alpha</td>
<td>Discrete particle velocity</td>
</tr>
<tr>
<td>c</td>
<td>Particle speed</td>
</tr>
<tr>
<td>C_\text{s}</td>
<td>Speed of sound</td>
</tr>
<tr>
<td>ξ</td>
<td>Particle velocity</td>
</tr>
</tbody>
</table>
$T$ Temperature

$\mu$ Dynamic Viscosity

$\mu_0$ Typical Dynamic

$\nu$ kinematic viscosity

$\epsilon$ Asymptotic expansion parameter in the Chapman-Enskog analysis

$\Lambda$ Collision matrix in moment space for velocity field

$\hat{\Lambda}$ Collision matrix in moment space for temperature field

$\Delta T$ Temperature difference

$\mathbf{S}$ Strain rate tensor

$Ma$ Mach number

$\mathbf{U}$ Fluid velocity

$u$ Cartesian component of velocity in the x-direction

$v$ Cartesian component of velocity in the y-direction

$x$ Cartesian component of position vector

$y$ Cartesian component of position vector

$B_r$ Brinkman number

$Pe$ Peclet number

$Re$ Reynold number

$I$ Identity matrix

$L$ Characteristic length

$\Omega_\alpha$ Discrete collision term in the LBE

$\omega_\alpha$ The Lattice weights

$\hat{\mathbf{g}}^{eq}$ The Lattice
CHAPTER I

THERMAL LATTICE BOLTZMANN MODEL

1.1 Introduction

Heat and mass transfer coupled with fluid flows is a widespread phenomena has many applications in engineering, energy, chemical reaction, geology etc. These phenomena are mathematically described by convection-diffusion equations and the Navier-Stokes equation. The Lattice Boltzmann method is a particle based method which when compared to traditional computational fluid dynamic methods has some distinct advantages such as ease of implementation of fully parallel algorithms, the capability to handle the complex boundary conditions, flexible and easy to modify the scheme and very fast with hardware acceleration (fast solver for NSE). The Lattice Boltzmann method offers a great potential for including kinetic and atomistic details into the computational models. In fact, this originates from the main purpose of LBM, which is to numerically solve the Lattice Boltzmann equation. This equation can describe the distribution of particles of a system in phase space at any thermodynamic state. The Lattice Boltzmann method does not solve the hydrodynamic and non-hydrodynamic conservation equation but rather models the streaming and collision for particles, i.e. relaxation toward local equilibrium. In other words, the idea of LBM is to construct simplified discrete microscopic dynamics to simulate the macroscopic model described by partial differential equations that model a physical system.

In this chapter, a definition of the Lattice Boltzmann method as a discrete
method and the concept of collision and streaming within the method are provided in section (1.2), then a brief description for each LBE collision model is given with the explanation of the main difference among these models in subsections (1.2.1), (1.2.2) and (1.2.3). Next, thermal flow approaches that are used to solve LBE are introduced in section (1.3) with a brief description for each approach.

Finally, the construction of the cascaded LB model for the evolution of the scalar field represented by the convection-diffusion equation (CDE) with a source term is provided in details in Sections (1.4) and (1.5)

1.2 Collision Models

The micro-dynamics of gases and fluids consists of the repetition of two processes collision and propagation (streaming). The macroscopic values of temperature, mass and momentum density are then calculated by a mean values over large spatial regions with thousands of nodes. The Lattice Boltzmann method as a kinetic based numerical method is designed to idealize the microscopic description that allows the recovery of the desired macroscopic equations throughout the artificial lattice (Boltzmann lattice). The movement of particles in the Lattice Boltzmann Method is assumed to be discrete in time and space, i.e. a set of directions with given velocities within the method is considered. Here, the collision of particles happens locally at a single node after a given time step. During the collision the particles are assumed to conserve density and momentum for isothermal flow and also the concentration or temperature for the convection diffusion flow then the particles are strictly streamed to their neighbors according to the allowed directions. Collisions in the LB models are typically represented by $\Omega(x,t)$, in which the particle distribution or moment approaches its equilibrium values over a characteristic time (relaxation time $\tau$). In other words, as the relaxation time constants determine the microscopic dynamics towards the local equilibrium, it also determines the macroscopic transport coefficients. The effectiveness of the collision operator determines the time to reach
Figure 1.1: D2Q9 lattice, Collision (right) and Streaming (left).

the equilibrium configuration by choosing a proper set of relaxation time constants and at the same time it keeps the numerical scheme stable. The LBE has several models, which differ from each other by the way they handle the collision step. In the following, a brief description of the most popular Lattice Boltzmann collision models is provided.

1.2.1 Single Relaxation Time (SRT) LBM

SRT LBM is a single relaxation time Lattice Boltzmann model contains with a simplified collision term. In fact, the Bhatragar-Gross-Krook (BGK) is a first form of the collision term introduced in [1] then extended to Lattice Boltzmann in [2, 3] which resulted in a considerable simplification of the LBE. The method has since been applied in the context of LBE by several authors for isothermal flow [4, 6, 7, 9, 17]. The SRT Lattice Boltzmann model was first introduced to simulate incompressible fluid flow more than two decades ago, then it has been widely employed to study thermal flow [14, 20, 21, 23, 51] because of its simplicity in structure. In this model, both collision and streaming processes are executed in phase space. We consider the D2Q9 SRT Lattice Boltzmann model. The evaluation equation of the SRT model for thermal flow is
Figure 1.2: D2Q9 lattice diagram, showing the 9 velocity directions.

\[
\mathbf{g}_\alpha(x + \mathbf{e}_\alpha \delta t, t + \delta t) - \mathbf{g}_\alpha(x, t) = \Omega(x, t), \quad (1.1)
\]

where \( \mathbf{g}_\alpha \) is the distribution function, \( \Omega(x, t) = -\frac{1}{\tau} [\mathbf{g}_\alpha(x, t) - g_{\alpha eq}(x, t)] \) is the collision operator, \( S_\alpha \) is the discrete source term. The equilibrium distribution function \( g_{\alpha eq} \) is given by

\[
g_{\alpha eq} = \omega_\alpha T \left[ 1 + \frac{e_{\alpha u}}{c_s^2} + \frac{(e_{\alpha u})^2}{2c_s^4} - \frac{u \cdot u}{2c_s^2} \right] \quad (1.2)
\]

The Lattice weights \( \omega_\alpha \) and the nine discrete velocities are given as

\[
\omega_\alpha = \begin{cases} 
\frac{4}{9} & \mathbf{e}_{\alpha}^2 = (0, 0), \alpha = 0 \\
\frac{1}{9} & \mathbf{e}_{\alpha}^2 = (\pm 1, 0), (0, \pm 1), \alpha = 1, 2, 3, 4 \\
\frac{1}{56} & \mathbf{e}_{\alpha}^2 = (\pm 1, \pm 1), \alpha = 5, 6, 7, 8 
\end{cases} \quad (1.3)
\]

where \( c_s \) is the lattice speed of sound is given by \( c_s^2 = \frac{c^2}{3} \). Typically for D2Q9 model, \( c_s^2 = 1/3 \), \( c \) is the lattice speed and is given by \( c = \frac{\Delta x}{\Delta t} \). Here, \( \Delta x = c_s \Delta t = 1, \Delta t = 1 \).

The temperature can be obtained in terms of transformed distribution function as
following

\[ T = \sum_{a=0}^{8} g_{a}^{\alpha}, \quad (1.4) \]

Through the Chapman-Enskog multi-scale expansion, the thermal diffusivity is found as:

\[ \alpha = \frac{1}{3} \left( \frac{1}{\lambda} - \frac{1}{2} \right) \quad (1.5) \]

where \( \lambda = 1/\tau \).

The shortcomings of BGK model are apparent in that the model suffers numerical instability due to the single relaxation time for all modes, i.e. All modes are relaxed on the same rate which leads to numerical instability at relatively small thermal diffusivity or at small viscosity for fluid flow. In addition, the Prandtl number is constant when we apply it to thermal flow and this model does not have sufficient parameters (one relaxation time \( \tau \) only) to describe anisotropic diffusion.

1.2.2 Multiple Relaxation Times (MRT) LBM

Here, different moments can relax at different rates, the collision process is mapped onto the raw moment space (fixed frame of reference) throughout an orthogonal transformation matrix while the streaming process is still executed in phase space. Certain Relaxation times can represent the hydrodynamic and non-hydrodynamic moments i.e. for temperature field, \( \lambda_{3} \) and \( \lambda_{5} \) are represent the thermal diffusivity (diffusion coefficient). [41, 46, 50, 51, 53, 55, 56].

\[ \alpha = \frac{1}{3} \left( \frac{1}{\lambda_{3,5}} - \frac{1}{2} \right) \quad (1.6) \]

The governing equation of the MRT model

\[ \overline{g}_{\alpha}(\vec{x} + \vec{e}_{\alpha} \delta_{t}, t + \delta_{t}) - \overline{g}_{\alpha}(\vec{x}, t + \delta_{t}) = \Omega(\vec{x}, t) + S_{\alpha}(\vec{x}, t). \quad (1.7) \]
Where $\Omega(x, t) = -M^{-1} \tilde{\Lambda} M \left[ \bar{g}_\alpha(\vec{x}, t) - g^{eq}_\alpha(\vec{x}, t) \right]$, $\tilde{\Lambda}$ is a diagonal collision matrix given by

$$\tilde{\Lambda} = diag(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_8).$$  

(1.8)

Where $M$ is the orthogonal matrix that is used to transform from velocity space to moment space as follows

$$\hat{\mathbf{g}} = M \mathbf{g}, \quad \hat{\mathbf{g}}^{eq} = M \mathbf{g}^{eq}, \quad \hat{\mathbf{S}} = M \mathbf{S},$$

(1.9)

The MRT LBE evaluation equation for temperature field Eq. (1.7) can be written in terms of moment space as

$$
\hat{\mathbf{g}}_\alpha(\vec{x} + \vec{e}_\alpha \delta t, t + \delta t) - \hat{\mathbf{g}}_\alpha(\vec{x}, t) = M^{-1} \left[ -\tilde{\Lambda} (\hat{\mathbf{g}} - \hat{\mathbf{g}}^{eq}) + \left( \mathcal{T} - \frac{1}{2} \tilde{\Lambda} \right) \hat{\mathbf{S}} \right] 
$$

(1.10)

The equilibrium distribution function for temperature $g^{eq}_\alpha$ is taken as

$$g^{eq}_\alpha = w_\alpha T \left[ 1 + \frac{e_{\alpha.u}^2}{c^2} + \frac{(e_{\alpha.u})^2}{2c^4} - \frac{u.u}{2c^2} \right]$$

(1.11)

$M$ is the orthogonal set of basis vectors matrix given by

$$M = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-4 & -1 & -1 & -1 & -1 & 2 & 2 & 2 \\
4 & -2 & -2 & -2 & -2 & 1 & 1 & 1 \\
0 & 1 & 0 & -1 & 0 & -1 & 1 & 1 \\
0 & -2 & 0 & 2 & 0 & 1 & -1 & -1 \\
0 & 0 & 1 & 0 & -1 & 1 & -1 & -1 \\
0 & 0 & -2 & 0 & 2 & 1 & 1 & -1 \\
0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\
\end{bmatrix}.$$  

(1.12)
It has been widely proved and accepted that the MRT LBM significantly improved the accuracy and numerical stability of Lattice Boltzmann schemes. In addition, the different relaxation time constants in MRT model are sufficient to cover the anisotropic diffusion coefficient tensor.

### 1.2.3 Cascaded LBE

The cascaded Lattice Boltzmann model is a multiple relaxation time model based on using central moments. Central moments are the moments that are obtained by shifting the partial velocity by the local fluid velocity i.e. the moments are calculated in a moving frame of reference while the moments in MRT LBM are computed in the frame of reference at rest and the collision process is mapped onto the raw moment. Here, no regular equilibrium distribution function is used, the equilibrium distribution function \( g^{MT} \) is obtained by making an analogy with Maxwell-Boltzmann distribution function \( f^M(\rho, \bar{u}, \bar{\xi}) \) in the continuous velocity space \( \bar{\xi} \) by replacing the density \( \rho \) with temperature \( T \) in our DDF formulation. That is,

\[
g^{MT} = \frac{T}{\rho} f^M(\rho, \bar{u}, \bar{\xi}) \quad \text{where} \quad f^M(\rho, \bar{u}, \bar{\xi}) = \frac{\rho}{2\pi c_s^2} \exp \left[ -\frac{(\bar{\xi} - \bar{u})^2}{2c_s^2} \right],
\]

The collision process is also mapped onto the central moment space throughout an orthogonal transformation matrix \( T \). The governing equation of the cascaded model

\[
\bar{g}_\alpha(x + \bar{\xi}_\alpha \delta t, t + \delta t) - \bar{g}_\alpha(x, t) = \Omega(\bar{x}, t) + S_\alpha(\bar{x}, t).
\]

(1.13)

Where \( \Omega_\alpha \) can be represented as \( \Omega_\alpha = (\mathbf{K} \cdot \hat{h})_\alpha \), \( \mathbf{h} \) the collision kernel \( \hat{\mathbf{A}} \) is a diagonal collision matrix given by

\[
\hat{\mathbf{A}} = \text{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_8).
\]

(1.14)
\( \mathcal{K} \) is the orthogonal set of basis vectors matrix given by

\[
\mathcal{K} = \begin{bmatrix}
1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 4 \\
1 & 1 & 0 & -1 & 1 & 0 & 0 & 2 & -2 \\
1 & 0 & 1 & -1 & -1 & 0 & 2 & 0 & -2 \\
1 & -1 & 0 & -1 & 1 & 0 & 0 & -2 & -2 \\
1 & 0 & -1 & -1 & -1 & 0 & -2 & 0 & -2 \\
1 & 1 & 1 & 2 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & -2 & 0 & 2 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 2 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 2 & 0 & -1 & 1 & -1 & 1
\end{bmatrix}.
\] (1.15)

Here, \( \lambda_1 \) and \( \lambda_2 \) represent the thermal diffusivity

\[
\alpha = \frac{1}{3} \left( \frac{1}{\lambda_{1,2}} - \frac{1}{2} \right)
\] (1.16)

The cascaded Lattice Boltzmann model is more stable than both SRT and MRT models and more Galilean invariant than the rest frame of reference models.

All the steps for the construction of the cascaded collision operator for the temperature field are provided in sec (1.4)

### 1.3 Thermal flow approaches

To capture or simulate thermal flow, the solution of the temperature field, whose evolution is represented by a convection-diffusion equation (CDE), for energy transport, coupled to the fluid velocity, which is represented by the Navier-Stokes (NS) equations. Classical numerical methods can become challenging to apply for the simulation of such flows, especially in the complex geometrics such as thermal flows in porous media. Within the LB framework, broadly, three different approaches that have been developed to construct the thermal (LBE) equation models. In all of these
methods, a distribution function \( f \) is applied to simulate the velocity field. The energy field is then simulated via either:

1) Multispeed approach (MS-LBE)
2) Hybrid approach
3) Double distribution functions (DDF)

1.3.1 Multispeed approach (MSLBE)

In this method, a distribution function \( f \) is applied to simulate both the velocity and energy field, i.e. using the same distribution function to solve both the NSE equations and CDE, this requires a large velocity set for the Lattice in order to recover the energy equation. The multi-speed Lattice Boltzmann is first introduced by [32]. The authors used thirteen velocities for two dimensions, LBE for fluid needs nine velocities and the extra four velocities to recover the energy equation, this leads to higher order moments than the regular LBE to account for the additional equation for the conservation of energy. In addition, complex terms and calculations due to the large set of velocities considered lead to numerical instability. Different authors later tried to implement this model with some modification to stabilize the method [33, 34, 35, 36, 47]. All these models are multi-speed models and suffer from numerical instability, are restricted to a limited temperature range and the treatment of boundary conditions with large number of discrete velocities sets is difficult.

1.3.2 Hybrid approach (HybLBE)

In the Hybrid approach, the LBE method is used to solve the flow field while any conventional numerical method such as finite difference or finite element scheme is used to solve the energy equation [31, 37, 38, 39]. The hybrid approach is more stable than multi-speed approach. However, this approach has the disadvantage that the simplicity of LBE is lost.
1.3.3 Double distribution functions (DDfLBE)

In the DDF approach, two separate distributions functions are employed, one for the flow field and the other one for the temperature field [28, 17, 45, 56]. Unlike the above two models, many of such limitations can be overcome by the DDF models [30, 49] and they have, hence, received significantly more attention recently. Most of the developments related to the DDF-LBE methods considered single relaxation time (SRT) collision [65]. Various aspects relevant to the LBE for correct representation of the CDE for the temperature field were identified. For example, the choice of the equilibrium distribution function with nonlinear velocity terms was important in this regard [23]. Concurrently, various boundary condition schemes for the DDF-LBE were developed [18, 46, 48, 53, 54, 60]. However, the use of SRT collision models, though simple in structure and characterized by the relaxation of all models at the same rate, is known to suffer from instability issues, particularly where the transport coefficients such as the fluid viscosity and thermal diffusivity become relatively small. This limits the ability to reach higher Reynolds or Peclet numbers. One possibility to address this issue is to consider using multi-relaxation time (MRT) models for DDF-LBE approach [50, 41, 46, 53, 55, 56]. In the MRT model,
the collision process is mapped onto the raw moment space through an orthogonal
transformation matrix, where different moments can relax at different rates. By
representing the relaxation times of the hydrodynamic and non-hydrodynamic
moments, the stability of the MRT model can be significantly improved. A further
improvement is to consider another type of MRT model, in which the collision step is
executed in terms of the relaxation of central moments, in which the particle
velocities are shifted by the local fluid velocity. Such type of collision model in a
moving frame of reference leads to a cascaded structure of the higher order moments
in terms of those at lower order following collision and hence is referred to as the
cascaded LB method [22].

1.4 Construction of the Cascaded Collision Operator for the Temperature Field

Our main goal in this investigation is to construct a cascaded LB model for the
evolution of the temperature field represented by the following convection-diffusion
equation (CDE) with a source term

\[
\frac{\partial T}{\partial t} + u \cdot \nabla T = \nabla \cdot (\alpha \nabla T) + G
\]  

(1.17)

where \( \alpha \) is the thermal diffusivity coefficient, \( T = T(x,y,t) \) and \( u = u(x,y,t) \) are local
temperature and velocity field, respectively. In addition \( G = G(x,y,t) \) is the local
source term, for example, due to internal heat generation or viscous dissipation. In
general, thermal transport can be significantly influenced by the presence of internal
heat generation, such as those related to nuclear or chemical reactions generating
local heating. Viscous heating effects due to shear stresses is another example. All of
these effects can be represented as a prescribed local source term \( G = G(x,y,t) \) in the
thermal transport equation. To handle such a general case, here we develop a new
cascaded LB model with a source term, which can recover the macroscopic equation
represented by CDE given in Eq. (1.17) above with second order accuracy. In
Eq. (1.17), the local velocity field $u = u(x, y, t)$ satisfied the Navies-Stokes equations (NSE) given by

$$\nabla \cdot u = 0, \quad (1.18a)$$

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho} \nabla P + \nu \nabla^2 u + F \quad (1.18b)$$

Where $p$ is the pressure of the fluid flow, $\nu$ is the kinematic viscosity of the fluid, $\rho$ is the reference density and $F = \rho a$ is the local external force field. The velocity field $u$ to be used in Eq. (1.17) is considered to be known, and can be obtained by solving another cascaded LBE constructed in previous work [22, 44]. In particular, the specific cascaded LBE with forcing term for obtaining the velocity field $u$ can be coupled to the new cascaded for the CDE to be developed in this work. In such a double distribution function DDF formulation, we refer the reader to the cascaded LBE with forcing term for the flow field presented in to [44] to maintain brevity and focus here on the construction of the cascaded LBE to solve for the temperature field $T = T(x, y, t)$, whose evolution is represented by Eq. (1.17)

The overall procedure to develop a thermal cascaded LBE involves the following: (i) prescribe a suitable choice of an orthogonal moment basis for the lattice velocity sets, (ii) specify formulations for the continuous central moments of equilibrium and source term and equate them to the corresponding discrete central moments involved in the cascaded LBE for the CDE, (iii) Transform the various discrete central moments in terms of various corresponding raw moments by using the binomial theorem, (iv) construct the collision kernel appearing in the cascaded collision operator for solving the CDE and the source term in the LB model.

First, we select a suitable moment basis for the two-dimensional, nine velocity (D2Q9) lattice. We consider the usual “bra” and the “ket” notations, i.e. $\langle . |$ and $| . \rangle$ to denote 9-dimensional row and column vectors, respectively. Then, we obtain
the following nine non-orthogonal basis vectors obtained from monomials \( e_{ax}^m e_{ay}^n \) at successively increasing orders:

\[
\mathcal{T} = \begin{bmatrix}
|1\rangle, |e_{ax}\rangle, |e_{ay}\rangle, |e_{ax}^2 + e_{ay}^2\rangle, |e_{ax}^2 - e_{ay}^2\rangle, |e_{ax} e_{ay}\rangle, |e_{ax} e_{ay}^2\rangle, |e_{ax}^2 e_{ay}\rangle
\end{bmatrix},
\]

where

\[
|T\rangle = (1, 1, 1, 1, 1, 1, 1, 1, 1)^T,
\]

\[
|e_{ax}\rangle = (0, 1, 0, -1, 0, 1, -1, 1, 1)^T,
\]

\[
|e_{ay}\rangle = (0, 0, 1, 0, -1, 1, 1, -1, 1)^T.
\]

The above nominal set of basis vectors is then transformed into an equivalent orthogonal set of basis vectors by means of the standard Gram-Schmidt procedure arranged in the increasing order of moments:

\[
|K_0\rangle = |1\rangle, \quad |K_1\rangle = |e_{ax}\rangle, \quad |K_2\rangle = |e_{ay}\rangle,
\]

\[
|K_3\rangle = 3 |e_{ax}^2 + e_{ay}^2\rangle - 4 |1\rangle, \quad |K_4\rangle = |e_{ax}^2 - e_{ay}^2\rangle, \quad |K_5\rangle = |e_{ax} e_{ay}\rangle,
\]

\[
|K_6\rangle = -3 |e_{ax} e_{ay}\rangle + 2 |e_{ay}\rangle, \quad |K_7\rangle = -3 |e_{ax} e_{ay}^2\rangle + 2 |e_{ax}\rangle,
\]

\[
|K_8\rangle = 9 |e_{ax} e_{ay}^2\rangle - 6 |e_{ax}^2 + e_{ay}^2\rangle + 4 |1\rangle.
\]

By grouping the above, set of vectors, we obtain an orthogonal transformation matrix \( \mathcal{K} \) as

\[
\mathcal{K} = [|K_0\rangle, |K_1\rangle, |K_2\rangle, |K_3\rangle, |K_4\rangle, |K_5\rangle, |K_6\rangle, |K_7\rangle, |K_8\rangle],
\]

(1.20)
which can be explicitly written as

\[
\mathcal{K} = \begin{bmatrix}
1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 4 \\
1 & 1 & 0 & -1 & 1 & 0 & 0 & 2 & -2 \\
1 & 0 & 1 & -1 & -1 & 0 & 2 & 0 & -2 \\
1 & -1 & 0 & -1 & 1 & 0 & 0 & -2 & -2 \\
1 & 0 & -1 & -1 & -1 & 0 & -2 & 0 & -2 \\
1 & 1 & 1 & 2 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & -2 & 0 & 2 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 2 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 2 & 0 & -1 & 1 & -1 & 1 \\
\end{bmatrix} .
\] (1.21)

Next, in order to construct a cascaded LB collision operator for representing the evaluation of the temperature field, we need to present the continuous moments of the equilibrium state and the source term. The continuous equilibrium central moments of order \((m + n)\) can be defined as

\[
\hat{\Pi}_{xmyn}^{eq} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^{MT}(\xi_x - u_x)^m(\xi_y - u_y)^n d\xi_x d\xi_y, \text{ which yields}
\]

\[
|\hat{\Pi}_{xmyn}^{eq}| = (\hat{\Pi}_{0}^{eq}, \hat{\Pi}_{1}^{eq}, \hat{\Pi}_{2}^{eq}, \hat{\Pi}_{1x}^{eq}, \hat{\Pi}_{1y}^{eq}, \hat{\Pi}_{2x}^{eq}, \hat{\Pi}_{x2}^{eq}, \hat{\Pi}_{y2}^{eq}, \hat{\Pi}_{xxy}^{eq}, \hat{\Pi}_{xyy}^{eq}, \hat{\Pi}_{xxyy}^{eq})^T,
\]

\[= (T, 0, 0, c_s^2 T, c_s^2 T, 0, 0, 0, c_s^4 T)^T. \] (1.22)

Here, the equilibrium distribution function \(g^{MT}\) is obtained by making an analogy with Maxwell-Boltzmann distribution function \(f^{M}(\rho, \bar{u}, \bar{\xi})\) in the continuous velocity space \(\bar{\xi}\) by replacing the density \(\rho\) with temperature \(T\) in our DDF formulation. That is,

\[
g^{MT} = \frac{T}{\rho} f^{M}(\rho, \bar{u}, \bar{\xi}) \text{ where } f^{M}(\rho, \bar{u}, \bar{\xi}) = \frac{\rho}{2 \pi c_s^2} \exp \left[-\frac{(\bar{\xi} - \bar{u})^2}{2 c_s^2}\right], \text{ where } c_s \text{ is the lattice speed of sound. Typically for D2Q9 model, } c_s^2 = 1/3.
\]
Similarly, defining the continuous central moments of the source term of order 
\((m + n)\) due to \(G = G(x, y, t)\) appearing in Eq. (1.17) as
\[
\hat{\Gamma}_{x^m y^n} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Delta g^G(\xi_x - u_x)^m(\xi_y - u_y)^n d\xi_x d\xi_y
\]
(1.23)
where \(\Delta g^G\) is the change in the distribution function due to source \(G\). Since the source \(G\) is expected influence only the lowest order (zeroth) moment, we can prescribed the following ansatz:
\[
|\hat{\Gamma}_{x^m y^n}\rangle = (\hat{\Gamma}_0, \hat{\Gamma}_x, \hat{\Gamma}_y, \hat{\Gamma}_{xx}, \hat{\Gamma}_{xy}, \hat{\Gamma}_{yy}, \hat{\Gamma}_{xxy}, \hat{\Gamma}_{xyy})^T,
\]
(1.24)
By using the above central moments our goal is to develop the collision operator and the source term of the cascaded LBE. For representing the transport of the temperature field the corresponding cascaded LBE using the trapezoidal rule to evaluate the source term of cascaded LBE to maintain the second order accuracy can be written as:
\[
g_\alpha(\vec{x} + \vec{e}_\alpha \delta t, t + \delta t) = g_\alpha(\vec{x}, t) + \Omega_\alpha(\vec{x}, t) + \frac{1}{2} \left[ S_\alpha(\vec{x}, t) + S_\alpha(\vec{x} + \vec{e}_\alpha, t + \delta t) \right].
\]
(1.25)
Here, the collision term \(\Omega_\alpha\) can be represented as \(\Omega_\alpha \equiv \Omega_\alpha(g, \hat{h}) = (K \cdot \hat{h})_{tt}\), where \(g \equiv |g_\alpha\rangle = (g_0, g_1, \cdots, g_8)^T\) is the vector of distribution functions and \(\hat{h} \equiv |\hat{h}_\alpha\rangle = (\hat{h}_0, \hat{h}_1, \cdots, \hat{h}_8)^T\) is the vector of unknown collision kernel to be obtained later. The discrete form of the source term \(S_\alpha\) in the cascaded LBE given above represents the influence of the source \(G\) in the velocity space and is defined as \(S \equiv |S_\alpha\rangle = (S_0, S_1, S_2, \cdots, S_8)^T\). Noting that Eq. (1.25) is semi-implicit, by using the standard variable transformation \(\bar{g}_\alpha = g_\alpha - \frac{1}{2} S_\alpha\) [13], the implicitness can be effectively removed. This yields
\[
\bar{g}_\alpha(\vec{x} + \vec{e}_\alpha \delta t, t + \delta t) = \bar{g}_\alpha(\vec{x}, t) + \Omega_\alpha(\vec{x}, t) + S_\alpha(\vec{x}, t).
\]
(1.26)
which can maintain second order accuracy in an effectively time explicit method.

In order to obtain the expressions for the structure of the cascaded collision operator $\hat{\mathbf{h}}$ and the source terms $\mathbf{S}_\alpha$ in the presence of a spatially and/or temporally source $G$, i.e. $G = G(x, y, t)$, we define the following set of discrete central moments.

$$\hat{\kappa}_{x^m y^n} = \sum_\alpha g_\alpha (e_{ax} - u_x)^m (e_{ay} - u_y)^n,$$

(1.27a)

$$\hat{\kappa}^{eq}_{x^m y^n} = \sum_\alpha g^{eq}_\alpha (e_{ax} - u_x)^m (e_{ay} - u_y)^n.$$

(1.27b)

$$\hat{\kappa}_{x^m y^n} = \sum_\alpha \bar{g}_\alpha (e_{ax} - u_x)^m (e_{ay} - u_y)^n.$$

(1.27c)

$$\hat{\kappa}^{eq}_{x^m y^n} = \sum_\alpha \bar{g}^{eq}_\alpha (e_{ax} - u_x)^m (e_{ay} - u_y)^n,$$

(1.27d)

$$\hat{\sigma}_{x^m y^n} = \sum_\alpha S_\alpha (e_{ax} - u_x)^m (e_{ay} - u_y)^n.$$  

(1.27e)

We then match the discrete central moments of the distribution functions and source terms with the corresponding continuous central moments at each order, i.e.

$$\hat{\kappa}^{eq}_{x^m y^n} = \hat{\Pi}^M_{x^m y^n},$$

(1.28a)

$$\hat{\sigma}_{x^m y^n} = \hat{\Pi}^F_{x^m y^n}.$$  

(1.28b)

Thus, we obtain

$$\langle \hat{\kappa}^{eq}_{x^m y^n} \rangle = (\hat{\kappa}^{eq}_0, \hat{\kappa}^{eq}_x, \hat{\kappa}^{eq}_y, \hat{\kappa}^{eq}_xx, \hat{\kappa}^{eq}_xy, \hat{\kappa}^{eq}_yy, \hat{\kappa}^{eq}_xxy, \hat{\kappa}^{eq}_xyy, \hat{\kappa}^{eq}_xxyy)^T,$$

(1.29)

$$= (T, 0, 0, c_s^2 T, c_s^2 T, 0, 0, 0, c_s^4 T)^T.$$

$$\langle \hat{\sigma}_{x^m y^n} \rangle = (\hat{\sigma}_0, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_{xx}, \hat{\sigma}_{xy}, \hat{\sigma}_{yy}, \hat{\sigma}_{xxy}, \hat{\sigma}_{xyy}, \hat{\sigma}_{xxyy})^T,$$

(1.30)

$$= (G, 0, 0, 0, 0, 0, 0, 0, 0)^T.$$  

Since the actual computations in the cascaded LBE are performed in terms of the various raw moments, we define the following set of discrete raw moments (denoted...
with a prime symbol):

\[
\hat{\kappa}'_{x^m y^n} = \sum_{\alpha} g_\alpha (e_{ax})^m (e_{ay})^n, \quad (1.31a)
\]

\[
\hat{\kappa}'_{x^m y^n}^{eq} = \sum_{\alpha} g_{eq}^\alpha (e_{ax})^m (e_{ay})^n, \quad (1.31b)
\]

\[
\hat{\kappa}'_{x^m y^n} = \sum_{\alpha} \tilde{g}_\alpha (e_{ax})^m (e_{ay})^n. \quad (1.31c)
\]

\[
\hat{\kappa}'_{x^m y^n}^{eq} = \sum_{\alpha} \tilde{g}_{eq}^\alpha (e_{ax})^m (e_{ay})^n, \quad (1.31d)
\]

\[
\hat{\sigma}'_{x^m y^n} = \sum_{\alpha} \sigma_\alpha (e_{ax})^m (e_{ay})^n. \quad (1.31e)
\]

By using Eqs. (1.27e), (1.30), (1.31e) and the binomial theorem, we obtain the following sets of discrete raw moments for the source term at different orders:

\[
\hat{\sigma}'_0 = \langle S_\alpha | T \rangle = \sum_{\alpha} S_\alpha = G, \quad (1.32a)
\]

\[
\hat{\sigma}'_x = \langle S_\alpha | e_{ax} \rangle = \sum_{\alpha} S_\alpha e_{ax} = u_x G, \quad (1.32b)
\]

\[
\hat{\sigma}'_y = \langle S_\alpha | e_{ay} \rangle = \sum_{\alpha} S_\alpha e_{ay} = u_y G, \quad (1.32c)
\]

\[
\hat{\sigma}'_{xx} = \langle S_\alpha | e_{ax}^2 \rangle = \sum_{\alpha} S_\alpha e_{ax}^2 = u_x^2 G, \quad (1.32d)
\]

\[
\hat{\sigma}'_{yy} = \langle S_\alpha | e_{ay}^2 \rangle = \sum_{\alpha} S_\alpha e_{ay}^2 = u_y^2 G, \quad (1.32e)
\]

\[
\hat{\sigma}'_{xy} = \langle S_\alpha | e_{ax} e_{ay} \rangle = \sum_{\alpha} S_\alpha e_{ax} e_{ay} = u_x u_y G, \quad (1.32f)
\]

\[
\hat{\sigma}'_{xy} = \langle S_\alpha | e_{ax}^2 e_{ay} \rangle = \sum_{\alpha} S_\alpha e_{ax}^2 e_{ay} = u_x^2 u_y G, \quad (1.32g)
\]

\[
\hat{\sigma}'_{xy} = \langle S_\alpha | e_{ax} e_{ay}^2 \rangle = \sum_{\alpha} S_\alpha e_{ax} e_{ay}^2 = u_x u_y^2 G, \quad (1.32h)
\]

\[
\hat{\sigma}'_{xyxy} = \langle S_\alpha | e_{ax}^2 e_{ay}^2 \rangle = \sum_{\alpha} S_\alpha e_{ax}^2 e_{ay}^2 = u_x^2 u_y^2 G. \quad (1.32i)
\]

In order to obtain the source terms in the velocity space, we first compute the source moments projected to the orthogonal moment space, i.e. \( \hat{m}_\beta = \langle K_\beta | S_\alpha \rangle \)
\[ m_0 = \langle K_0 | S_\alpha \rangle = G, \quad (1.33a) \]
\[ m_1 = \langle K_1 | S_\alpha \rangle = u_x G, \quad (1.33b) \]
\[ m_2 = \langle K_2 | S_\alpha \rangle = u_y G, \quad (1.33c) \]
\[ m_3 = \langle K_3 | S_\alpha \rangle = (3u_x^2 + 3u_y^2 - 4)G, \quad (1.33d) \]
\[ m_4 = \langle K_4 | S_\alpha \rangle = (u_x^2 - u_y^2)G, \quad (1.33e) \]
\[ m_5 = \langle K_5 | S_\alpha \rangle = u_x u_y G, \quad (1.33f) \]
\[ m_6 = \langle K_6 | S_\alpha \rangle = (-3u_x^2 u_y + 2u_y)G, \quad (1.33g) \]
\[ m_7 = \langle K_7 | S_\alpha \rangle = (-3u_x u_y^2 + 2u_x)G, \quad (1.33h) \]
\[ m_8 = \langle K_8 | S_\alpha \rangle = (9u_x^2 u_y^2 - 6(u_x^2 + u_y^2) + 4)G. \quad (1.33i) \]

Since there is only one conserved scalar for the thermal transport equation, the components of raw moments of sources \( \hat{\sigma}'_{x^\alpha y^\beta} \) are different from those appearing in the cascaded LBE with forcing terms [44]. Then using
\[ (K \cdot S)_\alpha = (\hat{m}_0, \hat{m}_1, \hat{m}_2, \ldots, \hat{m}_8)^T \] and inverting it by exploiting the orthogonality of \( K, \)
we get the following expressions for the source terms $S_\alpha$ in velocity space:

\[
S_0 = \frac{1}{9} [m_0 - m_3 + m_8],
\]

(1.34a)

\[
S_1 = \frac{1}{36} [4m_0 + 6m_1 - m_3 + 9m_4 + 6m_7 - 2m_8],
\]

(1.34b)

\[
S_2 = \frac{1}{36} [4m_0 + 6m_2 - m_3 - 9m_4 + 6m_6 - 2m_8],
\]

(1.34c)

\[
S_3 = \frac{1}{36} [4m_0 - 6m_1 - m_3 + 9m_4 - 6m_7 - 2m_8],
\]

(1.34d)

\[
S_4 = \frac{1}{36} [4m_0 - 6m_2 - m_3 - 9m_4 + 6m_6 - 2m_8],
\]

(1.34e)

\[
S_5 = \frac{1}{36} [4m_0 + 6m_1 + 6m_2 + 2m_3 + 9m_5 - 3m_6 - 3m_7 + m_8],
\]

(1.34f)

\[
S_6 = \frac{1}{36} [4m_0 - 6m_1 + 6m_2 + 2m_3 - 9m_5 - 3m_6 + 3m_7 + m_8],
\]

(1.34g)

\[
S_7 = \frac{1}{36} [4m_0 - 6m_1 - 6m_2 + 2m_3 + 9m_5 + 3m_6 + 3m_7 + m_8],
\]

(1.34h)

\[
S_8 = \frac{1}{36} [4m_0 + 6m_1 - 6m_2 + 2m_3 - 9m_5 + 3m_6 - 3m_7 + m_8].
\]

(1.34i)

In addition, in order to construct the cascaded collision operator for the solution of the temperature field, we need the raw moments of the collision kernel of different order, i.e. $\sum_\alpha (K \cdot \hat{h})_\alpha e^{m_x} e^{n_y} = \sum_\beta \langle K_\beta | e^{m_x} e^{n_y} \rangle \hat{h}_\beta$. Since the temperature field $T$ is a collision invariant, it follows that $\hat{h}_0 = 0$. Using this and considering the orthogonal
basis vector $k_\beta$ in Eq. (1.20), we get

\[ \sum_\alpha (\mathcal{K} \cdot \hat{h})_\alpha = \sum_\beta \langle K_\beta | T \rangle \hat{h}_\beta = 0, \]  
(1.35a)

\[ \sum_\alpha (\mathcal{K} \cdot \hat{h})_\alpha e_{\alpha x} = \sum_\beta \langle K_\beta | e_{\alpha x} \rangle \hat{h}_\beta = 6\hat{h}_1, \]  
(1.35b)

\[ \sum_\alpha (\mathcal{K} \cdot \hat{h})_\alpha e_{\alpha y} = \sum_\beta \langle K_\beta | e_{\alpha y} \rangle \hat{h}_\beta = 6\hat{h}_2, \]  
(1.35c)

\[ \sum_\alpha (\mathcal{K} \cdot \hat{h})_\alpha e_{\alpha x}^2 = \sum_\beta \langle K_\beta | e_{\alpha x}^2 \rangle \hat{h}_\beta = 6\hat{h}_3 + 2\hat{h}_4, \]  
(1.35d)

\[ \sum_\alpha (\mathcal{K} \cdot \hat{h})_\alpha e_{\alpha y}^2 = \sum_\beta \langle K_\beta | e_{\alpha y}^2 \rangle \hat{h}_\beta = 6\hat{h}_3 - 2\hat{h}_4, \]  
(1.35e)

\[ \sum_\alpha (\mathcal{K} \cdot \hat{h})_\alpha e_{\alpha x} e_{\alpha y} = \sum_\beta \langle K_\beta | e_{\alpha x} e_{\alpha y} \rangle \hat{h}_\beta = 4\hat{h}_5, \]  
(1.35f)

\[ \sum_\alpha (\mathcal{K} \cdot \hat{h})_\alpha e_{\alpha x}^2 e_{\alpha y} = \sum_\beta \langle K_\beta | e_{\alpha x}^2 e_{\alpha y} \rangle \hat{h}_\beta = 4\hat{h}_2 - 4\hat{h}_6, \]  
(1.35g)

\[ \sum_\alpha (\mathcal{K} \cdot \hat{h})_\alpha e_{\alpha x} e_{\alpha y}^2 = \sum_\beta \langle K_\beta | e_{\alpha x} e_{\alpha y}^2 \rangle \hat{h}_\beta = 4\hat{h}_1 - 4\hat{h}_7, \]  
(1.35h)

\[ \sum_\alpha (\mathcal{K} \cdot \hat{h})_\alpha e_{\alpha x}^2 e_{\alpha y}^2 = \sum_\beta \langle K_\beta | e_{\alpha x}^2 e_{\alpha y}^2 \rangle \hat{h}_\beta = 8\hat{h}_3 + 4\hat{h}_8. \]  
(1.35i)

Now, we are in a position to determine the structure of the cascaded collision operator with source terms to solve for the thermal transport equation represented by the CDE. The procedure can be briefly summarized as follows: Starting from the lowest order non-conserved post-collision central moments (i.e. the first order components in the present case), we tentatively set them equal to their corresponding equilibrium states. Once the expressions for the collision kernel $(\hat{h}_\beta) (\beta \geq 1)$ are determined, we discard these equilibrium assumption and multiply it with a corresponding relaxation parameters $(\hat{\lambda}_\beta)$ to allow for a relaxation process during collision [22, 44]. Here, only those terms that are not in the post-collision states for the lower order moments are multiplied by the relaxation parameters. Thus, we start from the first order post-collision central moment, i.e. $\langle s^p_\alpha | (e_{\alpha x} - u_x) \rangle$ and $\langle s^p_\alpha | (e_{\alpha y} - u_y) \rangle$ and tentatively set to $\hat{\kappa}^p_x$ and $\hat{\kappa}^p_y$, respectively:

\[ \hat{\kappa}^p_x = 0 \]  
(1.36)
Where from Eq. (1.54b) raw moments of the post-collision distribution function in terms of its pre-collision value, collision kernel, and source term are used to obtain the right hand sides of the above equation. That is,

\[
\langle g^p_x | e_{ax} \rangle = \langle \tilde{g}_a | e_{ax} \rangle + \langle (K \cdot \tilde{h})_a | e_{ax} \rangle + \langle S_a | e_{ax} \rangle = \dot{\tilde{\kappa}}_x + 6\tilde{h}_1 + \dot{\tilde{\sigma}}_x
\]

\[
\langle g^p_y | T \rangle = \langle \tilde{g}_a | T \rangle + \langle (K \cdot \tilde{h})_a | T \rangle + \langle S_a | T \rangle = \tilde{\kappa}'_y + \frac{1}{3}D\delta t(\partial_y T)
\]

Substituting the above expressions in Eq. (1.36) and rearranging, and solve for collision kernel we get the following tentative expression

\[
\tilde{h}_1 = \frac{1}{6} \left\{ \frac{\tilde{\kappa}^{eq'}_x - \tilde{\kappa}'_x}{2} - \frac{1}{2} \tilde{\sigma}'_x \right\}
\] (1.37)

Where \( \tilde{\kappa}^{eq'}_x = u_x T \). In order to provide further flexibility in adjusting the transport coefficient in the CDE, the raw moment equilibrium \( \tilde{\kappa}^{eq'}_x \) will be augmented with an extended moment equilibrium \( \frac{1}{2}D\delta t(\partial_y T) \), where \( D \) is the adjustable parameter. See Appendix B for the analysis of such a scheme. In addition, we apply a relaxation parameter \( \lambda_1 \) in the equation above Eq. (1.37) to reflect the collision as a relaxation process. Thus, we get

\[
\tilde{h}_1 = \frac{\lambda_1}{6} \left\{ \frac{\tilde{\kappa}^{eq'}_x - \tilde{\kappa}'_x}{2} - \frac{1}{2} \tilde{\sigma}'_x + \frac{1}{3}D\delta t(\partial_y T) \right\}
\] (1.38)

Similarly, setting \( \langle \tilde{g}^p_y | (e_{ay} - u_y) \rangle \) to \( \tilde{\kappa}^{eq}_y = 0 \) and using

\[
\langle \tilde{g}^p_x | e_{ay} \rangle = \langle \tilde{g}_a | e_{ay} \rangle + \langle (K \cdot \tilde{h})_a | e_{ay} \rangle + \langle S_a | e_{ay} \rangle = \dot{\tilde{\kappa}}'_y + 6\tilde{h}_2 + \dot{\tilde{\sigma}}_y
\]

and following the same procedure as above, we obtain

\[
\tilde{h}_2 = \frac{\lambda_2}{6} \left\{ \frac{\tilde{\kappa}^{eq'}_y - \tilde{\kappa}'_y}{2} - \frac{1}{2} \tilde{\sigma}'_y + \frac{1}{3}D\delta t(\partial_y T) \right\}
\] (1.39)

In the above, the temperature gradient needed in the extended moment equilibria can
be locally computed in terms of the first order non-equilibrium moments (see Appendix B for details). Next, consider the second order diagonal central moments and tentatively set them to their corresponding equilibrium states, i.e.

\[ \hat{\kappa}^{eq}_{xx} = c_s^2 T = \langle g^p_a | (e_{ax} - u_x)^2 \rangle = \langle g^p_a | (e_{ax}^2) \rangle - 2u_x \langle g^p_a | e_{ax} \rangle + u_x^2 \langle g^p_a | T \rangle. \]  

(1.40)

and

\[ \hat{\kappa}^{eq}_{yy} = c_s^2 T = \langle g^p_a | (e_{ay} - u_y)^2 \rangle = \langle g^p_a | (e_{ay}^2) \rangle - 2u_y \langle g^p_a | e_{ay} \rangle + u_y^2 \langle g^p_a | T \rangle. \]  

(1.41)

Then, using

\[ \langle g^p_a | e_{ax}^2 \rangle = \langle g^p_a | e_{ax}^2 \rangle + \langle (K \cdot \hat{h})_a | e_{ax}^2 \rangle + \langle S_a | e_{ax}^2 \rangle = \tilde{\kappa}_{xx} + 6\hat{h}_3 + 2\hat{h}_4 + \tilde{\sigma}_{xx} \]

\[ \langle g^p_a | e_{ay}^2 \rangle = \langle g^p_a | e_{ay}^2 \rangle + \langle (K \cdot \hat{h})_a | e_{ay}^2 \rangle + \langle S_a | e_{ay}^2 \rangle = \tilde{\kappa}_{yy} + 6\hat{h}_3 - 2\hat{h}_4 + \tilde{\sigma}_{yy} \]

substituting the above two expressions in Eqs. (1.40) and (1.41), respectively and rearranging, we get

\[ 6\hat{h}_3 + 2\hat{h}_4 = \frac{1}{3} T - u_x^2 T - \tilde{\kappa}_{xx} + 2u_x \tilde{\kappa}_x - \tilde{\sigma}_{xx} + 2u_x \tilde{\sigma}_x + \frac{1}{2} u_x^2 G + 12u_x\hat{h}_1. \]  

(1.42)

\[ 6\hat{h}_3 - 2\hat{h}_4 = \frac{1}{3} T - u_x^2 T - \tilde{\kappa}_{yy} + 2u_y \tilde{\kappa}_y - \tilde{\sigma}_{yy} + 2u_y \tilde{\sigma}_y + \frac{1}{2} u_y^2 G + 12u_y\hat{h}_2. \]  

(1.43)

Solving for \( \hat{h}_3 \) and \( \hat{h}_4 \) from the above two equations and then applying the relaxation parameters \( \lambda_3 \) and \( \lambda_4 \), respectively, for \( \hat{h}_3 \) and \( \hat{h}_4 \), while excluding the lower order collision kernel terms (i.e. \( \hat{h}_1 \) and \( \hat{h}_2 \)) as they are already in the post-collision state, we finally get

\[ \hat{h}_3 = \frac{\lambda_3}{12} \left\{ \frac{2}{3} T - (u_x^2 + u_y^2) T - (\tilde{\kappa}_{xx} + \tilde{\kappa}_{yy}) + 2u_x \tilde{\kappa}_x + 2u_y \tilde{\kappa}_y + \frac{1}{2} (\tilde{\sigma}_{xx} + \tilde{\sigma}_{yy}) \right\} 
+ u_x\hat{h}_1 + u_y\hat{h}_2, \]  

(1.44)
\[
\hat{h}_4 = \frac{\lambda_4}{4} \left\{- (u_x^2 - u_y^2) T - \left( \hat{\kappa}_{xx} - \hat{\kappa}_{yy} \right) + 2u_x \hat{\kappa}_x - 2u_y \hat{\kappa}_y + \frac{1}{2} \left( \hat{\sigma}_{xx} - \hat{\sigma}_{yy} \right) \right\} + 3u_x \hat{h}_1 - 3u_y \hat{h}_2,
\]

\[(1.45)\]

Clearly, the cascaded structure is already evident in the collision kernels of the second order moments, which is unlike that for the fluid flow LBE solver, where the cascaded structure starts to appear only at the third order moment collision kernels. This arises due to differences in the number of collision invariants between the two cascaded LBE models. Next, considering the post-collision state of the off-diagonal central moment as

\[
\hat{\kappa}_{eq}^{xy} = 0 = \langle g^p_a | (e_{ax} - u_x)(e_{ay} - u_y) \rangle = \langle g^p_a | e_{ax}e_{ay} \rangle - u_y \langle g^p_a | e_{ax} \rangle - u_x \langle g^p_a | e_{ay} \rangle + u_x u_y \langle g^p_a | T \rangle
\]

\[(1.46)\]

Using \(\langle g^p_a | e_{ax}e_{ay} \rangle = \hat{\kappa}_{eq}^{xy} + 4\hat{h}_5 + \hat{\sigma}_{xy}^{'}\) in the above equation and simplify it as a tentative expression for \(\hat{h}_5\) and then applying the relaxation parameter \(\lambda_5\) to those terms that are not yet in the post-collision states, we get

\[
\hat{h}_5 = \frac{\lambda_5}{4} \left\{ - \hat{\kappa}_{xy}^{'} - \hat{\kappa}_{xy} + u_x \hat{\kappa}_x + u_y \hat{\kappa}_y + \frac{1}{2} \hat{\sigma}_{xy}^{'} \right\} + \frac{3}{2} (u_x \hat{h}_2 + u_y \hat{h}_1)
\]

\[(1.47)\]

Now, consider the determination of the third order moment collision kernel. Setting tentatively

\[
\hat{\kappa}_{eq}^{xxy} = 0 = \langle g^p_a | (e_{ax} - u_x)^2(e_{ay} - u_y) \rangle
\]

\[(1.48)\]

\[
\hat{\kappa}_{eq}^{xyy} = 0 = \langle g^p_a | (e_{ax} - u_x)(e_{ay} - u_y)^2 \rangle
\]

\[(1.49)\]

and using

\[
\langle g^p_a | (e_{ax})^2(e_{ay}) \rangle = \hat{\kappa}_{xxy}^{'} + 4\hat{h}_2 - 4\hat{h}_6 + \hat{\sigma}_{xxy}^{'}
\]

\[
\langle g^p_a | (e_{ax})(e_{ay})^2 \rangle = \hat{\kappa}_{xyy}^{'} + 4\hat{h}_1 - 4\hat{h}_7 + \hat{\sigma}_{xyy}^{'}
\]
in Eqs. (1.48) and (1.49), respectively, and simplifying to obtain the tentative expressions for \( \lambda_6 \) and \( \lambda_7 \), respectively, to those terms that are not yet post-collision states, we obtain

\[
\hat{h}_6 = \frac{\lambda_6}{4} \left\{ -u_x^2 u_y T + \tilde{\kappa}'_{xxy} - u_y \tilde{\kappa}'_{xx} - 2u_x \tilde{\kappa}'_{xy} + u'_x \tilde{\kappa}_{xy} + 2u_x u_y \tilde{\kappa}'_{x} + \frac{1}{2} \tilde{\sigma}'_{xy} \right\}
+ \frac{1}{2} (2\hat{h}_2 + 3u_x^2 \hat{h}_2) - \frac{1}{4} u_y (6\hat{h}_3 + 2\hat{h}_4) + 3u_x u_y \hat{h}_1 - 2u_x \hat{h}_5,
\]

(1.50)

\[
\hat{h}_7 = \frac{\lambda_7}{4} \left\{ -u_x u_y^2 T + \tilde{\kappa}'_{xyy} - u_x \tilde{\kappa}'_{yy} - 2u_y \tilde{\kappa}'_{xy} + 2u_x u_y \tilde{\kappa}'_{y} + u'_y \tilde{\kappa}_{xy} + \frac{1}{2} \tilde{\sigma}'_{yy} \right\}
+ \frac{1}{2} (2\hat{h}_1 + 3u_y^2 \hat{h}_1) - \frac{1}{4} u_x (6\hat{h}_3 - 2\hat{h}_4) + 3u_x u_y \hat{h}_2 - 2u_y \hat{h}_5.
\]

(1.51)

Finally, by setting the post-collision state of the fourth order central moment to its corresponding equilibrium as a tentative step as

\[
\hat{\kappa}^\text{eq}_{xxyy} = \frac{1}{9} T = \langle \mathbf{g}^p_a | (e_{ax} - u_x)^2 (e_{ay} - u_y)^2 \rangle
\]

(1.52)

and applying \( \hat{\kappa}^\text{eq}_{xxyy} = \frac{1}{9} T = \langle \mathbf{g}^p_a | e_{ax} \rangle \langle e_{ay} \rangle = \hat{\kappa}^\text{eq}_{xxyy} + 8\hat{h}_3 + 4\hat{h}_8 + \hat{\sigma}'_{xxyy} \) and simplifying Eq. (1.52) and following the same procedure as above by applying the relaxation parameter \( \lambda_8 \), we get

\[
\hat{h}_8 = \frac{\lambda_8}{4} \left\{ \frac{1}{9} T - u_x^2 u_y^2 T + \left[ \tilde{\kappa}'_{xxyy} + 2u_x \tilde{\kappa}'_{xx} + 2u_y \tilde{\kappa}'_{xy} - u'_x \tilde{\kappa}_{xy} - u'_y \tilde{\kappa}_{xx} - 4u_x u_y \tilde{\kappa}_{xy} 
+ 2u_x u'_y \tilde{\kappa}_{x} + 2u'_x u_y \tilde{\kappa}_{y} \right] + \frac{1}{2} \tilde{\sigma}'_{xxyy} \right\}
- 2\hat{h}_3 - \frac{1}{2} u_y^2 (3\hat{h}_3 + \hat{h}_4) + 2u_x (3\hat{h}_1 - \hat{h}_7)
- \frac{1}{2} u_x^2 (3\hat{h}_3 - \hat{h}_4) + 2u_y (3\hat{h}_2 - \hat{h}_6) - 4u_x u_y \hat{g}_5 + 3u_x^2 u_y \hat{h}_2 + 3u_x^2 \hat{g}_7 \hat{h}_1.
\]

(1.53)

In addition, in order to maintain additional flexibility in the representation of the emergent transport coefficient (i.e. the thermal diffusivity of the CDE), we also
introduce extended moment equilibria involving temperature gradient terms with an
adjustable coefficient $D$ in the first order equilibrium moments. The resulting final
expressions of the collision kernel are given as follows:

\[
\begin{align*}
\hat{h}_0 &= 0, \quad (1.54a) \\
\hat{h}_1 &= \frac{\lambda_1}{6} \left\{ \hat{\kappa}_{x}^{eq'} - \hat{\kappa}_x' - \frac{1}{2} \hat{\sigma}_x' + \frac{1}{3} D \delta t (\partial_x T) \right\}, \quad (1.54b) \\
\hat{h}_2 &= \frac{\lambda_2}{6} \left\{ \hat{\kappa}_{y}^{eq'} - \hat{\kappa}_y' - \frac{1}{2} \hat{\sigma}_y' + \frac{1}{3} D \delta t (\partial_y T) \right\}, \quad (1.54c) \\
\hat{h}_3 &= \frac{\lambda_3}{12} \left\{ \frac{2}{3} T - (u_x^2 + u_y^2) T - (\hat{\kappa}_{xx}^{eq} + \hat{\kappa}_{yy}^{eq}) + 2 u_x \hat{\kappa}_x' + 2 u_y \hat{\kappa}_y' + \frac{1}{2} (\hat{\sigma}_{xx}' + \hat{\sigma}_{yy}') \right\} \\
& \quad + u_x \hat{h}_1 + u_y \hat{h}_2, \quad (1.54d) \\
\hat{h}_4 &= \frac{\lambda_4}{4} \left\{ - (u_x^2 - u_y^2) T - (\hat{\kappa}_{xx}^{eq} - \hat{\kappa}_{yy}^{eq}) + 2 u_x \hat{\kappa}_x' + 2 u_y \hat{\kappa}_y' + \frac{1}{2} (\hat{\sigma}_{xx}' - \hat{\sigma}_{yy}') \right\} \\
& \quad + 3 u_x \hat{h}_1 - 3 u_y \hat{h}_2, \quad (1.54e) \\
\hat{h}_5 &= \frac{\lambda_5}{4} \left\{ - \hat{\kappa}_{xy}^{eq'} - \hat{\kappa}_{xy}' + u_x \hat{\kappa}_y' + u_y \hat{\kappa}_x' + \frac{1}{2} \hat{\sigma}_{xy}' \right\} + \frac{3}{2} (u_x \hat{h}_2 + u_y \hat{h}_1), \quad (1.54f) \\
\hat{h}_6 &= \frac{\lambda_6}{4} \left\{ - u_x^2 u_y T + \hat{\kappa}_{xy}^{eq} - u_y \hat{\kappa}_{xx}' - 2 u_x u_y \hat{\kappa}_x' + u_x^2 \hat{\kappa}_y' + 2 u_x u_y \hat{\kappa}_x' + \frac{1}{2} \hat{\sigma}_{xy}' \right\} \\
& \quad + \frac{1}{2} (2 \hat{h}_2 + 3 u_x^2 \hat{h}_1) - \frac{1}{4} u_y (6 \hat{h}_3 + 2 \hat{h}_4) + 3 u_x u_y \hat{h}_1 - 2 u_x \hat{h}_5, \quad (1.54g) \\
\hat{h}_7 &= \frac{\lambda_7}{4} \left\{ - u_x u_y^2 T + \hat{\kappa}_{xyy}^{eq} - u_x \hat{\kappa}_{yy}' - 2 u_y u_x \hat{\kappa}_x' + u_x^2 \hat{\kappa}_y' + u_y^2 \hat{\kappa}_x' + \frac{1}{2} \hat{\sigma}_{xyy}' \right\} \\
& \quad + \frac{1}{2} (2 \hat{h}_1 + 3 u_y^2 \hat{h}_1) - \frac{1}{4} u_x (6 \hat{h}_3 - 2 \hat{h}_4) + 3 u_x u_y \hat{h}_2 - 2 u_y \hat{h}_5, \quad (1.54h) \\
\hat{h}_8 &= \frac{\lambda_8}{4} \left\{ \frac{1}{9} T - u_x^2 u_y^2 T + \left[ \hat{\kappa}_{xx}^{eq} + 2 u_x \hat{\kappa}_{xy}' + 2 u_y \hat{\kappa}_{xx}' - u_x^2 \hat{\kappa}_x' - u_y^2 \hat{\kappa}_x' + 4 u_x u_y \hat{\kappa}_{xy} \\
& \quad + 2 u_x^2 \hat{\kappa}_x' + 2 u_y^2 \hat{\kappa}_x' \right] + \frac{1}{2} \hat{\sigma}_{xyy}' \right\} - 2 \hat{h}_3 - \frac{1}{2} u_y^2 \hat{h}_4 + 2 u_x (3 \hat{h}_1 - \hat{h}_7) \\
& \quad - \frac{1}{2} u_x^2 (3 \hat{h}_3 - \hat{h}_4) + 2 u_y (3 \hat{h}_2 - \hat{h}_6) - 4 u_x u_y \hat{g}_8 + 3 u_x^2 u_y \hat{h}_2 + 3 u_x u_y^2 \hat{h}_1. \quad (1.54i)
\end{align*}
\]

Notice that $\hat{h}_1 \neq \hat{h}_2 \neq 0$, in the present case, which is unlike that for the casceded LBE
for fluid flow [44]. This difference arises from the fact that the casceded LBE for the
flow field has three collision invariants, i.e. mass and momentum, and hence its
corresponding zeroth and first order collision kernels are zero; On the other hand, in
the case of casceded LBE for the thermal transport equation, there is only one
collision invariant, i.e. temperature field, and therefore only its zeroth order collision kernel is \( \hat{h}_0 \) is zero. As a result of these differences the cascaded collision operator for the temperature field is markedly different from that for the flow field.

note that, \( \lambda_\beta, \beta = 1, 2, 3, \ldots, 8 \), are the relaxation parameters, satisfying the bounds \( 0 < \lambda_\beta < 2 \). Notice the cascaded structure for the second and higher order moment kernels, i.e. their dependence on the lower order moments for our thermal cascaded LBE. By contrast, the cascaded LBE for the fluid flow is significantly different, with the cascaded structure appearing only for third and higher order moments. When a Chapman-Enskog expansion (C-E) is performed on the above cascaded LB model (see Appendix B for details), it can be shown to recover the convection-diffusion thermal transport equation, with the relaxation parameters for the first order moments \( \lambda_1 \) and \( \lambda_2 \) and the adjustable coefficient \( D \) controlling the thermal diffusivity coefficient \( \alpha \) (see Eq. (1.17)):

\[
\alpha = \frac{1}{3} \left( \frac{1}{\lambda_j} - \frac{1}{2} - D \right) \delta t, \quad j = 1, 2. 
\] (1.55)

The rest of the parameters can be adjusted independently to improve numerical stability. In this work, \( \lambda_1 = \lambda_2 = (1/\tau_t) \) is selected based on the specified diffusivity, while the remaining relaxation parameters are set to be unity.

Moreover, the temperature gradients \( \partial_x T \) and \( \partial_y T \) appearing in the above (see Eqs. (1.54b) and (1.54c)) can be calculated locally in terms of the first order non-equilibrium moments. See the C-E analysis given in Appendix B for details. Thus, we have

\[
\frac{\partial T}{\partial x} = \frac{-3\lambda_1 \left[ \hat{\kappa}'_x - \hat{\kappa}^\text{eq}'_x \right]}{(1 - D\lambda_1)}, 
\] (1.56a)

\[
\frac{\partial T}{\partial y} = \frac{-3\lambda_2 \left[ \hat{\kappa}'_y - \hat{\kappa}^\text{eq}'_y \right]}{(1 - D\lambda_2)}. 
\] (1.56b)
By expanding collision and streaming steps: the thermal cascaded LBE given in Eq. (1.26) can be written in terms of the following distribution functions are given as follows:

\[ \tilde{g}^0(\vec{x}, t) = \tilde{g}(\vec{x}, t) + \Omega(\vec{x}, t) + S(\vec{x}, t), \quad (1.57a) \]

\[ \tilde{g}(\vec{x} + \vec{v}, t + 1) = \tilde{g}^p(\vec{x}, t). \quad (1.57b) \]

By expanding \( (K \tilde{h})_a \) in Eq. (1.57a), the explicit expressions for the post collision distribution functions are given as follows:

\[ \tilde{g}^p_0 = \tilde{g}_0 + [\hat{h}_0 - 4(\hat{h}_3 - \hat{h}_8)] + S_0, \quad (1.58a) \]
\[ \tilde{g}^p_1 = \tilde{g}_1 + [\hat{h}_0 + \hat{h}_1 - \hat{h}_3 + \hat{h}_4 + 2(\hat{h}_7 - \hat{h}_8)] + S_1, \quad (1.58b) \]
\[ \tilde{g}^p_2 = \tilde{g}_2 + [\hat{h}_0 + \hat{h}_2 - \hat{h}_3 - \hat{h}_4 + 2(\hat{h}_6 - \hat{h}_8)] + S_2, \quad (1.58c) \]
\[ \tilde{g}^p_3 = \tilde{g}_3 + [\hat{h}_0 - \hat{h}_1 - \hat{h}_3 + \hat{h}_4 - 2(\hat{h}_7 + \hat{h}_8)] + S_3, \quad (1.58d) \]
\[ \tilde{g}^p_4 = \tilde{g}_4 + [\hat{h}_0 - \hat{h}_2 - \hat{h}_3 - \hat{h}_4 - 2(\hat{h}_6 + \hat{h}_8)] + S_4, \quad (1.58e) \]
\[ \tilde{g}^p_5 = \tilde{g}_5 + [\hat{h}_0 + \hat{h}_1 + \hat{h}_2 + 2\hat{h}_3 + \hat{h}_5 - \hat{h}_6 - \hat{h}_7 + \hat{h}_8] + S_5, \quad (1.58f) \]
\[ \tilde{g}^p_6 = \tilde{g}_6 + [\hat{h}_0 - \hat{h}_1 + \hat{h}_2 + 2\hat{h}_3 - \hat{h}_5 - \hat{h}_6 + \hat{h}_7 + \hat{h}_8] + S_6, \quad (1.58g) \]
\[ \tilde{g}^p_7 = \tilde{g}_7 + [\hat{h}_0 - \hat{h}_1 + \hat{h}_2 + \hat{h}_3 - \hat{h}_5 + \hat{h}_6 + \hat{h}_7 + \hat{h}_8] + S_7, \quad (1.58h) \]
\[ \tilde{g}^p_8 = \tilde{g}_8 + [\hat{h}_0 + \hat{h}_1 - \hat{h}_2 + 2\hat{h}_3 - \hat{h}_5 + \hat{h}_6 - \hat{h}_7 + \hat{h}_8] + S_8. \quad (1.58i) \]

Finally, based on the solution of the thermal cascaded LBE given in Eqs. (1.57a) and (1.57b), the temperature field \( T \) can be obtained as

\[ T = \sum_{\alpha} g_\alpha = \sum_{\alpha} \tilde{g}_\alpha + \frac{1}{2} G \delta t. \quad (1.59) \]
1.5 Anisotropic Cascaded Model for the Temperature Field

Anisotropic diffusion is a common physical phenomenon and describe processes where the diffusion of some location is direction dependent, i.e. diffusion coefficients are location and/or direction dependent. In addition, many practical problems such as fluid flows in anisotropic media arising in environmental science and geophysics do have anisotropic diffusion coefficients. Most of the LBE models for CDE are limited to the description of the isotropic diffusion problems \[5, 25, 26\]

To incorporate full anisotropy with off-diagonal components of the thermal diffusion-coefficient tensor, we consider

\[
\hat{h}_1 = \frac{\lambda_{11}}{6} \left\{ \hat{k}^{\text{eq}}_x - \hat{k}_x - \frac{1}{2} \hat{\sigma}_x + \frac{1}{3} D \delta t (\partial_x T) \right\} + \frac{\lambda_{12}}{6} \left\{ \hat{k}^{\text{eq}}_y - \hat{k}_y - \frac{1}{2} \hat{\sigma}_y + \frac{1}{3} D \delta t (\partial_y T) \right\}
\]

(1.60)

\[
\hat{h}_2 = \frac{\lambda_{21}}{6} \left\{ \hat{k}^{\text{eq}}_x - \hat{k}_x - \frac{1}{2} \hat{\sigma}_x + \frac{1}{3} D \delta t (\partial_x T) \right\} + \frac{\lambda_{22}}{6} \left\{ \hat{k}^{\text{eq}}_y - \hat{k}_y - \frac{1}{2} \hat{\sigma}_y + \frac{1}{3} D \delta t (\partial_y T) \right\}
\]

(1.61)

where the off-diagonal components of the relaxation time matrix, as a result of the rotation of the principal axes, enable us to simulate the full anisotropy convection-diffusion.

Here, \( \Lambda \) is a collision matrix given by

\[
\Lambda = diag(\lambda_0, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, ..., \lambda_{18})
\]

(1.62)
1.6 Boundary Conditions

As mentioned in the previous sections, the Lattice Boltzmann method has many important advantages, one of these features is its capability to handle the complex boundary. Boundary conditions (BCs) have very important role for stability and the accuracy of any numerical solution for the lattice Boltzmann method, the discrete distribution functions on the boundary have to be taken care of to reflect the macroscopic BCs of the fluid. There are many types of boundary conditions that have been handled by LBE. There have been many types of thermal boundary conditions applied to CDE. [29]. The boundary conditions used in this study for simulating isotropic and anisotropic thermal convective flow consists of periodic boundary conditions and bounce-back boundary conditions. In this thesis, the halfway bounce-back scheme is employed to treat velocity boundary conditions while the general anti-bounce-back scheme [48] is adopted to deal with temperature boundary conditions and periodic boundary condition for inflow and outflow boundaries.
1.6.1 Halfway bounce-back boundaries

In the Lattice Boltzmann method the interaction of a fluid particle with a solid particle is performed using the bounce-back method. For the density distribution functions, bounce-back boundary conditions were applied on all solid boundaries, which means that incoming boundary population equal to out-going populations after the collision. In LBE distribution functions out of the domain are known from streaming process, Fig. 1.5 shows the unknown distribution functions (velocity and temperature) which are needed to be determined as dotted lines, which means that incoming boundary population equal to out-going populations after the collision. For example for east boundary, \( f_5 \), \( f_1 \), and \( f_8 \) stream into the wall, and are bounced back by setting \( f_3 = f_1 \), \( f_7 = f_5 \), and \( f_6 = f_8 \) as it can be seen in Fig. 1.5. The bounce-back method has been discussed extensively in the literature and a few versions have been developed. The most simple scheme is to place a wall halfway between a wall grid point and a fluid grid point and then "bounce-back" particles that stream into the wall. It has been found that the halfway bounce-back scheme is second order accurate with respect to grid spacing for regular boundary [27].
**Figure 1.5:** Boundary Conditions with unknown and known populations.

**Figure 1.6:** Location of boundary nodes.
1.6.1.1 Velocity Distribution Function

In order to obtain a no-slip boundary condition for a specified boundary, we use the “Bounce Back Method. The most simple scheme is to place a wall halfway between a wall grid point and a fluid grid point and then "bounce-back" particles that stream into the wall. For instance, $f_4$, $f_7$, and $f_8$ stream into the wall, and are bounced back by setting $f_5=f_7$, $f_2=f_4$, and $f_6=f_8$ (Fig. 1.6).

1.6.1.2 Temperature Distribution Function

The general anti-bounce-back scheme [48] is adopted here to deal with temperature boundary conditions, the thermal or concentration boundary conditions can be classified into three types, Dirichlet, Neumann and mixed boundary conditions as

$$b_1 \frac{\partial T}{\partial n} + b_2 T = b_3 \quad (1.64)$$

First: for Dirichlet boundary conditions:

For static boundary, i.e. $u_w = 0$

$$T = T_w = \frac{b_3}{b_2} \quad (1.65)$$

is constant at the boundary. To implement these boundary conditions with LBE, the distribution function is constructed as

$$\overline{g}_\alpha(x_f, t) = -\overline{g}_\alpha''(x_f, t) + 2\omega_\alpha T_w \quad (1.66)$$

for moving boundary, i.e. $u_w \neq 0$

$$\overline{g}_\alpha(x_f, t) = -\overline{g}_\alpha''(x_f, t) + 2\omega_\alpha T_w \times \left[ 1 + 4.5 \frac{(e_\alpha \cdot u_w)}{c^2} - 1.5 \frac{|u_w|^2}{c^2} \right] \quad (1.67)$$

Second: for Neumann boundary conditions:
\[ \frac{\partial T}{\partial n} = \frac{T_f - T_w}{-0.5\mathbf{n} \cdot \mathbf{e}_\alpha \delta x} \]  

(1.68)

Where \( T_f \) is the temperature at the fluid node neighboring the interface, \( \omega_\alpha \) is the Lattice weight and \( e_\alpha \) is the discrete velocity

1.6.2 Periodic Boundary Conditions

The periodic boundary is the simplest boundary conditions. In this type of boundary, the unknowns of one boundary can be directly related to the knowns of the other boundary. i.e, it is applied directly to calculate the unknown components of the particle distribution function of the nodes located at the boundaries of the domain, e.g. inlet and outlet as shown in Fig. 1.7.

For example, a periodic boundary condition is required on the inflow and
outflow boundaries as shown in Fig. 1.8. The periodic boundary condition on the inflow boundary

\[ g_\alpha(x_i, t + \delta t) = g_\alpha(x_n, t), \quad \alpha = 1, 5, 8 \]  \hspace{1cm} (1.69)

and on the outflow boundary

\[ g_\alpha(x_n, t + \delta t) = g_\alpha(x_i, t), \quad \alpha = 3, 6, 7 \]  \hspace{1cm} (1.70)

where \( x_i \) and \( x_n \) are the coordinates of the inflow and outflow boundaries
CHAPTER II

ISOTROPIC THERMAL FLOW

2.1 Numerical Results

Here, numerical simulations of some benchmark problems are conducted to validate the accuracy of our proposed cascaded LBE model for convective thermal flows. The test problems without source terms in the energy equation are thermal Poiseuille flow, thermal flow in a channel with wall injection, and natural convection in square cavity. Also, problems considered with variable source terms in the energy equation are a reaction-diffusion problem, and couette flow with temperature gradients (i.e. thermal Couette flow with viscous heat dissipation). In this study, the halfway bounce-back scheme is employed to treat velocity boundary conditions while the general anti-bounce-back scheme [48] is adopted to deal with temperature boundary conditions. In problems involving LBM solution of fluid flow all relaxation parameters are set to 1.0 except \( \omega_4 \) and \( \omega_5 \) (the relaxation rates for the first order moments) which are both equal to \( \tau_f^{-1} \). In the thermal model all relaxation parameters are set to 1.0 except \( \lambda_1 \) and \( \lambda_2 \) which are both equal to \( \tau_g^{-1} \).

2.2 Unsteady Reaction-Diffusion Problem: Variable Source Term

The unsteady reaction diffusion problem is a good problem to test the accuracy of the present LBE cascaded model for the equivalent energy equation with a variable source term. Such a system defined in the region \( 0 \leq x, y \leq l \), with macroscopic
The governing equations written as in [59]:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T + 2C \sin(\pi x/l) \sin(\pi y/l)$$  

(2.1)

Where \(G = 2C \sin(\pi x/l) \sin(\pi y/l)\) is the variable source term, \(l\) is the width of the region, \(C\) is a constant, and \(\alpha\) is the diffusion coefficient. The initial and boundary conditions of this system are:

\(T(x,y,0) = 0, T(0,y,t) = T(l,y,t) = 0,\)
\(T(x,0,t) = T(x,l,t) = 0.\)

The general bounce-back scheme [48] is employed to represent these boundary conditions.

The analytical solution of this problem is given by

$$T(x,y,t) = \frac{l^2}{\pi^2 \alpha} C \left[ 1 - \exp \left( -\frac{2\pi^2 t \alpha}{l^2} \right) \right] \sin(\pi x/l) \sin(\pi y/l)$$  

(2.2)

We conduct our numerical simulation with a grid resolution of \(61 \times 61, C = 10\)
and with the thermal diffusivity coefficients $\alpha = 10^{-3}$ and $\alpha = 10^{-4}$. The simulation results and the analytical solutions are compared at three different times $t = 50$, $t = 100$, and $t = 150$ as used in [59]. The relaxation time is set to $\tau_g = 0.503$ for $\alpha = 10^{-3}$, and $\tau_g = 0.5003$ for $\alpha = 10^{-4}$. Fig. 2.1, and Fig. 2.2 show the temperature profiles for both the values of diffusivity coefficients which indicate very good agreement with the analytical solutions for both thermal diffusion coefficients. We also examine the spatial accuracy of the present model. In this regard, a set of simulations are performed at four different grid sizes, i.e., $25 \times 25$, $51 \times 51$, $101 \times 101$, and $201 \times 201$ for both values of the diffusion coefficient, i.e. $\alpha = 10^{-3}$, and $\alpha = 10^{-4}$. The global relative error of temperature ($E_T$) used to measure the accuracy of the model is calculated as

$$E_T = \frac{||(T_c - T_a)||_2}{||T_a||_2}$$

(2.3)

Where $||.||_2$ is the Euclidean norm, $||(T_c - T_a)||_2 = \sqrt{\sum_i (T_{c,i} - T_{a,i})^2}$.
$||\langle T_a \rangle ||_2 = \sqrt{\sum_i (T_{a,i})^2}$, $T_c$ and $T_a$ are the computed and the analytical solutions respectively. The relative global error of temperature for each value of diffusivity coefficient are plotted in Fig. 2.3. It can be seen that the temperature global error decreases with increase in grid resolution with a slope of $-2$ in the log-log plot. Hence, our present cascaded LBM model with source term is second order accurate.

2.3 Thermal Flow in a Channel with Wall Injection

In this section, the present cascaded LBE model for convective thermal flow is employed to simulate the fully developed thermal flow in a channel, where the upper plate moves along the x-direction with velocity $U_p$, and a fluid is injected in the positive y-direction with a constant velocity $v_0$ through the stationary bottom wall. The upper wall is maintained at a higher temperature ($T_h$) and the bottom wall is fixed at a lower temperature ($T_c$). The computational domain of the problem is $0 \leq x, y \leq L$. In the steady state case, the analytical solutions for both velocity and
temperature fields are, respectively, given by.

\[ u_x(y) = \frac{\exp[Re(y/L)] - 1}{\exp(Re) - 1}, \]  \hspace{1cm} (2.4)

\[ T = T_c + \Delta T \frac{\exp[PrRe(y/L)] - 1}{\exp(PrRe) - 1} \]  \hspace{1cm} (2.5)

where \( Re \) is the Reynolds number defined by \( Re = \frac{Lv_0}{\nu} \), \( L \) is the width of the channel and \( \Delta T \) is the temperature difference. In our numerical test, we set \( U_p = v_0 = 0.01, T_h = 1, T_c = 0, Pr = 0.71 \), with a grid size 31 \times 61 at different Reynolds numbers \( Re = 5, 10, \) and 15. The relaxation rates for the flow and thermal equation cascaded LB solvers are obtained based on the value of \( Pr \) and \( Re \) at each case where \( \nu = \frac{Lv_0}{Re} \), and \( \alpha = \frac{\nu}{Pr} \). The rest of the relaxation rates are set to be 1.0. The general bounce-back scheme [48] is implemented at the upper and bottom plates for the temperature boundary conditions, a halfway bounce-back scheme is employed for the velocity boundary conditions, and periodic boundary conditions are imposed at the inlet and outlet of the channel. The profiles of velocity and temperature along the \( y \) direction at different Reynolds numbers and \( Pr = 0.71 \) are plotted in Figs. 2.4 and 2.5 respectively. It is found that the numerical results agree well with the analytical solutions for this test case. We also study the convergence rate by considering the following grid resolutions \( N_y = 31, 61, 91, \) and 121. In these simulations, we conduct the convergence study at Reynolds numbers: \( Re = 5, 10, \) and 15 for the above set of grid resolutions using \( \omega_4 = \omega_5 = \lambda_1 = \lambda_2 = \frac{1}{0.8} \), and all other relaxation parameters are set to be 1 with the corresponding values of the tunable parameter \( D \) as 0.05, 0.1, and 0.15 respectively. For the velocity field, the Reynolds number is set to be \( Re = 10 \) and relaxation rates are \( (\tau_f = \tau_g = 0.8) \). The relative global errors of velocity and temperature are plotted in Figs. 2.6 and 2.7. It can be seen that the relative errors have slopes of almost equal to 2.00, which a gain confirms that the present cascaded LBM model for thermal flow is second order accurate. In the above, the relative global error of temperature and velocity are defined,
respectively, by

\[ E_T = \frac{|| (T_c - T_a) ||_2}{|| T_a ||_2} \quad (2.6) \]

\[ E_u = \frac{|| (u_c - u_a) ||_2}{|| u_a ||_2} \quad (2.7) \]

where \( || \cdot ||_2 \) is the Euclidean norm, \( || (T_c - T_a) ||_2 = \sqrt{\sum_i (T_{c,i} - T_{a,i})^2} \),
\( || (u_c - u_a) ||_2 = \sqrt{\sum_i (u_{c,i} - u_{a,i})^2} \), \( || T_a ||_2 = \sqrt{\sum_i (T_{a,i})^2} \), \( || u_a ||_2 = \sqrt{\sum_i (u_{a,i})^2} \).

Here, \( T_c, u_c \) and \( T_a, u_a \) are the computed and the analytical solutions respectively.

2.4 Diffusion in 2D Poiseuille Flow

Next, we consider a 2D Poiseuille flow between two parallel plates in the streamwise direction driven by a constant body force \( F_x \). Both the upper and bottom walls are stationary and subjected to higher \( (T_h) \) and lower \( (T_c) \) uniform temperature respectively. The computational domain is \( 0 \leq x, y \leq L \). Where \( L \) is the channel width. A periodic boundary condition is applied at the entrance and the exit for both velocity and temperature fields, while the halfway bounce back scheme is
implemented at the solid boundaries (upper and bottom walls) for the velocity field to represent the no-slip boundary condition. The general bounce-back scheme [48] is employed to the solid boundaries for the temperature Dirichlet boundary conditions. The analytical solution for the velocity in Poiseuille flow (parabolic profiles) is given by

\[ u(y) = u_{\text{max}} (1 - (y/L)^2) \]  

(2.8)

where \( u_{\text{max}} = F_x L^2 / 2\nu \) is the maximum velocity occurring halfway between the plates, \( \nu \) is the kinematic viscosity related to the relaxation time \( \tau \). Here, \( L \) is the half distance between the two parallel plates. The analytical solution for the temperature in Poiseuille flow is given by

\[ T = T_c + \Delta T (y/L) \]  

(2.9)
Figure 2.6: Velocity relative global error of Couette flow with wall injection at $Re = 10$.

Figure 2.7: Temperature relative global error of Couette flow with wall injection at Reynolds numbers: $Re = 5,10,15$. 
where $\Delta T = T_h - T_c$ is the temperature difference. In our simulation, a grid size of $30 \times 60$ is employed. We consider two cases corresponding to different sets of Reynolds numbers $Re = u_{max} L / \nu$, Peclet numbers $Pe = u_{max} L / \alpha$ and Prandtl number $Pr = \nu / \alpha$. In the first case, we set $Pr = 0.71$, $Re = 10$ and $Pe = 7$. In the second case, we consider $Re = Pe = 10$, i.e. $Pr = 1$. Where, we consider $T_h = 1.1$, $T_c = 1$, and $\tau_f = 0.674$ in both cases. Fig. 2.8 presents a comparison of the velocity and temperature profiles for these cases. Excellent agreement with the analytical solution is seen.

**Figure 2.8:** Velocity and temperature profiles of Poiseuille flow with thermal diffusion at different values of $Re$ and $Pe$. Markers represent the Cascaded LBE results and lines represent the analytical solution.
2.5 Natural Convection in a Square Cavity

We now present a validation study involving coupled thermal convective flow. In this regard, our cascaded LBE model is employed to simulate natural convection in a square cavity. Here, the flow is driven by the buoyancy force due to the local temperature difference against a reference temperature in the present of gravity. The left wall is maintained at higher temperature $T_h$ and the right wall at lower temperature $T_c$, while the top and bottom walls are considered to be adiabatic. The macroscopic governing equations can be expressed as follows:

\begin{align}
\nabla \cdot u &= 0, \quad (2.10a) \\
\frac{\partial u}{\partial t} + u \cdot \nabla u &= -\frac{1}{\rho} \nabla P + \nu \nabla^2 u + F, \quad (2.10b) \\
\frac{\partial T}{\partial t} + u \cdot \nabla T &= \nabla \cdot (\alpha \nabla T). \quad (2.10c)
\end{align}

where $F$ is the body force which is based on the Boussinesq approximation and is given by

\begin{equation}
F = g\beta(T - T_0)\hat{j} \quad (2.11)
\end{equation}

Here, $\beta$ is the thermal expansion coefficient, $g$ is the acceleration due to gravity, $T_0 = (T_h + T_c)/2$ is the reference temperature, $\hat{j}$ is the unit vector in positive y-direction. This classical natural convection problem is governed by two non-dimensional parameters: The Prandtl number $Pr$ and the Rayleigh number $Ra$, which are given by

\begin{align}
Pr &= \frac{\nu}{\alpha} \\
Ra &= \frac{g\beta \Delta T H^3}{\nu \alpha} \quad (2.13)
\end{align}

where, $\Delta T = T_h - T_c$ is the temperature difference between hot and cold walls, and $H$ is the height of the square cavity.
The boundary conditions on the cavity walls can then be summarized as:

On the left wall:

\[ u_x = u_y = 0, \quad T = T_h = 21, \]  \hspace{1cm} (2.14)

On the right wall:

\[ u_x = u_y = 0, \quad T = T_c = 1, \]  \hspace{1cm} (2.15)

On the top wall:

\[ u_x = u_y = 0, \quad \frac{\partial T}{\partial y} = 0, \]  \hspace{1cm} (2.16)

On the bottom wall:

\[ u_x = u_y = 0, \quad \frac{\partial T}{\partial y} = 0. \]  \hspace{1cm} (2.17)

The general bounce-back scheme [48] is adopted to treat the thermal Dirichlet and Neumann boundary conditions. In these simulations, \( Pr = 0.71 \), the relaxation rates for fluid flow and temperature are set as \( 0.55, 0.57 \) respectively. The streamlines and the isotherms for the ranges of Rayleigh number \( Ra \) between \( 10^3 \) to \( 10^6 \) are shown in Fig. 2.12. Also, the vorticity contours for \( Ra = 10^3 - 10^6 \) are shown in Fig. 2.14. The streamlines, isotherms and vorticity contours are in very good correspondence and consistent with prior benchmark solution results [15, 16]. The natural convection flow patterns become more complex as \( Ra \) increases. In order to characterized this in more
detail, the temperature at the vertical and horizontal mid-planes of the square cavity, i.e. $x/H = 0.5$ and $y/H = 0.5$, respectively for various Rayleigh numbers ($Ra = 10^3 - 10^6$) are presented in Figs. 2.10 and 2.11. In these cases, the value of the factor $g \beta$ needed in the simulation is obtained as a function of Rayleigh number $Ra$ using

$$g \beta = \frac{v \alpha Ra}{\Delta TH^3}$$

(2.18)

The representative values of $g \beta$ corresponding to each $Ra$ is shown in Table 3.2. From Figs. 2.10 and 2.11, it is seen that the temperature contour lines become almost horizontal around the center of the cavity as the Rayleigh number $Ra$ increases. The streamlines become more packed next to the side wall as $Ra$ increases, i.e. the flow moves faster as natural convection is intensified. Finally, Table 3.6 shows quantitative comparison between the key parameters for this problem (average Nusselt number, maximum velocities magnitudes and their locations) between the present cascaded LBE results and benchmark data [15, 16]. Excellent agreement is seen (within 0.01 percent). The streamlines and isotherms for all Rayleigh numbers $Ra = (10^3 - 10^6)$ as shown in Fig.( 2.12) show that the computed results using the present cascaded LBM are in excellent agreement (within 0.01 percent) with the data by [15] and [16]. These numerical results for the comparison are presented in Table ( 3.6). In addition, the temperature contour lines become almost horizontal lines around the center of the cavity as the Rayleigh number increases and the stream lines become more packed next to the side wall as the Rayleigh number increases, i.e. the flow moves faster as natural convection is intensified.

<table>
<thead>
<tr>
<th>Rayleigh number $Ra$</th>
<th>Grid Size ($N_x \times N_y$)</th>
<th>$g \beta = \frac{Ra v \alpha}{\Delta TH^3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^3$</td>
<td>$128 \times 128$</td>
<td>$9.44 \times 10^{-9}$</td>
</tr>
<tr>
<td>$10^4$</td>
<td>$128 \times 128$</td>
<td>$9.44 \times 10^{-8}$</td>
</tr>
<tr>
<td>$10^5$</td>
<td>$128 \times 128$</td>
<td>$9.44 \times 10^{-7}$</td>
</tr>
<tr>
<td>$10^6$</td>
<td>$128 \times 128$</td>
<td>$9.44 \times 10^{-6}$</td>
</tr>
</tbody>
</table>
Figure 2.10: Temperature profiles along horizontal centerline of the cavity at various Rayleigh numbers: $Ra = 10^3, 10^4, 10^5,$ and $10^6$ computed using the cascaded LBM.

Figure 2.11: Temperature profiles along the vertical centerline of the cavity flow at various Rayleigh numbers $Ra = 10^3, 10^4, 10^5,$ and $10^6$ computed using the cascaded LBM.
2.6 Thermal Couette Flow with Viscous Heat Dissipation: Modeling Heat Source

Finally, we consider the simulation of Couette flow with temperature gradient to test the ability of the present thermal cascaded LB model with a source term to describe the viscous heat dissipation. We consider 2D thermal couette flow between two parallel plates, where the upper plate moves along $x$-direction with a velocity $U$, and at higher temperature $T_h$, whereas the bottom wall is stationary and maintained at lower temperature $T_c$; $L$ is the distance between the two plates. In this case the source term $G$ in the thermal energy equation, Eq. (1.17) represents the viscous heat dissipation and is given by

$$G = \frac{2\nu}{C_p} (S : S)$$  \hspace{1cm} (2.19)
Figure 2.13: Streamlines at different values of Rayleigh numbers $Ra = 10^3, 10^4, 10^5$ and $10^6$ for natural convection in a square cavity computed using the cascaded LBM.

$$ S = \frac{1}{2}(\nabla u + (\nabla u)^T) $$

(2.20)

Where $S$ is strain rate tensor, and $C_p$ is the specific heat at constant pressure. The macroscopic governing equations for momentum and energy can be written, respectively, as:

$$ \frac{\partial^2 u}{\partial y^2} = 0, $$

(2.21)

$$ \alpha \frac{\partial^2 T}{\partial y^2} + \frac{v}{C_p} \left[ \frac{\partial u}{\partial y} \right]^2 = 0. $$

(2.22)

The analytical solutions of the velocity and temperature are then given by:

$$ u(y) = U \frac{y}{L}, \quad v = 0 $$

(2.23)
**Figure 2.14**: Vorticity contours at various Rayleigh numbers $Ra = 10^3, 10^4, 10^5$ and $10^6$ for natural convection in a square cavity computed using the LBM.

\[
\frac{T - T_c}{T_h - T_c} = \frac{\nu}{L} + 0.5 Br \frac{y}{L} \left( 1 - \frac{y}{L} \right)
\]

Where $Ec = \frac{\nu^2}{Cp(T_h-T_c)}$ Eckert number, and $Pr = \nu/\alpha$ is the Prandtl number. The effect of viscous heat dissipation is controlled by the Brinkman number $Br = Ec.Pr$. In this simulation, we set $\tau_f = 0.9$, $T_c = 1$ with a grid resolution of $5 \times 41$, $Re = 10$, and $Pr = 0.71$ at Eckert numbers $7, 14,$ and $28$. For the solution of this problem, it is important to note that the convection diffusion equation with a source term (1.17) is coupled with Navier-Stokes equations (1.18b) and (1.18b). This is another cascaded LB model [44] is used to solve the N-S equations. The source term $G$ in Eq. (2.19) can be written as

\[
G = \frac{2\nu}{Cp} \left[ S_{xx}^2 + S_{yy}^2 + 2S_{xy}^2 \right]
\]
In this case, we have

\[ S_{xx} = \frac{\partial u}{\partial x} = 0, \quad S_{yy} = \frac{\partial v}{\partial y} = 0, \quad S_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]  

(2.26)

These strain rate components can be computed locally using the non-equilibrium moments in the cascaded LBE for the fluid flow as mentioned in [44]. Figure (2.15) shows the temperature profiles for different Eckert numbers. In addition, we carried out simulation for different values of Prandtl numbers, 0.25, 1.25, and 2.5 with the Eckert number being fixed at 8 with a grid size $5 \times 41$. Figure (2.16) shows the temperature profiles for different Prandtl numbers. In both these cases, very good agreement between the cascaded LBM results and analytical solutions seen.

Next, we study the convergence rate of the case with a fix $Ec = 8$ at different values of the Prandtl number: 0.25, 1.25, and 2.5. Here, we set $U = 0.07$, $\omega_4 = \omega_5 = \lambda_1 = \lambda_2 = \frac{1}{69}$, and all other relaxation parameters are set to be 1. The values of tunable parameter D at each Prandtl number are -1.2, 0.08, and 0.24 respectively. We also conduct the convergence study for the case where the Prandtl number is fixed at $Pr = 0.71$ while the Eckert number is changed as 7, 14, and 28, $\omega_4 = \omega_5 = 0.9, \lambda_1 = \lambda_2 = 0.94, D = 0$ and all other relaxation parameters are set to be 1. Three grid resolutions in y-direction ($Ny=41, 81, 161$) are employed to both cases of the convergence study. Figs. (2.17) and (2.18) show that the slope of the temperature relative global error is about 2, i.e. the present thermal cascaded LB model is of second order accuracy in space.

Couette flow with viscous heat dissipation was also used to test the ability of the present thermal cascaded Lattice Boltzmann model to simulate relatively high Peclet numbers at low grid resolution $5 \times 23$. The analytical solutions of temperature is given by Eq. (2.24), Here, the Brinkman number $Br$ is rewritten as $Br = \frac{PeEc}{Re}$

\[ \frac{T - T_c}{T_h - T_c} = \frac{y}{L} \left[ 1 + 0.5 \left( \frac{PeEc}{Re} \right) \left( 1 - \frac{y}{L} \right) \right] \]  

(2.27)
FIGURE 2.15: Temperature profiles in Couette flow at various values of Eckert number. Markers represent the cascaded LBE simulations and lines represent the analytical solutions.

FIGURE 2.16: Temperature profiles in Couette flow with different values of Prandtl number. Markers represent the cascaded LBE simulations and lines represent the analytical solutions.
Figure 2.17: Temperature global relative error at different Eckert numbers 7, 14, and 28 for thermal Couette flow with viscous heat dissipation.

Figure 2.18: Temperature global relative error at different Prandtl numbers 0.25, 1.25, and 2.5 for thermal Couette flow with viscous heat dissipation.
Where $Pe$ is Peclet number. In this case study, the temperature profiles for fixed Reynolds number ($Re = 10$), and Eckert number ($Ec = 0.1$) at different values of Peclet numbers $10, 10^2, 10^3, \text{and } 10^4$ are shown in Fig. 2.19. Here, we set $T_h = 1$, $T_c = 0$, $\omega_4 = \omega_5 = \lambda_1 = \lambda_2 = 1.063$, and all other relaxation parameters set to be 1. The values of tunable parameter D at each Peclet number are 0, 0.397, 0.4366, and 0.4406 respectively. Excellent agreement between cascaded LBM results and analytical solution is seen for relatively high Peclet number results. This result is indicative of the improved stability properties of the cascaded LBM as researchers utilizing SRT LBM [17] only presented results for Brinkman numbers up to $Br = 100$.

The convergence study was done for this problem at different values of Peclet numbers $10, 10^2, \text{and } 10^3$ for grid resolutions $Ny=81, 161, \text{and } 321$, $\omega_4 = \omega_5 = \lambda_1 = \lambda_2 = 1.11$, and all other relaxation parameters are set to be 1; the values of tunable parameter D at each Peclet number are set to be 0, 0.36, and 0.696, respectively. The relative global error of temperature against the grid resolutions is shown in Fig. 2.20. It is evident that the slope of the temperature relative global error is about $-2$, i.e the present thermal cascaded LB model is again of second order accuracy in space at relatively high Peclet numbers $10, 10^2, 10^3, 10^4, 10^5$ and $10^6$ (corresponding to Brinkman numbers $Br = 10^{-1}, 1, 10^2, 10^3$ and $10^4$).

2.7 Summary and Conclusion

In this chapter, we presented the isotropic thermal cascaded (LB) MRT model based on central moments and including a source term. This model solves the convection-diffusion equation (CDE) for the temperature field within the double distribution function framework for the D2Q9 lattice, where the fluid motion is represented by another cascaded LB model constructed in in [44]. The collision operator for the thermal field has significantly different cascaded structure for its collision kernel where compared to that for the flow field due to the differences in the number of collision invariants between them. A consistent second order scheme to
FIGURE 2.19: Temperature profiles in thermal Couette flow at various values of Peclet numbers: $Pe = 10, 10^2, 10^3, 10^4, 10^5$ and $10^6$. Markers represent the Cascaded LBE results and lines represent the analytical solution.

FIGURE 2.20: Temperature global relative error at different Peclet numbers: $Pe = 10, 10^2$, and $10^3$ for thermal Couette flow with viscous heat dissipation.
incorporate the effect of locally varying heat sources by means of a variable transformation for the thermal cascaded LB model is also discussed. A Chapman-Enskog analysis of the thermal cascaded LB model shows its consistency with the CDE including a source term. It also provides expressions for temperature gradients in the augmented moment equilibria in terms of locally known non-equilibrium moments. The new thermal cascaded LBE is validated for a number of benchmark problems, including thermal Poiseuille flow, thermal Couette flow and natural convection in a square cavity. Comparison of the temperature profiles under different conditions for these problems, as well as the average Nusselt number at different Rayleigh numbers in the case of the natural convection within a square cavity, with prior benchmark results demonstrate high accuracy of the isotropic thermal cascaded LBE model. Furthermore, it is shown numerically that the model is second order accurate in space for a range of thermal convective flow problems.
TABLE 2.2: Comparison between numerical results obtained using the cascaded LBM and the published results (de Vahl Davis(1983) and Hortmann et al(1990)) at different Rayleigh numbers ($Ra = 10^3 - 10^6$).

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<thead>
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</tr>
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CHAPTER III

ANISOTROPIC THERMAL FLOW

3.1 Numerical Results

In this section, numerical tests are carried out to validate the accuracy of our proposed cascaded LBE model for anisotropic convective diffusion thermal flows. The applicability of the anisotropic LBE model is validated by numerical simulations including the convection and diffusion of a Gaussian Hill, solving anisotropic convection diffusion equation with variable diffusion tensor and variable source term, and anisotropic natural convection in a square cavity. In this study, the periodic boundary conditions and bounce-back are employed for temperature boundary conditions.

3.1.1 Convection and Diffusion of a Gaussian Hill: No Source Term

The following convection-diffusion equation (CDE) without a source term

\[
\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi = \nabla \cdot (\alpha \nabla \phi)
\]  

(3.1)

Here, the initial distribution of the scalar variable is given as

\[
\phi(x, 0) = \frac{\phi_0}{2\pi \sigma_0^2} \exp \left(-\frac{x^2}{2\pi \sigma_0^2}\right)
\]  

(3.2)
Where $\sigma_0^2$ is the initial variance, $\phi_0 = 2\pi \sigma_0^2$ is the total concentration. The analytical solution of this problem is given by

$$
\phi(x, t) = \frac{\phi_0}{2\pi \sqrt{||\sigma_t||}} \exp \left( -\frac{1}{2} \sigma_t^{-1} : ([x - ut](x - ut)) \right)
$$

(3.3)

Where $\sigma_t = \sigma_0^2 I + 2t\alpha$. $||\sigma_t||$ is the absolute value of the determinant of $\sigma_t$, while $\sigma_t^{-1}$ is the inverse of $\sigma_t$. The computational domain is chosen to be $[-1, 1] \times [-1, 1]$.

Three types of diffusion tensors are considered

$$
\alpha = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times 10^{-4}, \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \times 10^{-4}, \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \times 10^{-4}
$$

Which represent the isotropic convection diffusion, diagonal anisotropic convection diffusion, and full anisotropic convection diffusion problems respectively.

The periodic boundary condition is employed to this problem. We conduct our numerical simulation with a grid resolution of $151 \times 151$, $\sigma_0 = 0.05$. First, we examine the pure diffusion for the three diffusion tensors given above by setting $u = (0, 0)$ at time $t_m$ as shown in Fig. 3.2 and Fig. 3.3 then we test the convection diffusion case by set $u = (0.05, 0.05)$ at time $t_m$, and $0.5t_m$ as shown in Fig. 3.4. The time $t_m$ is determined by $\phi_{max}(x, t_m) = 0.5\phi_{max}(x, 0)$ as

$$
t_m = \frac{\sigma_0^2}{4} \left( \sqrt{(\alpha_{11} + \alpha_{22})^2 + 12||\alpha|| - \alpha_{11} - \alpha_{22}} \right)
$$

(3.4)

Where $||\alpha|| = |\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}|$, $t_m$ is equal to 6.25, 5.5375, 5.8547 for the isotropic, diagonal anisotropic, and full anisotropic case respectively.

The simulation results and the analytical solutions are compared at the three different diffusion tensors as in [59]. The relaxation rates are taken $\lambda_{11} = \lambda_{22} = 1.96$, $\lambda_{12} = \lambda_{21} = 0$ for isotropic case, $\lambda_{11} = 1.96$, $\lambda_{22} = 1.85$, $\lambda_{12} = \lambda_{21} = 0$ for diagonal anisotropic case, and $\lambda_{11} = 1.96$, $\lambda_{22} = 1.85$, $\lambda_{12} = \lambda_{21} = 0.036$ for full anisotropic case, with $\delta x^2/\delta t = 0.03$. The good agreement with the analytical solutions is shown for all thermal diffusion coefficients.
Figure 3.1: Distribution of the scalar variable $\phi$ at the time $t = t_m$ and $u = v = 0.0$. 

(a) Isotropic diffusion, $t = t_m$

(b) Diagonally anisotropic diffusion, $t = t_m$
Figure 3.2: Distribution of the scalar variable \( \phi \) for diffusion \( u = (0, 0) \) of a Gaussian hill computed using the LBM.
We also test the convergence rate of the present model. For this purpose, a set of simulations are conducted at four different grid sizes, i.e., $101 \times 101$, $151 \times 151$, $201 \times 201$, and $301 \times 301$. The global relative error of temperature ($E_T$) used to measure the accuracy of the model is calculated as

$$E_T = \frac{||(T_c - T_a)||_2}{||T_a||_2}$$  \hspace{1cm} (3.5)$$

Where $||.||_2$ is the Euclidean norm, $||(T_c - T_a)||_2 = \sqrt{\sum_i (T_{c,i} - T_{a,i})^2}$, $||T_a||_2 = \sqrt{\sum_i (T_{a,i})^2}$, $T_c$ and $T_a$ are the computed and the analytical solutions respectively. The relative global error of temperature for the isotropic, diagonal anisotropic, and full anisotropic case are plotted in Fig. 3.8. It can be seen that the temperature global error decreases with increase in grid resolution with a slope of 2 in the log-log plot. Hence, our present cascaded LBM model with source term is
second order accurate.

(a) Isotropic diffusion    (b) Diagonally anisotropic

(c) Fully anisotropic diffusion

**Figure 3.4:** Contours of the scalar variable $\phi$ at $u = (0.05, 0.05)$ computed using LBE

### 3.1.2 Convection-Diffusion of a Gaussian Hill: Stability Test

We consider the convection diffusion equation, Eq. (1.17), where $u = u_o \hat{i} + v_o \hat{j}$ is a prescribed 2-D uniform velocity field and subjected to the Gaussian hill initial condition.

$$T(x, y, 0) = \frac{T_o}{2\pi\sigma_o^2} \exp\left(\frac{-[x^2 + y^2]}{2\sigma_o^2}\right),$$  \hspace{1cm} (3.6)
where the parameter $\sigma_0$ controls the width of the profile. The analytical solution of this problem is given by

$$T(x, y, t) = \frac{T_o}{2\pi(\sigma_o^2 + 2\alpha t)} \exp \left( -\frac{[(x - u_0 t)^2 + (y - v_0 t)^2]}{2\sigma_o^2 + 2\alpha t} \right).$$

(3.7)

We set $\sigma_0 = 0.05$ and advect the profile with the diagonal velocity vector $u_o = v_o = 0.25c_s$. We choose $T_o = 2\pi\sigma_0^2$ so that the initial profile has a peak magnitude of 1.0. Periodic boundary conditions for the temperature are employed. In what follows we vary the fluid diffusivity to compare the stability characteristics of the cascaded central moment LBM with the SRT and MRT implementations of the LBM. We consider the MRT method in [59] and we set the tunable parameters in both methods to be zero. In the three methods the fluid diffusivity is given by
\[ \alpha = c_s^2 \left( \frac{1}{\lambda} - \frac{1}{2} \right) \] where \( \lambda \) is the relaxation rate of the first order moments corresponding to the equilibrium moments \( u_x T \) and \( u_y T \) in the cascaded and MRT LBM. We use a \( 521 \times 521 \) grid and vary the diffusivity by varying the relaxation time \( \tau_g = \frac{1}{\lambda} \). Comparison of the cascaded and MRT LBM methods is complicated by the large number of relaxation parameters associated with each method. We set the relaxation time for the first order moments to \( \tau_g \). We then set all other relaxation times to 1.0. With this choice of parameters we are relaxing the energy fluxes (first order moments) at the same rate in both methods. The higher order moments are also relaxed at the same rate. This choice of parameters is not necessarily optimal for either method but it does give us a rational basis for comparison.

Finally, we consider numerical stability results from the SRT, MRT, and cascaded LBM at various values of the relaxation time \( \tau_g \). Table 3.1 provides the numerical stability and global relative error (given by Eq. (3.5) indicates the accuracy of the three methods is similar. The table also indicates that the SRT LBM is not stable at the smaller diffusivities and that the MRT is eventually not stable at the still smaller diffusivity for which the cascaded LBM is stable. It is likely that relaxation rates for the higher moments of the MRT and cascaded LB methods may be found that result in more stable behavior but this short study concurs with other work that indicate superior stability characteristics of the cascaded LBM [61, 63].

**Table 3.1:** Global relative error and stability characteristics after 1000 time increments for SRT, MRT, and Cascaded LBM for different relaxation times for the convection-diffusion of a Gaussian hill problem at Mach number 0.25.

<table>
<thead>
<tr>
<th>( \tau_g )</th>
<th>0.55</th>
<th>0.51</th>
<th>0.501</th>
</tr>
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<tbody>
<tr>
<td>( \alpha )</td>
<td>( 1.67 \times 10^{-2} )</td>
<td>( 3.33 \times 10^{-3} )</td>
<td>( 3.33 \times 10^{-4} )</td>
</tr>
<tr>
<td>SRT</td>
<td>0.0097</td>
<td>unstable</td>
<td>unstable</td>
</tr>
<tr>
<td>MRT</td>
<td>0.0086</td>
<td>0.0096</td>
<td>unstable</td>
</tr>
<tr>
<td>Cascaded</td>
<td>0.0101</td>
<td>0.0108</td>
<td>0.0110</td>
</tr>
</tbody>
</table>
3.1.3 Anisotropic Convection Diffusion Equation: Variable Source Term

The following anisotropic convection-diffusion equation (CDE) with a constant velocity and variable diffusion tensor $\mathbf{\alpha}$ and with a distributed source term

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = \nabla \cdot (\mathbf{\alpha} \cdot \nabla \phi) + G$$

(3.8)

Where $\mathbf{u} = (u_x, u_y)$ and $G = G(x, y, t)$ is the source term. The initial distribution of the scalar variable is given by

$$\phi(x, y, 0) = \sin(2\pi x)\sin(2\pi y).$$

(3.9)

The analytical solution of this problem is given by

$$\phi = \exp \left[ \left( 1 - 12\pi^2 \mathbf{\alpha} \right) t \right] \sin(2\pi x)\sin(2\pi y).$$

(3.10)

The source term is defined as

$$G = \exp \left[ \left( 1 - 12\pi^2 \mathbf{\alpha} \right) t \right] \left\{ \sin(2\pi x)\sin(2\pi y) + 4\mathbf{\alpha} \cos(4\pi x)\sin^2(2\pi y) \\
+ 2\pi \left[ u_x \cos(2\pi x)\sin(2\pi y) + u_y \sin(2\pi x)\cos(2\pi y) \right] \right\}.$$  

(3.11)

The computational domain is chosen to be $[0, 1] \times [0, 1]$. Here, the diffusion tensor is a function of space $x, y$. The diffusion tensor $\mathbf{\alpha}$ is given by a diagonal matrix

$$\mathbf{\alpha} = \alpha \times \begin{pmatrix} 2 - \sin(2\pi x)\sin(2\pi y) & 0 \\
0 & 1 \end{pmatrix}$$

(3.12)

Where $\alpha$ is constant and is given to be $\alpha = 1.0 \times 10^{-3}$. Here, the diffusion tensor $\mathbf{\alpha}$ represents the diagonal anisotropic convection diffusion. The periodic boundary condition is employed to this problem. We conduct our numerical simulation with a grid resolution of $101 \times 101$ as shown in Fig. 3.7, we test the convection diffusion case.
by set $u = (0.1, 0.1)$ at time $t = 3$ and $Pe = 100$. The relaxation time is set to
$$\tau_{11} = 3 \times 10^{-3} (2 - \sin(2\pi x)\sin(2\pi y)) + 0.5, \tau_{22} = 0.54, \tau_{12} = \tau_{21} = 0.$$ Fig. 3.7 shows
the temperature distribution for the given values of diffusivity coefficients which
show very good agreement with the analytical solutions. We also examine the spatial
accuracy of the present model. In this regard, a set of simulations are performed at
five different grid sizes, i.e., $201 \times 201, 301 \times 301, 401 \times 401, 501 \times 501$ and $521 \times 521$
for given value of the diffusion tensor. The relative global error of temperature for the
given value of diffusivity coefficient are plotted in Fig. 3.9. It can be seen that the
temperature global error decreases with increase in grid resolution with a slope of $-2$
in the log-log plot. Hence, our present cascaded LBM model with source term is
second order accurate.

3.2 Natural Convection in a Square Cavity: Anisotropic Case

Natural convection heat transfer and fluid flow in cavities are important subjects of
investigation due to their effect on many engineering applications and nature
phenomena, such as thermal power, petrochemical industries, aerospace,
construction and solar collectors. In this section, the effect of anisotropy is performed
for a heated cavity for the following values of Rayleigh number
($Ra = 10^3, 10^4, 10^5, 10^6, 10^7$ and $10^8$). In this regard, we define the $x$ thermal
diffusivity as $\alpha_x$ and $y$ thermal diffusivity as $\alpha_y$ cascaded LBE model is employed to
simulate natural convection in a square cavity. Here, the flow is driven by the
buoyancy force due to the local temperature difference against a reference
temperature in the present of gravity. The left wall is maintained at higher
temperature $T_h$ and the right wall at lower temperature $T_c$, while the top and bottom
walls are considered to be adiabatic. The macroscopic governing equations can be
Figure 3.6: Distribution of the scalar variable $\phi$ at the time $t = 3$ and $Pe = 100$. 
Figure 3.7: Contours of the scalar variable $\phi$ at the time $t = 3$ and $Pe = 100$. 

(a) Numerical Solution

(b) Analytical solution
**FIGURE 3.8:** Temperature global relative errors at different grid sizes for diffusion of a Gaussian hill.

**FIGURE 3.9:** Temperature global relative error with variable diffusion tensor and source term.
expressed as follows:

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \quad (3.13a) \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (3.13b) \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + F, \quad (3.13c) \\
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} &= \alpha_x \frac{\partial^2 T}{\partial x^2} + \alpha_y \frac{\partial^2 T}{\partial y^2}. \quad (3.13d)
\end{align*}
\]

where \( F \) is the body force which is based on the Boussinesq approximation and is given by

\[ F = g \beta (T - T_0) \hat{j} \quad (3.14) \]

Here, \( \beta \) is the thermal expansion coefficient, \( g \) is the acceleration due to gravity, \( T_0 = (T_h + T_c) / 2 \) is the reference temperature, \( \hat{j} \) is the unit vector in positive y-direction. This classical natural convection problem is governed by two non-dimensional parameters: The Prandtl number \( Pr \)

\[ Pr = \frac{\nu}{\alpha} \]

and the Rayleigh number \( Ra \), which are given by

\[
\begin{align*}
Ra_x &= \frac{g \beta \Delta T H^3}{\nu \alpha_x} \quad (3.16) \\
Ra_y &= \frac{g \beta \Delta T H^3}{\nu \alpha_y} \quad (3.17)
\end{align*}
\]

where, \( \Delta T = T_h - T_c \) is the temperature difference between hot and cold walls, and \( H \) is the height of the square cavity.
\( \alpha_x \) is the thermal diffusivity and \( \alpha_y \) is the thermal diffusivity. The choice for the anisotropic as follows.

The thermal diffusivity given by \( \alpha_x = 0.5\alpha_y \) and \( \alpha_x = 2\alpha_y \) for different Rayleigh numbers \( (Ra_y = 10^3, 10^4, \ldots, 10^8) \). The average of Nusselt number for the right wall is given by

\[
Nu = \frac{1}{\alpha_x \Delta T} \int_0^H \bar{q}_x(x,y) dy
\]  

(3.18)

where \( \bar{q}_x(x,y) = uT(x,y) - \alpha_x \frac{\partial T}{\partial x} \) is the local heat flux in \( x \)-direction.

The boundary conditions on the cavity walls can then be summarized as:

On the left wall:

\[
u_x = u_y = 0, \quad T = T_h = 21,
\]  

(3.19)

On the right wall:

\[
u_x = u_y = 0, \quad T = T_c = 1,
\]  

(3.20)

On the top wall:

\[
u_x = u_y = 0, \quad \frac{\partial T}{\partial y} = 0,
\]  

(3.21)

On the bottom wall:

\[
u_x = u_y = 0, \quad \frac{\partial T}{\partial y} = 0.
\]  

(3.22)

The general bounce-back scheme [48] is adopted to treat the thermal Dirichlet and Neumann boundary conditions. In this simulations, \( Pr = 0.71 \), the relaxation rates for fluid flow and temperature are set as 0.55, 0.57 respectively. The streamlines and the isotherms for the ranges of Rayleigh number \( Ra_y \) between \( 10^3 \) to \( 10^6 \) are shown in Fig. 2.12. Also, the vorticity contours for \( Ra_y = 10^3 - 10^8 \) are shown in Fig. 2.14. The streamlines, isotherms and vorticity contours are in very good correspondence and consistent with prior benchmark solution results [15, 16]. The natural convection flow patterns become more complex as \( Ra \) increases. In order to characterized this in more detail, the temperature at the vertical and horizontal mid-planes of the square cavity,
i.e. \( x/H = 0.5 \) and \( y/H = 0.5 \), respectively for various Rayleigh numbers \((Ra_y = 10^3 - 10^8)\) are presented in Figs. 2.10 and 2.11. In these cases, the value of the factor \( g\beta \) needed in the simulation is obtained as a function of Rayleigh number \( Ra \) using

\[
g\beta = \frac{\nu a_y R a_y}{\Delta TH^3} \quad (3.23)
\]

The representative values of \( g\beta \) corresponding to each \( Ra \) is shown in Table 3.2. From Figs. 2.10 and 2.11, it is seen that the temperature contour lines become almost horizontal around the center of the cavity as the Rayleigh number \( Ra \) increases. The streamlines become more packed next to the side wall as \( Ra \) increases, i.e. the flow moves faster as natural convection is intensified. Finally, Table 3.6 shows quantitative comparison between the key parameters for this problem (average Nusselt number, maximum velocities magnitudes and their locations) between the present cascaded LBE results and benchmark data [15, 16]. Excellent agreement is seen (within 0.01 percent). The streamlines and isotherms for all Rayleigh numbers \( Ra_y = (10^3 - 10^8) \)

### Table 3.2: Values of \( g\beta \) corresponding to each Rayleigh number.

<table>
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<tr>
<th>Rayleigh number ( Ra )</th>
<th>Grid Size ((N_x \times N_y))</th>
<th>( g\beta = \frac{Ra_y \nu a_y}{\Delta TH^3} )</th>
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as shown in Fig. (2.12) show that the computed results using the present cascaded LBM are in excellent agreement (within 0.01 percent) with the data by [15] and [16]. These numerical results for the comparison are presented in Table (3.6). In addition, the temperature contour lines become almost horizontal lines around the center of the cavity as the Rayleigh number increases and the stream lines become more packed next to the side wall as the Rayleigh number increases, i.e. the flow moves faster as natural convection is intensified.

73
3.3 Summary and Conclusion

In this chapter, we presented the numerical results for simulation of anisotropic thermal cascaded (LB) MRT model based on central moments and including a source term. This model solves the anisotropic convection-diffusion equation (CDE) for the temperature field within the double distribution function framework for the D2Q9 lattice, where the fluid motion is represented by another cascaded LB model constructed in [44]. A Chapman-Enskog analysis of the anisotropic thermal cascaded
Figure 3.10: Streamlines at different values of Rayleigh numbers $Ra = 10^3, 10^4, 10^5, 10^6, 10^7$ and $10^8$ for natural convection in a square cavity computed using the cascaded LBM. (Left) $\alpha_x = \alpha_y / 2$, (center) $\alpha_x = \alpha_y$, (right) $\alpha_x = 2\alpha_y$.

LB model shows its consistency with the CDE including a source term. It also provides expressions for temperature gradients in the augmented moment equilibria in terms of locally known non-equilibrium moments.

The new thermal cascaded LBE is validated for a number of benchmark problems, including convection-diffusion of a Gaussian Hill, solving anisotropic convection diffusion equation with variable diffusion tensor and variable source term, and anisotropic natural convection in a square cavity. Comparison of the temperature profiles under different conditions for these problems, as well as the average Nusselt number at different Rayleigh numbers in the case of the anisotropic natural convection within a square cavity, with prior benchmark results demonstrate high accuracy of the anisotropic thermal cascaded LBE model. Furthermore, it is shown numerically that the model is second order accurate in space for a range of anisotropic thermal convective flow problems.

Finally, a stability test of the present cascaded LBE model is conducted to compare our model by single relaxation time (SRT) LB model and multiple relaxation
times (MRT) LB model using the diffusion in a Gaussian Hill as a test problem by varying the fluid diffusivity to compare the stability characteristics of the cascaded central moment. This stability study concurs with other work that indicates superior stability characteristics of the cascaded LB method. The present anisotropic thermal cascaded LB model exhibits improved stability characteristics over the SRT LB model and the conventional multiple MRT LB model.
Figure 3.11: Vorticity at different values of Rayleigh numbers $Ra = 10^3, 10^4, 10^5, 10^6, 10^7$ and $10^8$ for natural convection in a square cavity computed using the cascaded LBM. (Left) $\alpha_x = \alpha_y / 2$, (center) $\alpha_x = \alpha_y$, (right) $\alpha_x = 2\alpha_y$. 
Table 3.3: Numerical results obtained using the cascaded LBM for anisotropic natural convection in a square cavity at different Rayleigh numbers ($Ra = 10^3, 10^4, 10^5, 10^6, 10^7$ and $10^8$).

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Table 3.6: Comparison between numerical results obtained using the cascaded LBM and the published results (de Vahl Davis(1983) and Hortmann et al(1990)) for isotropic natural convection in a square cavity at different Rayleigh numbers ($Ra = 10^3 - 10^8$).

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<td>\Psi</td>
<td>_{max}$</td>
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<td>$</td>
<td>\Psi</td>
<td>_{max}$</td>
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Figure 3.12: Isothermals at different values of Rayleigh numbers $Ra = 10^3, 10^4, 10^5$ and $10^6$ for natural convection in a square cavity computed using the cascaded LBM.

- (Left) $\alpha_x = \alpha_y / 2$
- (center) $\alpha_x = \alpha_y$
- (right) $\alpha_x = 2\alpha_y$
Figure 3.13: First component of velocity ($u$) at different values of Rayleigh numbers $Ra = 10^3, 10^4, 10^5, 10^6, 10^7$ and $10^8$ for natural convection in a square cavity computed using the cascaded LBM. (Left) $\alpha_x = \alpha_y/2$, (center) $\alpha_x = \alpha_y$, (right) $\alpha_x = 2\alpha_y$
Figure 3.14: Second component of velocity (v) at different values of Rayleigh numbers $Ra = 10^3, 10^4, 10^5, 10^6, 10^7$ and $10^8$ for natural convection in a square cavity computed using the cascaded LBM. (Left) $\alpha_x = \alpha_y/2$, (center) $\alpha_x = \alpha_y$, (right) $\alpha_x = 2\alpha_y$
Figure 3.15: Pressure at different values of Rayleigh numbers $Ra = 10^3, 10^4, 10^5, 10^6, 10^7$ and $10^8$ for natural convection in a square cavity computed using the cascaded LBM. (Left) $\alpha_x = \alpha_y / 2$, (center) $\alpha_x = \alpha_y$, (right) $\alpha_x = 2\alpha_y$.
FIGURE 3.16: Temperature profiles along horizontal centerline of the cavity at various Rayleigh numbers: $Ra = 10^3, 10^4, 10^5, 10^6, 10^7, \text{ and } 10^8$ computed using the cascaded LBM.
Figure 3.17: Temperature profiles along the vertical centerline of the cavity flow at various Rayleigh numbers $Ra = 10^3, 10^4, 10^5, 10^6, 10^7, \text{ and } 10^8$ computed using the cascaded LBM.
CHAPTER IV

FLOW WITH VARIABLE VISCOSITY

4.1 Numerical Results

In this section, numerical test is carried out to validate the accuracy of our proposed cascaded LBE model for variable viscosity thermal flows. We consider a very basic flow configuration to isolate the effect of viscosity variation, where the fluid flows between parallel plates, i.e., the flows is through a thin channel. In this study, the periodic boundary condition is employed for temperature boundary conditions.

4.1.1 Couette Flow with shear heating: Variable viscosity

Here, we consider incompressible fluid with an exponential viscosity variation flows through the narrow channel of with H and length L in the x-direction, the flow is driven by shear or velocity on surface $z = H$. The problem configuration is shown in FIG. (4.1). The top wall is maintained at temperature $T_u$ and the bottom wall at temperature $T_1$. The macroscopic governing equations can be expressed as follows:

\[\mu = e^{-\beta T}\]  \hspace{1cm} (4.1)

\[\frac{\partial u}{\partial z} = Ae^{\beta T}\]  \hspace{1cm} (4.2)

\[\frac{\partial^2 T}{\partial z^2} + B_rA^2e^{\beta T} = 0.\]  \hspace{1cm} (4.3)

92
Where $Br = \frac{\mu_0 U^2}{\alpha \Delta T}$ is the Brinkman number, $\Delta T = T_u - T_l$, $A$ is the shear stress, $\beta$ and $\mu_0$ are constant.

The analytical solutions of the velocity and temperature are then given by:

$$u(z) = \frac{\sqrt{2}e^{0.5\beta T_m}}{\sqrt{\beta Br}} \left( \tanh \left[ \frac{A\sqrt{\beta Br}}{\sqrt{2}e^{-0.5\beta T_m}} - \tanh^{-1} \left( \sqrt{1 - e^{-\beta T_m}} \right) \right] + \sqrt{1 - e^{-\beta T_m}} \right)$$

(4.4)

$$T = T_m + \frac{1}{\beta} \ln \left( 1 - \tanh^2 \left[ \tanh^{-1} \left( \sqrt{1 - e^{-\beta T_m}} \right) - \frac{A\sqrt{\beta Br}}{\sqrt{2}e^{-0.5\beta T_m}} \right] \right)$$

(4.5)

Where $T_m$ is the maximum temperature, i.e. The temperature where the temperature gradient is zero.

We conduct our numerical simulation with a grid resolution of $5 \times 23$

$T_u = 1, T_l = 0, \tau_t = 0.65, U = 1, \text{ and } \tau_f = 3e^{-T} + 0.5$ for different Brinkman numbers $Br = Br_0, 5, 10, \text{ and } 20$. The simulation results and the analytical solutions for temperature field are compared at the four different Brinkman numbers $Br = 3.43, 5, 10, \text{ and } 20$, while the simulation results and the analytical solutions for temperature field are compared at the three different Brinkman numbers $Br = 20$. 
**Figure 4.2:** Temperature profiles for velocity driven flow at various Brinkman numbers: $Br = 3.43, 5, 10,$ and $20$. Markers represent the Cascaded LBE results and lines represent the analytical solution.

**Figure 4.3:** Velocity profiles for velocity driven flow at various Brinkman numbers: $Br = 3.43, 10,$ and $20$. Markers represent the Cascaded LBE results and lines represent the analytical solution.
Br = 3.43, 10, and 20. The good agreement with the analytical solutions is shown for all cases for temperature and velocity in FIG. (4.2) and FIG. (4.3) respectively.

4.2 Summary and Conclusion

The new central moment cascaded LB model was adapted to simulate of fluid flow with temperature-dependent viscosity for isotropic case, where the fluid viscosity is exponentially varying with temperature. The simulation results of Couette flow with shear heating confirmed the validity of the present central moment LB model to incorporate correctly fluid flow with variable viscosity at different Brinkman numbers.
A new cascaded central moment based lattice Boltzmann (LB) method for solving low Mach number convective thermal flows with source terms in two-dimensions in a double distribution function framework is presented. For the passive temperature field, which satisfies a convection diffusion equation (CDE) along with a source term to represent internal/external local heat source, a new cascaded collision kernel is presented. Due to the use of a single conserved variable in the thermal energy equation, the cascaded structure in its collision operator begins from the first order moments and evolves to higher order moments. This is markedly different from the collision operator for the fluid flow equations, constructed in previous work, where the cascaded formulation starts at the second order moments in its collision kernel. A consistent implementation of the spatially and temporally varying source terms in the thermal cascaded LB method representing the heat sources in the CDE that maintains second order accuracy via a variable transformation is discussed. In addition, the first order equilibrium moments in this model are augmented with spatial temperature gradient terms obtained locally and involving a tunable coefficient to maintain additional flexibility in the representation of the transport coefficient for the temperature field. The consistency of the thermal cascaded LB method including a source term with the macroscopic convection-diffusion equation is demonstrated by means of a Chapman-Enskog analysis. The emergent tunable diffusivity is shown to be dependent on the relaxation times of the first order moments as well as the tunable parameter in the additional gradient terms in our
cascaded multiple-relaxation-time formulation. The new model is tested on a set of benchmark problems such as the thermal Poiseuille flow, thermal Couette flow with either wall injection or including viscous dissipation and natural convection in a square cavity. The validation studies show that the thermal cascaded LB method with source term is in very good agreement with analytical solutions or numerical results reported for benchmark problems. In addition, the numerical results show that our new thermal cascaded LB model maintains second order spatial accuracy.

The new LBE model is modified to simulate anisotropic fluids that are characterized by different diffusion coefficients along different directions. The applicability of the LBE model is validated by numerical simulations including the convection and diffusion of a Gaussian Hill, solving anisotropic convection diffusion equation with variable diffusion tensor and variable source term, and anisotropic natural convection in a Square Cavity. The validation study shows that the anisotropic thermal cascaded LB model with source term is in very good agreement with the analytical solutions or numerical results reported for the benchmark problems. In addition, the numerical results show that our new anisotropic thermal cascaded LB model maintains second order accuracy as does the isotropic model.

A stability test of the present cascaded LBE model is conducted to compare our model by single relaxation time (SRT) LB model and multiple relaxation times (MRT) LB model using the diffusion in a Gaussian Hill as a test problem by varying the fluid diffusivity to compare the stability characteristics of the cascaded central moment. This stability study concurs with other work that indicates superior stability characteristics of the cascaded LB method.

Finally, the central moment cascaded LB model was adapted to simulate of fluid flow with temperature-dependent viscosity, where the fluid viscosity is exponentially varying with temperature. The simulation results of couette flow with shear heating confirmed the validity of the present central moment LB model to incorporate correctly fluid flow with variable viscosity at different Brinkman numbers.


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APPENDIX I

CHAPMAN-ENSKOG ANALYSIS OF THE ISOTROPIC THERMAL CASCADED LBM

In this section, a Chapman-Enskog (C-E) analysis of the thermal cascaded LBM is presented and the results of the analysis provide the macroscopic emergent equations, viz; the convection-diffusion equation (CDE) with a source term given in Eq. (1.17) earlier. In this regard, we consider the strategy of rewriting the central moment LBM in terms of the relaxation to a generalized equilibrium in the rest frame of reference. To facilitate analysis and its establish consistency to the CDE, it is sufficient to consider terms only up to second order in Mach number in such an equivalent formulation [24, 44]. In this regard, we consider performing calculations in terms of various raw moments (designned with "prime symbols") with respect to the non-orthogonal moment basis vectors collected in the matrix $\mathcal{T}$ and given in Eq. (1.19). First, the base raw moment equilibrium $\hat{\kappa}_{x^m y^n}^{eq}$ can be obtained from the corresponding central moment equilibria $\hat{\kappa}_{x^m y^n}^{eq}$ given in equation Eq. (1.29). This
\[ \hat{\kappa}_0^{eq} = T, \quad (1.1a) \]
\[ \hat{\kappa}_x^{eq} = u_x T, \quad (1.1b) \]
\[ \hat{\kappa}_y^{eq} = u_y T, \quad (1.1c) \]
\[ \hat{\kappa}_{xx}^{eq} = c_s^2 T + u_x^2 T, \quad (1.1d) \]
\[ \hat{\kappa}_{yy}^{eq} = c_s^2 T + u_y^2 T, \quad (1.1e) \]
\[ \hat{\kappa}_{xy}^{eq} = u_x u_y T, \quad (1.1f) \]
\[ \hat{\kappa}_{xxy}^{eq} = c_s^2 T + u_x^2 u_y T, \quad (1.1g) \]
\[ \hat{\kappa}_{xyy}^{eq} = c_s^2 T + u_y^2 u_x T, \quad (1.1h) \]
\[ \hat{\kappa}_{xxyy}^{eq} = c_s^4 T + c_s^2 (u_x^2 + u_y^2) T + u_x^2 u_y^2 T. \quad (1.1i) \]

Similarly, the raw moment for the source terms can be obtained from their corresponding central moments Eq. (1.30), which are presented in Eq. (1.32a). Now, for convenience, the various raw moments can be related to their corresponding starts in the velocity space via the non-orthogonal transformation matrix \( T \). We now define raw moments of distribution functions (including the transformed one), equilibrium and sources for convenience as

\[ \hat{g} = T g, \quad \hat{\tilde{g}} = T \tilde{g}, \quad \hat{S} = T S, \quad (1.2) \]

where \( \hat{\cdot} \) represents column vectors in (raw) moment space and the transformation matrix \( T \) is given in Eq. (1.19). That is,

\[ \hat{g} = (\hat{g}_0, \hat{g}_1, \hat{g}_2, \ldots, \hat{g}_8)^\dagger = \left( \hat{\kappa}_0', \hat{\kappa}_x', \hat{\kappa}_y', \hat{\kappa}_{xx}', \hat{\kappa}_{yy}', \hat{\kappa}_{xy}', \hat{\kappa}_{xxy}', \hat{\kappa}_{xyy} \right)^\dagger, \]
\[ \hat{\tilde{g}} = (\hat{\tilde{g}}_0, \hat{\tilde{g}}_1, \hat{\tilde{g}}_2, \ldots, \hat{\tilde{g}}_8)^\dagger = \left( \hat{\tilde{\kappa}}_0', \hat{\tilde{\kappa}}_x', \hat{\tilde{\kappa}}_y', \hat{\tilde{\kappa}}_{xx}', \hat{\tilde{\kappa}}_{yy}', \hat{\tilde{\kappa}}_{xy}', \hat{\tilde{\kappa}}_{xxy}', \hat{\tilde{\kappa}}_{xyy} \right)^\dagger, \]
\[ \hat{S} = (\hat{S}_0, \hat{S}_1, \hat{S}_2, \ldots, \hat{S}_8)^\dagger = \left( \hat{\sigma}_0', \hat{\sigma}_x', \hat{\sigma}_y', \hat{\sigma}_{xx}', \hat{\sigma}_{yy}', \hat{\sigma}_{xy}', \hat{\sigma}_{xxy}', \hat{\sigma}_{xyy} \right)^\dagger. \]
In addition, in order to maintain flexibility in the specification of the transport coefficient (i.e. the thermal diffusivity) appearing in the emergent CDE, we specify the raw moment equilibrium \( \hat{g}^{eq} \) by augmenting the base moment equilibria given in Eq. (2.1i) with an extended first order moment equilibria involving the components of the temperature gradients with an adjustable coefficient (designated as D below). That is

\[
\hat{g}^{eq} = T \hat{g}^{eq}
\]  

(1.3)

Here

\[
\hat{g}^{eq} = \hat{g}^{eq(0)} + \hat{g}^{eq(1)}
\]  

(1.4)

Where, \( \hat{g}^{eq(0)} \) and \( \hat{g}^{eq(1)} \) are the base and extended moment equilibria, respectively, and are given as follows:

\[
\hat{g}^{eq(0)} = \left( \hat{g}^{eq(0)}, \hat{g}^{eq(0)}, \ldots, \hat{g}^{eq(0)} \right)^\dagger = \left( \hat{\kappa}_0, \hat{\kappa}_x, \hat{\kappa}_y, \hat{\kappa}_xx + \hat{\kappa}_{yy}, \hat{\kappa}_{xy}, \hat{\kappa}_{xxy}, \hat{\kappa}_{xyy}, \hat{\kappa}_{xxyy} \right)^\dagger
\]  

(1.5)

\[
\hat{g}^{eq(1)} = \left( \hat{g}^{eq(1)}, \hat{g}^{eq(1)}, \hat{g}^{eq(1)}, \ldots, \hat{g}^{eq(1)} \right)^\dagger = \left( 0, c_s^2 D \partial_x T, c_s^2 D \partial_y T, 0, 0, 0, 0, 0, 0 \right)
\]  

(1.6)

We can then rewrite post-collision state of the thermal cascaded LBE in Eq. (??) in terms of the following:

\[
\bar{g}^p = \bar{g} + T^{-1} \left[ -\Lambda \left( \hat{g} - \hat{g}^{eq} \right) + \left( I - \frac{1}{2} \Lambda \right) \hat{S} \right]
\]  

(1.7)
where $\Lambda$ is a diagonal collision matrix given by

$$\Lambda = \text{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_8).$$  \hfill (1.8)

Now, applying a Chapman-Enskog multiscale expansion by expanding the raw moments $\hat{g}$ and the time derivative in terms of a small perturbation parameter $\epsilon = \delta t$ (which will be set to 1 at the end of the analysis) using the following multi-scale expansions:

$$\hat{g} = \sum_{n=0}^{\infty} \epsilon^n \hat{g}^{(n)} = \hat{g}^{(0)} + \epsilon \hat{g}^{(1)} + \epsilon^2 \hat{g}^{(2)} , \quad (1.9a)$$

$$\hat{g}^{eq} = \sum_{n=0}^{\infty} \epsilon^n \hat{g}^{eq(n)} = \hat{g}^{eq(0)} + \epsilon \hat{g}^{eq(1)} , \quad (1.9b)$$

$$\partial_t = \sum_{n=0}^{\infty} \epsilon^n \partial_t^n = \partial_t^0 + \epsilon \partial_t^1 + \epsilon^2 \partial_t^2, \quad \nabla = \epsilon \nabla \quad (1.9c)$$

Notice that in the above we have used the moment equilibria $\hat{g}^{eq}$ in terms of the sum of the base moment equilibria $\hat{g}^{eq(0)}$ and the extended moment equilibria $\hat{g}^{eq(1)}$. In addition, a Taylor expansion is used for the representation of the streaming operator, which is carried out in its natural velocity space:

$$g(\vec{x} + \vec{e}_a \epsilon, t + \epsilon) = \sum_{n=0}^{n} \frac{\epsilon^n}{n!} (\partial_t + \vec{e}_a \cdot \vec{\nabla})^n g(\vec{x}, t). \quad (1.10)$$

$$O(\epsilon^0): \quad \hat{g}^{(0)} = \hat{g}^{eq}, \quad (1.11a)$$

$$O(\epsilon^1): \quad (\partial_{t0} + \hat{E}_i \partial_i) \hat{g}^{(0)} = -\Lambda \left[ \hat{g}^{(1)} - \hat{g}^{eq(1)} \right] + \hat{S}_x \quad (1.11b)$$

$$O(\epsilon^2): \quad \partial_{t1} \hat{g}^{(0)} + (\partial_{t0} + \hat{E}_i \partial_i) \left[ I - \frac{\Lambda}{2} \right] \hat{g}^{(1)} = -\Lambda \hat{g}^{(2)} , \quad (1.11c)$$

where $\hat{E}_i = T(e_{ai} I) T^{-1}, \quad i \in x, y$. In order derive the macroscopic CDE, up to the first order moment components in $O(\epsilon)$ (Eq. 2.11b) are relevant, which are listed as

107
follows:

\[ \partial_t T + \partial_x (Tu_x) + \partial_y (Tu_y) = G, \quad (1.12a) \]
\[ \partial_t (Tu_x) + \partial_x \left( \frac{1}{3}T + Tu_x^2 \right) + \partial_y (Tu_x u_y) = -\lambda_1 \hat{\delta}_1^{(1)} + \frac{1}{3}D\lambda_1 \partial_x T + u_x G, \quad (1.12b) \]
\[ \partial_t (Tu_y) + \partial_x (Tu_x u_y) + \partial_y \left( \frac{1}{3}T + Tu_y^2 \right) = -\lambda_2 \hat{\delta}_2^{(1)} + \frac{1}{3}D\lambda_2 \partial_y T + u_y G, \quad (1.12c) \]

Similarly, the second-order moment equations can be derived from Eq. (2.11c), which can be written as

\[ \partial_t T + \partial_x \left[ \left( 1 - \frac{\lambda_1}{2} \right) \hat{\delta}_1^{(1)} \right] + \partial_y \left[ \left( 1 - \frac{\lambda_2}{2} \right) \hat{\delta}_2^{(1)} \right] + \partial_x \left[ \frac{\lambda_1}{6} D\partial_x T \right] + \partial_y \left[ \frac{\lambda_2}{6} D\partial_y T \right] = 0, \quad (1.13) \]

Now, combining Eqs. (2.12a) with \( \epsilon \) times Eq. (2.13) and setting \( \partial_t = \partial_{t_0} + \epsilon \partial_{t_1} \), we get the dynamical equations for the conserved or hydrodynamic moments after setting the parameter \( \epsilon \) to unity. That is,

\[ \partial_t T + \partial_x (Tu_x) + \partial_y (Tu_y) = -\epsilon \partial_x \left[ \left( 1 - \frac{\lambda_1}{2} \right) \hat{\delta}_1^{(1)} \right] - \epsilon \partial_y \left[ \left( 1 - \frac{\lambda_2}{2} \right) \hat{\delta}_2^{(1)} \right] - \epsilon \partial_x \left[ \frac{\lambda_1}{6} D\partial_x T \right] - \epsilon \partial_y \left[ \frac{\lambda_2}{6} D\partial_y T \right] + G, \quad (1.14) \]

In the above equation, Eq. (2.14), we need the first order non-equilibrium raw moments \( \hat{\delta}_1^{(1)} \) and \( \hat{\delta}_2^{(1)} \). They can be obtained from Eqs. (2.12b) and (2.12c), respectively. Thus,

\[ \hat{\delta}_1^{(1)} = \frac{1}{\lambda_1} \left[ \left\{ \frac{\lambda_1}{3} D\partial_x T - \partial_{t_0} (u_x T) - \partial_x \left( \frac{1}{3}T + u_x^2 T \right) - \partial_y (u_x u_y T) \right\} + u_x G \right], \quad (1.15) \]

\[ \hat{\delta}_2^{(1)} = \frac{1}{\lambda_2} \left[ \left\{ \frac{\lambda_2}{3} D\partial_y T - \partial_{t_0} (u_y T) - \partial_x (u_x u_y T) - \partial_y \left( \frac{1}{3}T + u_y^2 T \right) \right\} + u_y G \right], \quad (1.16) \]
With the help of the first-order incompressible Navier Stokes equation (first-order hydrodynamic moment equations and continuity) [56, 44]

\[ \partial_t u + u \cdot \nabla u = -\nabla P + a \]

we have \( \partial_{t0} (\rho u_x^2) \approx 2F_x u_x, \partial_{t0} (\rho u_y^2) \approx 2F_y u_y \) and \( \partial_{t0} (\rho u_x u_y) \approx F_x u_y + F_y u_x \).

Substituting for these terms in Eqs. (2.15)-(2.16), and representing the components of momentum for brevity as The two non-equilibrium moments can be simplified, we get

\[ \hat{g}^{(1)}_1 = \frac{1}{3} \left\{ D - \frac{1}{\lambda_1} \right\} \partial_x T, \quad (1.17) \]

\[ \hat{g}^{(1)}_2 = \frac{1}{3} \left\{ D - \frac{1}{\lambda_2} \right\} \partial_y T \quad (1.18) \]

Now, by substituting these simplified expressions for the non-equilibrium moments, Eq. (2.17) and Eq. (2.18) into Eq. (2.14) it follows that

\[ \partial_t T + \partial_x (Tu_x) + \partial_y (Tu_y) = \partial_x \left[ (\alpha_1) \partial_x T \right] + \partial_y \left[ (\alpha_2) \partial_y T \right] + G \quad (1.19) \]

Which represents the convection-diffusion equation (CDE) with source term. The coefficients \( \alpha_1 \) and \( \alpha_2 \) represent the thermal diffusivity and are related to their relaxation parameters \( \lambda_1 \) and \( \lambda_2 \), and the adjustable parameter \( D \) in the extended moment equilibria:

\[ \alpha_1 = \frac{1}{3} \left( \frac{1}{\lambda_1} - \frac{1}{2} - D \right), \quad \alpha_2 = \frac{1}{3} \left( \frac{1}{\lambda_2} - \frac{1}{2} - D \right) \quad (1.20) \]

For isotropy of the thermal diffusion \( \alpha = \alpha_1 = \alpha_2 \), it follows that \( \lambda_1 = \lambda_2 \), and the rest of the higher order relaxation parameters can be tuned to improve the numerical
stability. Finally, the temperature gradients $\partial_x T$ and $\partial_y T$ needed in the extended moment equilibria Eq. (2.6) can be computed locally from Eq. (2.17) and Eq. (2.18), i.e. the first order non-equilibrium moments. Using $\hat{g}^{(1)}_1 = \hat{g}_1 - \hat{g}^{eq}(0)$ and $\hat{g}^{(1)}_2 = \hat{g}_2 - \hat{g}^{eq}(0)$ in Eqs. (2.17) and (2.18), and rearranging, we get

\[
\partial_x T = \frac{3\lambda_1 \left[ \hat{\kappa}'_x - \hat{\kappa}^{eq}_x \right]}{(D\lambda_1 - 1)},
\]

(1.21a)

\[
\partial_y T = \frac{3\lambda_2 \left[ \hat{\kappa}'_y - \hat{\kappa}^{eq}_y \right]}{(D\lambda_2 - 1)}
\]

(1.21b)
APPENDIX II

CHAPMAN-ENSKOG ANALYSIS OF THE ANISOTROPIC THERMAL CASCADED LBM

In this section, a Chapman-Enskog (C-E) analysis of the anisotropic thermal cascaded LBM is presented and the results of the analysis provide the macroscopic emergent equations, vis; the convection-diffusion equation (CDE) with a source term given in Eq. (1.17) earlier. In this regard, we consider the strategy of rewriting the central moment LBM in terms of the relaxation to a generalized equilibrium in the rest frame of reference. To facilitate analysis and its establish consistency to the CDE, it is sufficient to consider terms only up to second order in Mach number in such an equivalent formulation [24, 44]. In this regard, we consider performing calculations in terms of various raw moments (designned with "prime symbols") with respect to the non-orthogonal moment basis vectors collected in the matrix $\mathcal{T}$ and given in Eq. (1.19). First, the base raw moment equilibrium $\tilde{\kappa}_{x^m y^n}^{eq}$ can be obtained from the corresponding central moment equilibria $\tilde{\kappa}_{x^m y^n}^{eq}$ given in equation Eq. (1.29). This
reads as

\[\hat{\kappa}_{0}^{eq} = T,\]  
\[\hat{\kappa}_{x}^{eq} = u_{x}T,\]  
\[\hat{\kappa}_{y}^{eq} = u_{y}T,\]  
\[\hat{\kappa}_{xx}^{eq} = c_{s}^{2}T + u_{x}^{2}T,\]  
\[\hat{\kappa}_{yy}^{eq} = c_{s}^{2}T + u_{y}^{2}T,\]  
\[\hat{\kappa}_{xy}^{eq} = u_{x}u_{y}T,\]  
\[\hat{\kappa}_{xxy}^{eq} = c_{s}^{2}u_{y}T + u_{x}^{2}u_{y}T,\]  
\[\hat{\kappa}_{xyy}^{eq} = c_{s}^{2}u_{x}T + u_{y}^{2}u_{x}T,\]  
\[\hat{\kappa}_{xxyy}^{eq} = c_{s}^{4}T + c_{s}^{2}(u_{x}^{2} + u_{y}^{2})T + u_{x}^{2}u_{y}^{2}T.\]  

Similarly, the raw moment for the source terms can be obtained from their corresponding central moments Eq. (1.30), which are presented in Eq. (1.32a). Now, for convenience, the various raw moments can be related to their corresponding starts in the velocity space via the non-orthogonal transformation matrix \(T\). We now define raw moments of distribution functions (including the transformed one), equilibrium and sources for convenience as

\[\hat{g} = Tg, \quad \hat{\tilde{g}} = \tilde{T}g, \quad \hat{S} = \tilde{T}S,\]  

where \(\hat{\cdots}\) represents column vectors in (raw) moment space and the transformation matrix \(T\) is given in Eq. (1.19). That is,

\[\hat{g} = (\hat{g}_{0}, \hat{g}_{1}, \hat{g}_{2}, \ldots, \hat{g}_{8})^\dagger = (\hat{\kappa}_{0}^{\prime}, \hat{\kappa}_{x}^{\prime}, \hat{\kappa}_{y}^{\prime}, \hat{\kappa}_{xx}^{\prime} + \hat{\kappa}_{yy}^{\prime}, \hat{\kappa}_{xy}^{\prime}, \hat{\kappa}_{xyy}^{\prime}, \hat{\kappa}_{xxy}^{\prime}, \hat{\kappa}_{xxyy}^{\prime})^\dagger,\]  

\[\hat{\tilde{g}} = (\hat{\tilde{g}}_{0}, \hat{\tilde{g}}_{1}, \hat{\tilde{g}}_{2}, \ldots, \hat{\tilde{g}}_{8})^\dagger = (\hat{\sigma}_{0}^{\prime}, \hat{\sigma}_{x}^{\prime}, \hat{\sigma}_{y}^{\prime}, \hat{\sigma}_{xx}^{\prime} + \hat{\sigma}_{yy}^{\prime}, \hat{\sigma}_{xy}^{\prime}, \hat{\sigma}_{xyy}^{\prime}, \hat{\sigma}_{xxy}^{\prime}, \hat{\sigma}_{xxyy}^{\prime})^\dagger,\]  

\[\hat{S} = (\hat{S}_{0}, \hat{S}_{1}, \hat{S}_{2}, \ldots, \hat{S}_{8})^\dagger = (\hat{\sigma}_{0}^{\prime}, \hat{\sigma}_{x}^{\prime}, \hat{\sigma}_{y}^{\prime}, \hat{\sigma}_{xx}^{\prime} + \hat{\sigma}_{yy}^{\prime}, \hat{\sigma}_{xy}^{\prime}, \hat{\sigma}_{xyy}^{\prime}, \hat{\sigma}_{xxy}^{\prime}, \hat{\sigma}_{xxyy}^{\prime})^\dagger.\]
In addition, in order to maintain flexibility in the specification of the transport coefficient (i.e. the thermal diffusivity) appearing in the emergent CDE, we specify the raw moment equilibrium \( \hat{g}^{eq} \) by augmenting the base moment equilibria given in Eq. (2.1i) with an extended first order moment equilibria involving the components of the temperature gradients with an adjustable coefficient (designated as D below). That is

\[
\hat{g}^{eq} = T \hat{g}^{eq}
\]  

(2.3)

Here

\[
\hat{g}^{eq} = \hat{g}^{eq(0)} + \hat{g}^{eq(1)}
\]  

(2.4)

Where, \( \hat{g}^{eq(0)} \) and \( \hat{g}^{eq(1)} \) are the base and extended moment equilibria, respectively, and are given as follows:

\[
\hat{g}^{eq(0)} = \left( \begin{array}{c}
\hat{g}^{eq(0)}_0, \\
\hat{g}^{eq(0)}_1, \\
\vdots, \\
\hat{g}^{eq(0)}_8
\end{array} \right) = \left( \begin{array}{c}
\hat{\kappa}^{eq}, \\
\hat{\kappa}_x, \\
\hat{\kappa}_y, \\
\hat{\kappa}_{xx}, \\
\hat{\kappa}_{yy} - \hat{\kappa}_{xy}, \\
\hat{\kappa}_{xxy}, \\
\hat{\kappa}_{xyy}, \\
\hat{\kappa}_{xxxy}, \\
\hat{\kappa}_{xxyy}
\end{array} \right)
\]  

(2.5)

\[
\hat{g}^{eq(1)} = \left( \begin{array}{c}
\hat{g}^{eq(1)}_0, \\
\hat{g}^{eq(1)}_1, \\
\hat{g}^{eq(1)}_2, \\
\hat{g}^{eq(1)}_3, \\
\vdots, \\
\hat{g}^{eq(1)}_8
\end{array} \right) = \left( \begin{array}{c}
0, c_s^2 D \partial_x T, c_s^2 D \partial_y T, 0, 0, 0, 0, 0, 0
\end{array} \right)
\]  

(2.6)

We can then rewrite post-collision state of the thermal cascaded LBE in Eq. (??) in terms of the following:

\[
\bar{g}^p = \bar{g} + T^{-1} \left[ -\Lambda \left( \hat{g} - \hat{g}^{eq} \right) + \left( \mathcal{I} - \frac{1}{2} \Lambda \right) \hat{S} \right]
\]  

(2.7)
where \( \Lambda \) is a diagonal collision matrix given by

\[
\Lambda = \text{diag}(\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_8).
\] (2.8)

Now, applying a Chapman-Enskog multiscale expansion by expanding the raw moments \( \hat{g} \) and the time derivative in terms of a small perturbation parameter \( \epsilon = \delta t \) (which will be set to 1 at the end of the analysis) using the following multi-scale expansions:

\[
\hat{g} = \sum_{n=0}^{\infty} \epsilon^n \hat{g}^{(n)} = \hat{g}^{(0)} + \epsilon \hat{g}^{(1)} + \epsilon^2 \hat{g}^{(2)},
\] (2.9a)

\[
\hat{g}^{eq} = \sum_{n=0}^{\infty} \epsilon^n \hat{g}^{eq(n)} = \hat{g}^{eq(0)} + \epsilon \hat{g}^{eq(1)},
\] (2.9b)

\[
\partial_t = \sum_{n=0}^{\infty} \epsilon^n \partial_t^n = \partial_{t_0} + \epsilon \partial_{t_1} + \epsilon^2 \partial_{t_2}, \quad \nabla = \epsilon \nabla
\] (2.9c)

Notice that in the above we have used the moment equilibria \( \hat{g}^{eq} \) in terms of the sum of the base moment equilibria \( \hat{g}^{eq(0)} \) and the extended moment equilibria \( \hat{g}^{eq(1)} \). In addition, a Taylor expansion is used for the representation of the streaming operator, which is carried out in its natural velocity space:

\[
g(\vec{x} + \vec{v} \epsilon, t + \epsilon) = \sum_{n=0}^{n} \frac{\epsilon^n}{n!} (\partial_t + \vec{v} \cdot \nabla)^n g(\vec{x}, t).
\] (2.10)

\[
O(\epsilon^0) : \quad \hat{g}^{(0)} = \hat{g}^{eq},
\] (2.11a)

\[
O(\epsilon^1) : \quad (\partial_{t_0} + \vec{E}_i \partial_i) \hat{g}^{(0)} = -\Lambda \left[ \hat{g}^{(1)} - \hat{g}^{eq(1)} \right] + \hat{S}_\alpha
\] (2.11b)

\[
O(\epsilon^2) : \quad \partial_{t_1} \hat{g}^{(0)} + (\partial_{t_0} + \vec{E}_i \partial_i) \left[ \mathcal{I} - \frac{\Lambda}{2} \right] \hat{g}^{(1)} = -\Lambda \hat{g}^{(2)},
\] (2.11c)

114
where $\hat{E}_i = T(e_{ai}; T^{-1}, i \in x, y$. In order derive the macroscopic CDE, up to the first order moment components in $O(\epsilon)$ (Eq. 2.11b) are relevant, which are listed as follows:

\begin{equation}
\partial_t T + \partial_x (Tu_x) + \partial_y (Tu_y) = G, \tag{2.12a}
\end{equation}

\begin{equation}
\partial_t (Tu_x) + \partial_x \left( \frac{1}{3} T + Tu_x^2 \right) + \partial_y (Tu_x u_y) = -\lambda_1 \hat{g}^{(1)} - \lambda_{12} \hat{g}^{(1)} + \frac{1}{3} D \lambda_1 \partial_x T \\
+ \frac{1}{3} D \lambda_{12} \partial_y T + u_x G, \tag{2.12b}
\end{equation}

\begin{equation}
\partial_t (Tu_y) + \partial_x (Tu_x u_y) + \partial_y \left( \frac{1}{3} T + Tu_y^2 \right) = -\lambda_2 \hat{g}^{(1)} - \lambda_{21} \hat{g}^{(1)} + \frac{1}{3} D \lambda_2 \partial_y T \\
+ \frac{1}{3} D \lambda_{21} \partial_x T + u_y G \tag{2.12c}
\end{equation}

Similarly, the second-order moment equations can be derived from Eq. (2.11c), which can be written as

\begin{equation}
\partial_t T + \partial_x \left[ \left( 1 - \frac{\lambda_1}{2} \right) \hat{g}^{(1)} - \frac{\lambda_{12}}{2} \hat{g}^{(1)} \right] + \partial_y \left[ \left( 1 - \frac{\lambda_2}{2} \right) \hat{g}^{(1)} - \frac{\lambda_{21}}{2} \hat{g}^{(1)} \right] \\
+ \partial_x \left[ \frac{\lambda_1}{6} D \partial_x T + \frac{\lambda_{12}}{6} D \partial_y T \right] + \partial_y \left[ \frac{\lambda_2}{6} D \partial_y T + \frac{\lambda_{21}}{6} D \partial_x T \right] = 0 \tag{2.13}
\end{equation}

Now, combining Eqs. (2.12a) with $\epsilon$ times Eq. (2.13) and setting $\partial_t = \partial_{t_0} + \epsilon \partial_{t_1}$, we get the dynamical equations for the conserved or hydrodynamic moments after setting the parameter $\epsilon$ to unity. That is,

\begin{equation}
\partial_t T + \partial_x (Tu_x) + \partial_y (Tu_y) = -\epsilon \partial_x \left[ \left( 1 - \frac{\lambda_1}{2} \right) \hat{g}^{(1)} - \frac{\lambda_{12}}{2} \hat{g}^{(1)} \right] - \epsilon \partial_y \left[ \left( 1 - \frac{\lambda_2}{2} \right) \hat{g}^{(1)} - \frac{\lambda_{21}}{2} \hat{g}^{(1)} \right] \\
- \epsilon \partial_x \left[ \frac{\lambda_1}{6} D \partial_x T + \frac{\lambda_{12}}{6} D \partial_y T \right] - \epsilon \partial_y \left[ \frac{\lambda_2}{6} D \partial_y T + \frac{\lambda_{21}}{6} D \partial_x T \right] + G \tag{2.14}
\end{equation}

In the above equation, Eq. (2.14), we need the first order non-equilibrium raw moments $\hat{g}_1^{(1)}$ and $\hat{g}_2^{(1)}$. They can be obtained from Eqs. (2.12b) and (2.12c).
respectively. Thus,

\[ \hat{g}_1^{(1)} = \frac{1}{\lambda_1} \left[ \left\{ \frac{\lambda_1}{3} D \partial_x T + \frac{\lambda_{12}}{3} D \partial_y T - \partial_{t_0} (u_x T) - \partial_x \left( \frac{1}{3} T + u_x^2 T \right) - \partial_y (u_x u_y T) \right\} 
\quad + u_x G - \lambda_{12} \hat{g}_2^{(1)} \right], \]  

(2.15)

\[ \hat{g}_2^{(1)} = \frac{1}{\lambda_2} \left[ \left\{ \frac{\lambda_2}{3} D \partial_y T + \frac{\lambda_{21}}{3} D \partial_x T - \partial_{t_0} (u_y T) - \partial_x (u_x u_y T) - \partial_y \left( \frac{1}{3} T + u_y^2 T \right) \right\} 
\quad + u_y G - \lambda_{21} \hat{g}_1^{(1)} \right], \]  

(2.16)

With the help of the first-order incompressible Navier Stokes equation (first-order hydrodynamic moment equations and continuity) [56, 44]

\[ \partial_t u + u \cdot \nabla u = - \nabla P + a \]

we have \( \partial_{t_0} (\rho u_x^2) \approx 2F_x u_x, \partial_{t_0} (\rho u_y^2) \approx 2F_y u_y \) and \( \partial_{t_0} (\rho u_x u_y) \approx F_x u_y + F_y u_x \).

Substituting for these terms in Eqs. (2.15)-(2.16), and solving for \( \hat{g}_1^{(1)} \) and \( \hat{g}_2^{(1)} \). The two non-equilibrium moments can be simplified as

\[ \hat{g}_1^{(1)} = \frac{1}{3} \left\{ D - \frac{\lambda_2}{\lambda_1 \lambda_2 - \lambda_{12} \lambda_{21}} \right\} \partial_x T + \frac{1}{3} \left\{ \frac{\lambda_{12}}{\lambda_1 \lambda_2 - \lambda_{12} \lambda_{21}} \right\} \partial_y T, \]  

(2.17)

\[ \hat{g}_2^{(1)} = \frac{1}{3} \left\{ \frac{\lambda_{21}}{\lambda_1 \lambda_2 - \lambda_{12} \lambda_{21}} \right\} \partial_x T + \frac{1}{3} \left\{ D - \frac{\lambda_1}{\lambda_1 \lambda_2 - \lambda_{12} \lambda_{21}} \right\} \partial_y T \]  

(2.18)

Now, by substituting these simplified expressions for the non-equilibrium moments, Eq. (2.17) and Eq. (2.18) into Eq. (2.14) it follows that

\[ \partial_t T + \partial_x (Tu_x) + \partial_y (Tu_y) = \partial_x \left[ (a_{11}) \partial_x T \right] + \partial_x \left[ (a_{12}) \partial_y T \right] \]
\[ \quad + \partial_y \left[ (a_{21}) \partial_x T \right] + \partial_y \left[ (a_{22}) \partial_y T \right] + G \]  

(2.19)
Which represents the convection-diffusion equation (CDE) with source term. The coefficients $\alpha_{11}, \alpha_{12}, \alpha_{21}$ and $\alpha_{22}$ represent the thermal diffusivity and are related to their relaxation parameters $\lambda_1, \lambda_{12}, \lambda_{21}$ and $\lambda_{22}$, and the adjustable parameter $D$ in the extended moment equilibria:

\[
\begin{align*}
\alpha_{11} &= \frac{1}{3} \left( \frac{\lambda_2}{|A|} - \frac{1}{2} - D \right), \\
\alpha_{12} &= \frac{1}{3} \left( -\frac{\lambda_{12}}{|A|} \right), \\
\alpha_{21} &= \frac{1}{3} \left( -\frac{\lambda_{21}}{|A|} \right), \\
\alpha_{22} &= \frac{1}{3} \left( \frac{\lambda_1}{|A|} - \frac{1}{2} - D \right)
\end{align*}
\]  
(2.20)

Where $A = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}$, $|A| = |\lambda_1 \lambda_2 - \lambda_{12} \lambda_{21}|$. For diagonal anisotropic case where the diffusion tensor is diagonal matrix (i.e. $\lambda_{12} = \lambda_{21} = 0$), the relaxation parameters can be simplified as

\[
\begin{align*}
\alpha_{11} &= \frac{1}{3} \left( \frac{1}{\lambda_1} - \frac{1}{2} - D \right), \\
\alpha_{22} &= \frac{1}{3} \left( \frac{1}{\lambda_2} - \frac{1}{2} - D \right)
\end{align*}
\]  
(2.21)