PROBLEMS IN STRUCTURAL AND EXTREMAL GRAPH THEORY

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A thesis submitted to the

Faculty of the Graduate School of the

University of Colorado in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Applied Mathematics

2015

This thesis for the Doctor of Philosophy degree by

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April 24, 2015

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Problems in Structural and Extremal Graph Theory

Thesis directed by Associate Professor Michael J. Ferrara

ABSTRACT

The research in this dissertation can be placed into two broad categories: structural and extremal graph theory. Extremal graph theory seeks to determine maximum and minimum values of graph parameters that ensure (or prohibit) particular graph properties. For instance, we may consider the threshold for the minimum degree of a graph G that guarantees the existence of a hamiltonian cycle, which is a cycle that spans the vertices of G. Structural graph theory considers the substructures of a graph, such as paths, cycles, and matchings, and considers conditions that guarantee the existence (or non-existence) of such graph structures. For example, we may use the fact that a graph G does not contain any odd cycles to force G to be bipartite. The work in this dissertation considers three distinct problems.

A *rainbow matching* in an edge-colored graph is a matching in which all the edges have distinct colors. In 2011, Wang asked if there is a function $f(\delta)$ such that a properly edge-colored graph G with minimum degree δ and order at least $f(\delta)$ must have a rainbow matching of size δ. We answer this question in the affirmative; an extremal approach yields that $f(\delta) = 98\delta/23 < 4.27\delta$ suffices. Furthermore, we give an $O(\delta(G)|V(G)|^2)$-time algorithm that generates such a matching in a properly edge-colored graph of order at least 6.5δ.

A 2-*factor* in a graph is a spanning 2-regular subgraph, or equivalently a spanning collection of disjoint cycles. We investigate the existence of 2-factors with a bounded number of odd cycles in a graph. We extend results of Ryjáček, Saito, and Schelp [Closure, 2-factors, and cycle coverings in claw-free graphs, *J. Graph Theory*, **32**

(1999), no. 2, 109-117] and show that the number of odd components of a 2-factor in a claw-free graph is stable under Ryjáček's closure operation. Additionally a 2-factor with no odd cycles is equivalent to a pair of disjoint perfect matchings, and we consider conditions for this specific case. Most significantly, we show that a $\{K_{1,4}, P_4\}$-free graph of even order contains disjoint perfect matchings or is a member of a particular family of graphs.

A sequence $\pi = (d_1, \ldots, d_n)$ is *graphic* if there is a simple graph G with vertex set $\{v_1, \ldots, v_n\}$ such that the degree of v_i is d_i. We say that graphic sequences $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$ *pack* if there exist edge-disjoint n-vertex graphs G_1 and G_2 such that for $j \in \{1, 2\}$, $d_{G_j}(v_i) = d_i^{(j)}$ for all $i \in \{1, \ldots, n\}$. We prove several extremal degree sequence packing theorems that parallel central results and open problems from the graph packing literature. Specifically, the main result of this chapter implies a degree sequence packing analogue to the Bollobás-Eldridge-Catlin graph packing conjecture [Packings of graphs and applications to computational complexity, *J. Combin. Theory Ser. B*, **25** (1978), 105–124; Embedding subgraphs and coloring graphs under extremal degree conditions, *Ph.D. Thesis*, Ohio State University, 1976] along with a degree sequence version of the Sauer-Spencer Theorem [Edge disjoint placement of graphs, *J. Combin. Theory Ser. B*, **25** (1978), 295–302].

These results are related in part to discrete tomography, a branch of discrete imaging science, in which the goal is to reconstruct discrete objects using data acquired from low-dimensional projections. Specifically, in the *k-color discrete tomography problem* the goal is to color the entries of an $m \times n$ matrix using k colors so that each row and column receive a prescribed number of entries of each color. This problem is equivalent to packing the degree sequences of k bipartite graphs with parts of sizes m and n. Here we modify our techniques to prove several Sauer-Spencer-type theorems that have direct applications to the 2-color discrete tomography problem.

The form and content of this abstract are approved. I recommend its publication.

Approved: Michael J. Ferrara

DEDICATION

To my family, for their unending support, confidence, and prayers.

ACKNOWLEDGMENT

There are so many people who I must thank for their support throughout my time as a graduate student. First and foremost, my advisor, Mike Ferrara, who has always said that no matter what at least one of he or I would believe I could complete my PhD. This has been true, and more often than not, he has had to pull the weight on believing I could complete this. I also am indebted to the other members of my committee: Mike Jacobson, Paul Wenger, Flo Pfender, and Ellen Gethner. Their help has come in support of my research, outreach opportunities, and reminders to enjoy graduate school.

I owe thanks to the NSF Grant #074324 "UCD GK-12 Transforming Experiences Learning Communities Project" and the Lynn Bateman Memorial Fellowship for funding parts of this research.

There are countless others who have helped me on my way. Firstly, Art Busch, who introduced me to graph theory; Sogol Jahanbekam who made research far more fun (even if more disorganized); and my other co-authors such as James Shook. My fellow graduate students have greatly helped over the years and I cannot thank them enough: Henc Bouwmeester, Jeff Larsen, Eric Sullivan, and Breeann Flesch gave me encouragement, wisdom, and friendship; the beginning of this journey began with studying for courses and prelims with Mark Mueller, Cathy Erbes, Phil DeOrsey, Brad Lowery, and Tim Morris; Axel Brandt, Daniel Yorgov, Devon Sigler, Reid Messersmith, Stephanie Patterson, Scott Walsh, Tuguluke Abulitibu, Brent Thomas and TK Miller have provided rides home, listening ears, and happy hour companions over the years; finally, thank you to Nathan Graber, Luke Nelsen, and Eric (2) Sullivan for making my final year such fun. Words cannot express my gratitude to Gary Olson and Nora Jennings, two of my best friends – their support through prayers, movie nights, dinners out, etc. are unmatched. My entire family has been with me in this journey, which has been invaluable. In particular, my brother has always supported my decisions; my sister, brother-in-law, and nephews, who found time to visit often and motivated me to never give up; and my parents, who have given me encouragement in every possible way. Lastly, anyone who has enjoyed dessert over the years, thank you.

TABLE OF CONTENTS

FIGURES

Figure

1. Introduction

1.1 Definitions and Notation

A *graph* G is an ordered pair of disjoint sets (V, E), such that E is a subset of $V \times V$, of unordered pairs of vertices in V. We call V the *vertex set* and E the *edge set*. A graph G has *order* $|V|$ and *size* $|E|$. For two vertices, u and v in V, if (u, v) is in E, u and v are *adjacent*, and for simiplicity we often write $uv \in E$. The number of edges incident to a vertex v is the *degree* of v, denoted $d_G(v)$. The set of vertices adjacent in G to v is the *neighborhood* of v, denoted $N_G(V)$. Thus, we have that $|N_G(v)| = d_G(v)$. We denote the minimum degree over all vertices in G as $\delta(G)$, and likewise the maximum degree over all vertices in G as $\Delta(G)$. When every vertex of G has degree r, which is to say that $\Delta(G) = \delta(G) = r$, G is *r-regular*. We will often drop the subscripts on $d(v)$ and $N(v)$, among others, when the context is clear.

All graphs considered in this dissertation are simple, with no loops or multiple edges between vertices. Thus each edge e is incident to exactly two distinct vertices, its *endpoints*, u and v. Additionally, for every pair of vertices, u and v, there is at most one edge uv in E.

The list of degrees of the vertices of a graph G is the *degree sequence* of G. A sequence of nonnegative integers $\pi = (d_1, d_2, \ldots, d_n)$ is *graphic* if there is a (simple) graph G of order n having degree sequence π. In this case, G is said to *realize* or be a *realization of* π, and we write $\pi = \pi(G)$. If a sequence π consists of the terms $d_1, \ldots, d_t$ having multiplicities $\mu_1, \ldots, \mu_t$, then we may write $\pi = ({d_1}^{\mu_1}, \ldots, {d_t}^{\mu_t})$.

For the purposes of this dissertation, we are often interested in considering various subgraphs. A *subgraph* H of G satisfies $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is a *spanning* subgraph of G. A spanning, r-regular subgraph of a graph G is called an *r-factor* of G. Of particular importance for this dissertation are 1-factors and 2-factors. Specifically, a 1-factor is also called a *perfect matching* and is a set of independent edges of a graph G which spans the vertices of G. A

2-factor of a graph G is a set of disjoint cycles that spans G. A subgraph H of G in which for each pair u and v in $V(H)$, uv is in $E(H)$ if and only if uv is in $E(G)$ is an *induced subgraph*. A graph on n vertices in which every possible edge appears is a *complete graph*, denoted K_n. When the subgraph induced by $V(H) \subseteq V(G)$ forms a complete graph, we say that $V(H)$ is a *clique* in G. On the other hand, a subset of $V(G)$ containing no edges forms an *independent set* in G. An independent set of edges in a graph G (which does not necessarily span $V(G)$) is a *matching*. A *path* on n vertices, denoted P_n is a set of vertices $V(P_n) = \{v_1, \ldots, v_n\}$ with edge set $E(G) = \{v_i v_{i+1} : i \in \{1, \ldots, n-1\}\}$. A *cycle* of order n, denoted C_n is a path P_n along with the edge $v_1 v_n$. Notice that $K_3 = C_3$, and we will often refer to this particular complete graph as a *triangle*. A graph with no cycles is a *tree*, and the set of degree 1 vertices of a tree are its *leaves*.

Given a graph G, when $V(G)$ can be partitioned into two independent sets, X and Y, G is *bipartite*, and therefore $E(G)$ consists only of edges xy where $x \in X$ and $y \in Y$. For instance, cycles on an even number of verices are bipartite graphs, as are matchings and trees. When a bipartite graph G contains every possible edge between X and Y, G is a *complete bipartite graph*, denoted $K_{n,m}$, where $|X| = n$ and $|Y| = m$.

A (vertex) coloring of a graph G is a function $c : V(G) \to \{1, \ldots, k\}$. A coloring is proper if every pair of adjacent vertices receives distinct colors, so that for any edge uv in E, $c(u) \neq c(v)$. The minimum number of colors required to properly color a graph G is the *chromatic number* of G, denoted $\chi(G)$. Likewise, an *edge-coloring* of a graph G is a function $c : E(G) \to \{1, \ldots, k\}$. An edge coloring of a graph is *proper* if the colors on edges sharing an endpoint are distinct, that is for all $v \in V(G)$, $c(vu) \neq c(vw)$ for all vertices u and w in $N(v)$. The minimum number of colors required to properly edge-color a graph G is the *chromatic index* of G denoted $\chi'(G)$. Finally, an edge-colored graph is *rainbow* if all edges have distinct colors.

Any undefined notation and definitions can be found in [87].

1.2 Overview of Structural and Extremal Graph Theory

Extremal graph theory seeks to determine maximum and minimum values of graph parameters that ensure or prohibit particular graph properties. For instance, we may consider the threshold for the number of edges in a graph G which ensures G contains some particular subgraph. The *extremal number*, denoted $ex(n, H)$ is the maximum number of edges in a graph G on n vertices such that G does not contain a copy of H. Equivalently, we may ask how many edges guarantee that a graph G on n vertices with at least $f(n)$ edges contains a copy of H. In 1941, Turán [82] showed the extremal number for complete graphs on $k+1$ vertices. The *Turán graph*, denoted $\mathscr{T}_n^r$, is the complete r-partite graph on n vertices, in which each part is as balanced as possible, having $\left\lfloor \frac{n}{r} \right\rfloor$ or $\left\lfloor \frac{n}{r} \right\rfloor + 1$ vertices.

Theorem 1.1 [82] *For $r, n \geq 2$,*

$$ex(n, K_{r+1}) = \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2},$$

and $\mathscr{T}_n^r$ is the unique graph that has the extremal number of edges and does not contain a copy of K_{r+1}.

As an extension of Theorem 1.1, Erdős and Stone [30] showed the following.

Theorem 1.2 [30] *If G is a graph of order n with at least $\left(1 - \frac{1}{k} + \varepsilon\right)\frac{n^2}{2}$ edges, then for n sufficiently large, G contains a copy of the complete $(k+1)$-partite graph with t vertices in each part.*

Later Erdős and Simonovits [29] gave the general extremal number for any graph H in a family of graph $\mathcal{F}$, the proof of which highly relies upon Theorem 1.2.

Theorem 1.3 [29] *Let $\mathcal{F}$ be a family of graphs with $p = \min\{\chi(H) : H \in \mathcal{F}\}$, then*

$$ex(n, \mathcal{F}) = \left(1 - \frac{1}{p+1}\right)\binom{n}{2} + o(n^2).$$

Structural graph theory considers the substructures of a graph, such as paths, cycles, and matchings and considers conditions that guarantee the existence (or non-existence) of such graph structures. In structural graph theory, the substructures of a graph are the focus. These problems can have ties to extremal graph theory, as in Theorems 1.1, 1.2, and 1.3.

One way to ensure the existence of particular substructures, such as cycles, is to forbid particular subgraphs. Given a family $\mathcal{F}$ of graphs, a graph G is said to be *$\mathcal{F}$-free* if G contains no member of $\mathcal{F}$ as an induced subgraph. A graph G is *perfect* if for every induced subgraph H of G, the chromatic number of H is equal to the size of the largest clique of H. In 1961, Berge [6] made two conjectures regarding perfect graphs, which have both been shown to be true. The weaker conjecture, that a graph is perfect if and only if its complement is perfect was proven by Lovász in 1972 [65]. In 2006, the Strong Perfect Graph Theorem was proven by Chudnovsky, Robertson, Seymour, and Thomas [14], which characterizes perfect graphs in terms of forbidden subgraphs.

Theorem 1.4 The Strong Perfect Graph Theorem [14] *A graph G is perfect if and only if G does not contain an induced subgraph which is an odd cycle or the complement of an odd cycle.*

Structural and extremal problems often have overlap in research; for instance knowing that a graph is bipartite leads to a natural question of the extremal number for bipartite graphs. The problems considered here fall into three main chapters, and the problems we will explore can be broadly organized into structural and extremal graph theory, or some mix of these.

In Chapter 2, we discuss the threshold for when a properly edge-colored graph G contains a perfect matching of size $\delta(G)$. Specifically, we answer a question posed by Wang [83], if there exists a function f such that if G has order $n \geq f(\delta(G))$ then G is guaranteed to contain a rainbow matching of size δ. This problem has gained interest in part due to Ryser's Conjecture [75], which states that every proper edge-coloring of $K_{2k+1,2k+1}$ contains a rainbow perfect matching. This was originally formulated in terms of transversals in Latin squares, and is still open. Here we consider properly edge-colored graphs and present an algorithm which produces a rainbow matching of the desired size, $\delta(G)$, when the graph has enough vertices relative to the minimum degree, $\delta(G)$.

Chapter 3 discusses the cycle structure of 2-factors of graphs, especially under forbidden subgraph conditions. To gain local structure, we consider *claw-free* graphs, those graphs that have the claw, $K_{1,3}$, as a forbidden subgraph. We denote by $\langle v; x, y, z\rangle$ the claw with central vertex v and leaves x, y, and z. In 1997, Ryjáček [73] introduced a closure concept which iterativly adds edges to a graph, and in a claw-free graph, each iteration of this closure maintains the property of being claw-free. The closure of a claw-free graph also preserves the length of the longest cycle and in general has been shown to be a useful tool in studying a variety of structural properties of claw-free graphs. We also discuss forbidden subgraph conditions to ensure that a graph contains a 2-factor with no odd cycles. In particular, we consider forbidden subgraphs that guarantee that a graph contains a 2-factor, and consider what additional conditions ensure that there is a 2-factor with no odd cycles.

In Chapter 4, we discuss extremal problems related to graphic sequences. Two n-vertex graphs G_1 and G_2 *pack* if they can be expressed as edge-disjoint subgraphs of K_n. Sauer and Spencer [76] gave a classic result in graph packing, that if the maximum degrees of G_1 and G_2 are Δ_1 and Δ_2 respectively, if $\Delta_1\Delta_2 < \dfrac{n}{2}$ then G_1 and G_2 pack. Bollobás and Eldridge [7] and independently Catlin [12] conjectured the stronger condition that $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ would imply that G_1 and G_2 pack. This conjecture remains open, but many partial results have been shown to hold (see for example [1, 4, 17, 55]). We considered the graphic sequence analogue to graph packing. Two graphic sequences $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$ *pack* provided there are edge-disjoint graphs G_1 and G_2 such that $d_{G_1}(v_i) = d_i^{(1)}$ and $d_{G_2}(v_i) = d_i^{(2)}$, for i in $\{1, \ldots, n\}$. Our results imply a degree sequence analogue to the Bollobás-Eldridge-Catlin graph packing conjecture [7, 12], which in turn implies a degree sequence analogue to the Sauer-Spencer graph packing theorem [76].

2. Rainbow Matchings in Properly Edge-Colored Graphs

2.1 Introduction

Rainbow matchings are of particular interest given their connection to transversals of Latin squares. A *Latin square* of order n is an $n \times n$ matrix in which a set of n elements appears in each row and in each column so that no element is repeated in any row or column. A *transversal* in a Latin square $\mathscr{L}$ is a set of n entries that spans the rows and columns of $\mathscr{L}$ and covers each of the n distinct elements. Each Latin square can be converted to a properly edge-colored complete bipartite graph, and a transversal of the Latin square is a rainbow perfect matching in the graph. Specifically, let $K_{n,n}$ represent a Latin square $\mathscr{L}$ of order n where one part of $K_{n,n}$ is a set of vertices $\{1_R, \ldots, n_R\}$ for each row and the other part is a set of vertices $\{1_C, \ldots, n_C\}$ for each column. An edge $i_R j_C$ receives a color corresponding to the element in the (i, j) entry of $\mathscr{L}$. Ryser's conjecture [75] that every Latin square of odd order has a transversal can be seen as the beginning of the study of rainbow matchings. Stein [79] later conjectured that every Latin square of order n has a transversal of size $n - 1$; equivalently every proper edge-coloring of $K_{n,n}$ has a rainbow matching of size $n - 1$. The connection between Latin transversals and rainbow matchings in $K_{n,n}$ has inspired additional interest in the study of rainbow matchings in triangle-free graphs. A thorough survey of rainbow matching and other rainbow subgraphs in edge-colored graphs appears in [53].

Several results have been attained for rainbow matchings in arbitrarily edge-colored graphs. The *color degree* of a vertex v in an edge-colored graph G, written $\hat{d}(v)$, is the number of distinct colors on edges incident to v. We let $\hat{\delta}(G)$ denote the minimum color degree among the vertices in G. Wang and Li [84] proved that every edge-colored graph G contains a rainbow matching of size at least $\left\lceil \frac{5\hat{\delta}(G)-3}{12} \right\rceil$, and conjectured that a rainbow matching of size $\left\lceil \frac{\hat{\delta}(G)}{2} \right\rceil$ exists if $\hat{\delta}(G) \geq 4$. LeSaulnier et al. [62] then proved that every edge-colored graph G contains a rainbow matching of size $\left\lfloor \frac{\hat{\delta}(G)}{2} \right\rfloor$. Finally, Kostochka and Yancey [59] proved the conjecture of Wang and Li in full, and also proved that triangle-free graphs have rainbow matchings of size $\left\lceil \frac{2\hat{\delta}(G)}{3} \right\rceil$.

Since the edge-colored graphs generated by Latin squares are properly edge-colored, it is of interest to consider rainbow matchings in properly edge-colored graphs. In this direction, LeSaulnier et al. [62] proved that a properly edge-colored graph G satisfying $|V(G)| \neq \delta(G) + 2$ that is not K_4 has a rainbow matching of size $\left\lceil \frac{\delta(G)}{2} \right\rceil$. Wang then asked if there is a function f such that a properly edge-colored graph G with minimum degree δ and order at least $f(\delta)$ must contain a rainbow matching of size δ [83]. As a first step towards answering this question, Wang showed that a graph G with order at least $8\delta/5$ has a rainbow matching of size $\left\lfloor \frac{3\delta(G)}{5} \right\rfloor$.

Since there are $n \times n$ Latin squares with no transversals when n is even (see [9, 86]), for such a function f it is clear that $f(\delta) > 2\delta$ when δ is even. Furthermore, since a maximum matching in $K_{\delta,n-\delta}$ has only δ edges (provided $n \geq 2\delta$), there is no function for the order of G depending on $\delta(G)$ that can guarantee a rainbow matching of size greater than $\delta(G)$.

In this chapter we answer Wang's question from [83] in the affirmative, proving that a linear number of vertices in terms of the minimum degree suffices.

Theorem 2.1 [21] *If G is a properly edge-colored graph satisfying*

$$|V(G)| \geq \frac{98\delta(G)}{23},$$

then G contains a rainbow matching of size $\delta(G)$.

Independently, Wang, Zhang, and Liu [85] also answering Wang's question in the affirmative, proved that a properly edge-colored graph G with at least $\frac{1}{4}(\delta(G)^2 + 4\delta(G) - 4)$ vertices has a rainbow matching of size $\delta(G)$.

The proof of Theorem 2.1 utilizes extremal techniques akin to those that appear in [59, 62, 83] and [84]. We also implement a greedy algorithmic approach to demonstrate that it is possible to efficiently construct a rainbow matching of size δ in a properly edge-colored graph with minimum degree δ having order at least 6.5δ. To our knowledge, an algorithmic approach of this type has not been previously employed in the study of rainbow matchings.

Theorem 2.2 [21] *If G is a properly edge-colored graph with minimum degree δ satisfying*

$$|V(G)| > \frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta},$$

then there is an $O(\delta(G)|V(G)|^2)$-time algorithm that produces a rainbow matching of size δ in G.

As a contrast, Itai, Rodeh, and Tanimota [52] proved that determining if an edge-colored graph G has a rainbow matching of size k is NP-complete, even if G is bipartite. More recently, Le and Pfender [61] have shown that the problem of determining the maximum size of a rainbow matching in a properly edge-colored graph is NP-hard, even when restricted to properly edge-colored paths.

2.2 Proof of Theorem 2.1

Let G be a properly edge-colored n-vertex graph with minimum degree δ and $n \geq \frac{98\delta}{23}$. The theorem holds easily if $\delta \leq 2$, so we may assume that $\delta \geq 3$. By way of contradiction, let G be a counterexample with δ minimized; thus G does not contain a rainbow matching of size δ. Further, we may assume that $|E(G)|$ is minimized, so in particular the vertices of degree greater than δ form an independent set, as otherwise we could delete an edge without lowering the minimum degree. We break the proof into a series of simple claims.

Let $\Delta(G) = d_1 \geq d_2 \geq \ldots \geq d_n = \delta$ with $d(v_i) = d_i$ be the degree sequence of G.

Lemma 2.3 *For $1 \leq k \leq 2\delta/3$, there exists an $i \leq k$ such that $d_i \leq 3\delta - k - 2i$.*

Proof: Suppose that for some $k \le 2\delta/3$, $d_i \ge 3\delta + 1 - k - 2i$ for all $1 \le i \le k$. It follows that $d_i > \delta$ for $i \le k$, and therefore $\{v_1, \ldots, v_k\}$ is an independent set. Delete the vertices $v_1, v_2, \ldots, v_k$ from G, and note that $\delta(G \setminus \{v_1, \ldots, v_k\}) \ge \delta - k$. By the minimality of G, there exists a rainbow matching M_k of size $\delta - k$ in $G \setminus \{v_1, \ldots, v_k\}$.

The vertex v_k has at most $2(\delta - k)$ neighbors in M_k, and is incident to at most $\delta - k$ edges colored with colors occurring in M_k. Thus, v_k has a neighbor $w_k \notin V(M_k)$ such that the color of $v_k w_k$ does not occur in M_k, and we can extend M_k by adding the edge $v_k w_k$; call the resulting rainbow matching M_{k-1}. Note that $w_k \neq v_i$ for $i \le k$ as $\{v_1, \ldots v_k\}$ is an independent set.

The jth iteration of this process produces a rainbow matching M_{k-j} of size $\delta - k + j$ that contains $\{v_k, \ldots, v_{k-j+1}\}$. Hence v_{k-j} has at most $2(\delta - k) + j$ neighbors in M_{k-j} and is incident to at most $\delta - k + j$ edges colored with colors occurring in M_{k-j}. Thus there is a vertex $w_{k-j} \in N(v_{k-j})$ such that the edge $v_{k-j} w_{k-j}$ extends M_{k-j} to a $(\delta - k + j + 1)$-edge rainbow matching $M_{k-(j+1)}$. Continuing in this fashion extends the matching M_k by k edges, yielding a rainbow matching of size δ, a contradiction finishing the proof. ■

As a corollary of Lemma 2.3, we obtain the following.

Lemma 2.4 *For $1 \le k \le 2\delta/3$, we have $\sum_{i=1}^{k} d_i \le k(3\delta - 2 - k)$, with equality only if $d_1 = d_k = 3\delta - 2 - k$.*

Proof: We proceed by induction on k. For $k = 1$, the statement follows from Lemma 2.3. Now let $k > 1$ and let $i \le k$ such that $d_i \le 3\delta - k - 2i$. By induction,

$$\sum_{j=1}^{i-1} d_j \le (i-1)(3\delta - 1 - i), \text{ and}$$

$$\sum_{j=i}^{k} d_j \le (k - i + 1) d_i \le (k - i + 1)(3\delta - k - 2i).$$

Thus,

$$\sum_{j=1}^{k} d_j \leq (i-1)(3\delta - 1 - i) + (k - i + 1)(3\delta - k - 2i)$$
$$= 3k\delta - k^2 - k + 1 - i(k + 2 - i) \leq k(3\delta - 2 - k)$$

and equality holds only if $i = 1$ and $d_1 = d_k = 3\delta - 2 - k$. ■

Let C be a maximum color class in G and let $|C| = a$. By the minimality of G, there exists a rainbow matching $M = \{x_i y_i : 1 \leq i \leq \delta - 1\}$ of size $\delta - 1$ in $G - C$. Without loss of generality, we may assume that $c(x_i y_i) = i$ for $1 \leq i \leq \delta - 1$ and the edges in C have color δ. Let $W = V(G) \backslash V(M)$; observe that $|W| = n - 2(\delta - 1)$. If there is an edge e in $G[W]$ with $c(e) \notin \{1, \ldots, \delta - 1\}$ then we can add e to M to obtain a rainbow matching of size δ. Thus we may assume that every edge whose color is not in $\{1, \ldots, \delta - 1\}$ has an endpoint in $V(M)$. An edge uv is *good* if its color is not in $\{1, \ldots, \delta - 1\}$ and one of its endpoints is in W. A vertex $v \in V(M)$ is *good* if v is incident with at least seven good edges.

Claim 2.5 *For $i \in \{1, \ldots, \delta - 1\}$, if x_i is incident with at least three good edges, then no good edge is incident with y_i, and vice versa.*

Proof: Suppose that $y_i u$ is a good edge. If x_i is incident with at least three good edges, then x has a neighbor w such that vw is a good edge, $w \neq u$, and $c(x_i w) \neq c(y_i u)$. Thus $(M \cup \{x_i w, y_i u\}) \backslash \{x_i y_i\}$ is a rainbow matching of size δ, a contradiction. ■

By Claim 2.5, we may assume without loss of generality that $\{x_1, \ldots, x_r\}$ is the set of good vertices for some $r \geq 0$. Let $W' = W \cup \{y_1, \ldots, y_r\}$.

Claim 2.6 *No edge uv in $G[W']$ has color in $\{1, \ldots, r\}$.*

Proof: By way of contradiction, assume that there is an edge uv in $G[W']$

such that $c(uv) \in \{1, \ldots, r\}$. Let M' be the subset of M consisting of the edge with color $c(uv)$ and any edges with an endpoint in $\{u, v\}$. There are at most three such edges (the edge with color $c(uv)$ and possibly one for each endpoint); without loss of generality, let $M' = \{x_1y_1, \ldots, x_ty_t\}$ (here $1 \leq t \leq 3$). Note that x_j is a good vertex for $1 \leq j \leq t$. Thus there are distinct vertices $w_1, \ldots, w_t$ such that x_jw_j is a good edge for $1 \leq j \leq t$ and the colors on the edges $uv, x_1w_1, \ldots, x_tw_t$ are distinct. Thus $(M \cup \{uv, x_1w_1, \ldots, x_tw_t\}) \backslash \{x_1y_1, \ldots, x_ty_t\}$ is a rainbow matching of size δ, a contradiction. ■

An edge uv is *nice* if its color is not in $\{r+1, \ldots, \delta-1\}$ and one of its endpoints is in W'. Note that every good edge is nice. Recall that every good edge has an endpoint in $V(M)$. By Claim 2.5 and Claim 2.6, no nice edge lies in $G[W']$. Hence, every nice edge joins vertices in W' and $V(G)\backslash W'$. A vertex $v \in V(M)\backslash\{x_1, \ldots, x_r, y_1, \ldots, y_r\}$ is *nice* if v is incident with at least seven nice edges. Note that if there is no good vertex (i.e. $r = 0$), then the definitions of good and nice vertices are the same and so there is also no nice vertex. Next, we prove analogues of Claim 2.5 and Claim 2.6 for nice vertices and edges.

Claim 2.7 *For $i \in \{r+1, \ldots, \delta-1\}$, if x_i is incident with at least three nice edges, then no nice edge is incident with y_i, and vice versa.*

Proof: Suppose y_iu is a nice edge for some $i \in \{r+1, \ldots, \delta-1\}$. If x_i is incident to at least three nice edges, then x_i has a neighbor v such that x_iv is a nice edge, $v \neq u$, and $c(x_iv) \neq c(y_iu)$. Let M' be the subset of M consisting of edges with an endpoint in $\{u, v\}$ or a color in $\{c(x_iv), c(y_iu)\}$. There are at most four such edges (possibly one with each endpoint and one with each color); without loss of generality, let $M' = \{x_1y_1, \ldots, x_ty_t\}$ (here $0 \leq t \leq 4$). Note that x_j is a good vertex for $1 \leq j \leq t$. Thus there are distinct vertices $w_1, \ldots, w_t$ such that x_jw_j is a good edge for $1 \leq j \leq t$ and the colors on the edges $x_iv, y_iu, x_1w_1, \ldots, x_tw_t$ are distinct.

Thus $(M \cup \{x_iv, y_iu, x_1w_1, \ldots, x_tw_t\})\backslash\{x_iy_i, x_1y_1, \ldots, x_ty_t\}$ is a rainbow matching of size δ, a contradiction. ■

By Claim 2.7, we may assume that $\{x_{r+1}, x_{r+2}, \ldots, x_{r+s}\}$ is the set of nice vertices for some $s \geq 0$.

Claim 2.8 *No edge uv in $G[W']$ has color in $\{1, \ldots, r+s\}$.*

Proof: By Claim 2.6, the claim holds if $s = 0$. Assume that $s \geq 1$, and consequently $r \geq 1$. Without loss of generality, suppose that there is an edge uv in $G[W']$ with $c(uw) = r+1$. Because x_{r+1} is nice, it has a neighbor v' in W' such that $x_{r+1}v'$ is a nice edge and $v' \neq u, v$. Let M' be the subset of M consisting of those edges with an endpoint in $\{u, v, v'\}$ or color $c(x_{r+1}v')$. Again there are at most four edges in M' and we let $M' = \{x_1y_1, \ldots, x_ty_t\}$. As before, $w_1, \ldots, w_t$ are distinct vertices such that x_jw_j is a good edge for $1 \leq j \leq t$ and the colors on uv, $x_1w_1, \ldots, x_tw_t$ are distinct, thus $(M \cup \{uv, x_{r+1}v', x_1w_1, \ldots, x_tw_t\})\backslash\{x_{r+1}y_{r+1}, x_1y_1, \ldots, x_ty_t\}$ is a rainbow matching of size δ, a contradiction. ■

Next, we count the number of nice edges in G.

Claim 2.9 *There are at most*

$$\max\left\{(3\delta - 8 - r + s)r + 6(\delta - 1), \left(\frac{7\delta}{3} - 7 + s\right)r + 6(\delta - 1)\right\}$$

nice edges in G.

Proof: Recall that $V(G) \setminus W' = \{x_1, \ldots, x_{\delta-1}, y_{r+1}, \ldots, y_{\delta-1}\}$ and every nice edge joins vertices from W' and $V(G) \setminus W'$. If $r \leq 2\delta/3$, then the set of good vertices is incident to at most $r(3\delta - 2 - r)$ nice edges by Lemma 2.4. Similarly, if $r \geq 2\delta/3$, then the set of good vertices is incident to at most $r(3\delta - 2 - \lfloor 2\delta/3 \rfloor) \leq r(7\delta/3 - 1)$ nice edges. For $i \in \{r+1, \ldots, r+s\}$, Claim 2.7 implies that x_i is incident to at most $r+6$ nice edges and y_i is incident to none. For $i \in \{r+s+1, \ldots, \delta-1\}$, by Claim 2.7

there are at most six nice edges with an endpoint in $\{x_i, y_i\}$. Therefore, the number of nice edges is at most

$$(3\delta - 2 - r)r + (r+6)s + 6(\delta - 1 - r - s) = (3\delta - 8 - r + s)r + 6(\delta - 1)$$

if $r \leq 2\delta/3$, and

$$\left(\frac{7\delta}{3} - 1\right) r + (r+6)s + 6(\delta - 1 - r - s) = \left(\frac{7\delta}{3} - 7 + s\right) r + 6(\delta - 1)$$

if $r \geq 2\delta/3$. ■

Recall that C is the color class with color δ, $|C| = a$, and C is a maximum size color class. Therefore there are at least $2(a - \delta + 1)$ vertices in W incident to an edge in C. Since every edge in C has an endpoint in $V(M)$ it follows that there are at least $2(a - \delta + 1)$ vertices in $V(M)$ joined to W by edges in C. Without loss of generality, let $\{r+s+1, \ldots, r+s+t\}$ be the set of indices $i \in \{r+s+1, \ldots, \delta - 1\}$ such that x_i or y_i is incident to an edge in C. By Claim 2.5 and Claim 2.7, we have

$$t \geq a - \delta + 1 - \frac{r+s}{2} \text{ and } r + s + t \leq \delta - 1. \tag{2.1}$$

Claim 2.10 *For $i \in \{r+s+1, \ldots, r+s+t\}$, there is at most one edge of color i in $G[W]$.*

Proof: Suppose uv and $u'v'$ are edges of color i in $G[W]$ for some $i \in \{r+s+1, \ldots, r+s+t\}$. Without loss of generality, we may assume that there exists $w \in W$ such that $c(x_i w) = \delta$ and $w \neq u, v$. Hence, $(M \cup \{uv, x_i w\}) \setminus \{x_i y_i\}$ is a rainbow matching of size δ, a contradiction. ■

Now we count the number of nice edges from W' to $V(G) \setminus W'$. Recall that each color class in G contains at most a edges. By Claim 2.8, there is no edge in $G[W']$ of color $i \in \{r+1, \ldots, r+s\}$. Thus, for $i \in \{r+1, \ldots, r+s\}$ there are at most $a - 1$

vertices in W' that are incident with an edge of color i. Since every color class has size at most a, for $i \in \{r+s+1, \ldots, \delta-1\}$ there are at most $2(a-1)$ vertices in W' that are incident with an edge of color i. Furthermore for $i \in \{r+s+1, \ldots, r+s+t\}$, Claim 2.10 implies that there are at most a vertices in W that are incident with an edge of color i. Since $W' \setminus W = \{y_1, \ldots, y_r\}$, it follows that for $i \in \{r+s+1, \ldots, r+s+t\}$ there are at most $\min\{a+r, 2(a-1)\}$ vertices in W' that are incident with an edge of color i. It then follows, using the fact that $|W'| = |W| + r = n - 2(\delta - 1) + r$ and (2.1), that the number of nice edges from W' to $V(G) \setminus W'$ is at least

$$
\begin{aligned}
& \delta|W'| - (a-1)(2\delta - 2 - 2r - s - 2t) - \min\{a+r, 2(a-1)\}t \\
= \; & \delta n - 2\delta(\delta - 1) + \delta r - (a-1)(2\delta - 2 - 2r - s) + \max\{a - r - 2, 0\}t.
\end{aligned}
$$

We first consider the case when $r \leq 2\delta/3$. Applying the upper bound of $(3\delta - 8 - r + s)r + 6(\delta - 1)$ nice edges from Claim 2.9, we obtain

$$
\begin{aligned}
\delta n \leq \quad & (2\delta - 8 - r + s)r + 2(\delta + 3)(\delta - 1) + (a-1)(2\delta - 2 - 2r - s) \\
& - \max\{a - r - 2, 0\}t. \qquad (2.2)
\end{aligned}
$$

To finish the proof we bound the right hand side of (2.2) to obtain a contradiction. Note that the coefficient of t is nonpositive. Thus, the right hand side of (2.2) is maximized when t is minimized. By (2.1), $t \geq \max\{a - \delta + 1 - (r+s)/2, 0\}$.

If $a \leq \delta - 1 + (r+s)/2$, then we let $t = 0$. The coefficient of a is nonnegative, and thus (2.2) is maximized when a is maximized; hence we assume $a = \delta - 1 + (r+s)/2$. Substituting for a yields a negative quadratic in s that is maximized when $s = 1 - r/2$. Since s is a nonnegative integer and $s = 0$ if $r = 0$, (2.2) is maximized at $s = 0$, which yields

$$\delta n \leq 2(2\delta+1)(\delta-1)+(\delta-5-2r)r.$$

This is maximized when $r=(\delta-5)/4$, yielding $n \leq \frac{33\delta}{8}-\frac{13}{4}+\frac{9}{8\delta}$, a contradiction.

If $a > \delta-1+r/2$, we let $t=a-\delta+1-r/2$. Since $t>0$, it follows that $r+s \leq \delta-2$. Thus $a-r-2 > \delta-3-(r-s)/2 \geq \delta-3-(\delta-2)/2 = \delta/2-2$. As $\delta \geq 3$ and $a-r-2$ is an integer, $\max\{a-r-2,0\} = a-r-2$. Therefore the coefficient of s in (2.2) is nonpositive and we may assume that $s=0$. Consequently, (2.2) becomes

$$\delta n \leq (3\delta-1-r/2-a)a+(\delta-6-3r/2)r+2(\delta^2-1).$$

The right hand side is maximized when $a=\frac{3\delta}{2}-\frac{1}{2}-\frac{r}{4}$, so

$$\delta n \leq \frac{1}{16}(-28+68\delta^2-24\delta+4\delta r-92r-23r^2),$$

which is maximized when $r=2\delta/23-2$. This yields

$$n \leq \frac{98\delta}{23}-2+\frac{4}{\delta},$$

a contradiction.

To complete the proof of the theorem, we are left with the case $r \geq 2\delta/3$. Similar to (2.2), since we have at most $(7\delta/3-7+s)\,r+6(\delta-1)$ nice edges in G by Claim 2.9, we have

$$\begin{aligned}\delta n \leq\ & (4\delta/3-7+s)\,r+2(\delta+3)(\delta-1)+(a-1)(2\delta-2-2r-s)\\ & -\max\{a-r-2,0\}t. \end{aligned} \tag{2.3}$$

Again, the right hand side of (2.3) is maximized when t is minimized.

If $a \le \delta - 1 + (r+s)/2$, then (2.3) is maximized when $t = 0$ and $a = \delta - 1 + r/2$. Again we may assume that $s = 0$, yielding

$$\delta n \le -2 + 4\delta^2 - 2\delta + r\left(\frac{\delta}{3} - 4 - r\right).$$

This is maximized when $r = \delta/6 - 2$, yielding $n \le \frac{145\delta}{36} - \frac{8}{3} + \frac{6}{\delta}$, a contradiction.

If $a > \delta - 1 + (r+s)/2$, we let $t = a - \delta + 1 - (r+s)/2$ and again we may assume that $s = 0$. Then, (2.3) becomes

$$\delta n \le \frac{1}{6}\left(-6a^2 + 3a(6\delta - 2) + 12\delta^2 - (3a + 30 + 3r - 2\delta)r - 12\right),$$

which is maximized when r is minimized. Since we have assumed that $r \ge 2\delta/3$, we have $r = 2\delta/3$ and we are back in the case $r \le 2\delta/3$, finishing the proof of the theorem. ■

2.3 Proof of Theorem 2.2

We proceed by induction on $\delta(G)$. The result is trivial if $\delta(G) = 1$. We assume that G is a graph with minimum degree $\delta > 1$ and order greater than $\frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta}$.

Lemma 2.11 *If G has a color class containing at least $2\delta - 1$ edges, then G has a rainbow matching of size δ.*

Proof: Let C be a color class with at least $2\delta - 1$ edges. By induction, there is a rainbow matching M of size $\delta - 1$ in $G - C$. There are $2\delta - 2$ vertices covered by the edges in M, thus one of the edges in C has no endpoint covered by M, and the matching can be extended. ■

It is useful to note that we also have the following, which is identical to Lemma 2.3 when $k = 1$.

Lemma 2.12 *If G satisfies $\Delta(G) > 3\delta - 3$, then G has a rainbow matching of size δ.*

We begin by preprocessing the graph so that each edge is incident to at least one vertex with degree δ. To achieve this, arbitrarily order the edges in G and process them in order. If both endpoints of an edge have degree greater than δ when it is processed, delete that edge. In the resulting graph, every edge is incident to a vertex with degree δ. Furthermore, by Lemma 2.12 we may assume that $\Delta(G) \leq 3\delta - 3$; thus the degree sum of the endpoints of any edge is bounded above by $4\delta - 3$. After preprocessing, we begin the greedy algorithm.

In the ith step of the algorithm, a smallest color class is chosen (without loss of generality, color i), and then an edge e_i of color i is chosen such that the degree sum of the endpoints is minimized. All the remaining edges of color i and all edges incident with the endpoints of e_i are deleted. The algorithm terminates when there are no edges in the graph.

We assume that the algorithm fails to produce a matching of size δ in G; suppose that the rainbow matching M generated by the algorithm has size k. We let R denote the set of vertices that are not covered by M.

Let c_i denote the size of the smallest color class at step i. Since at most two edges of color $i+1$ are deleted in step i (one at each endpoint of e_i), we observe that $c_{i+1} + 2 \geq c_i$. Otherwise, at step i color class $i+1$ has fewer edges. Let step h be the last step in the algorithm in which a color class that does not appear in M is completely removed from G. It then follows that $c_h \leq 2$, and in general $c_i \leq 2(h-i+1)$ for $i \in [h]$. Let f_i denote the number of edges of color i deleted in step i with both endpoints in R. Since $f_i < c_i$, we have $f_i \leq 2(h-i)+1$ for $i \in [h]$. Note that after step h, there are exactly $k-h$ colors remaining in G. By Lemma 2.11, color classes contain at most $2\delta - 2$ edges, and therefore the last $k-h$ steps remove at most $(k-h)(2\delta-2)$ edges. Furthermore, for $i > h$, the degree sum of the endpoints of e_i is at most $2(\delta - 1)$.

For $i \in [h]$, let x_i and y_i be the endpoints of e_i, and let $d_i(v)$ denote the degree of a vertex v at the beginning of step i. Let $\tau_i = \max\{0, d_i(x_i) + d_i(y_i) - 2\delta\}$; note that $2\delta \le 2\delta + \tau_i \le 4\delta - 3$. Thus, at step i, at most $2\delta + \tau_i + f_i - 1$ edges are removed from the graph. Since the algorithm removes every edge from the graph, we conclude that

$$|E(G)| \le (k-h)(2\delta - 2) + \sum_{i=1}^{h}(2\delta + \tau_i + f_i - 1). \tag{2.4}$$

We now compute a lower bound for the number of edges in G. Since the degree sum of the endpoints of e_i in G is at least $2\delta + \tau_i$, we immediately obtain the following inequality:

$$\frac{n\delta + \sum_{i\in[h]} \tau_i}{2} \le |E(G)|.$$

If $f_i > 0$ and $\tau_i > 0$, then there is an edge with color i having both endpoints in R. Since this edge was not chosen in step i by the algorithm, the degree sum of its endpoints is at least $2\delta + \tau_i$, and one of its endpoints has degree at least $\delta + \tau_i$. For each value of i satisfying $f_i > 0$, we wish to choose a representative vertex in R with degree at least $\delta + \tau_i$. Since there are f_i edges with color i having both endpoints in R, there are f_i possible representatives for color i. Since a vertex in R with high degree may be the representative for multiple colors, we wish to select the largest system of distinct representatives.

Suppose that the largest system of distinct representatives has size t, and let T be the set of indices of the colors that have representatives. For each color $i \in T$ there is a distinct vertex in R with degree at least $\delta + \tau_i$. Thus we may increase the edge count of G as follows:

$$\frac{n\delta + \sum_{i\in[h]} \tau_i + \sum_{i\in T} \tau_i}{2} \le |E(G)|. \tag{2.5}$$

We let $\{f_i^{\downarrow}\}$ denote the sequence $\{f_i\}_{i\in[h]}$ sorted in nonincreasing order. Since

$f_i \leq 2(h-i)+1$, we conclude that $f_i^{\downarrow} \leq 2(h-i)+1$. Because there is no system of distinct representatives of size $t+1$, the sequence $\{f_i^{\downarrow}\}$ cannot majorize the sequence $\{t+1, t, t-1, \dots, 1\}$. Hence there is a smallest value $p \in [t+1]$ such that $f_p^{\downarrow} \leq t+1-p$. Therefore, the maximum value of $\sum_{i=1}^{h} f_i^{\downarrow}$ is bounded by the sum of the sequence $\{2h-1, 2h-3, \dots, 2(h-p)+3, t+1-p, \dots, t+1-p\}$. Summing we attain

$$\sum_{i\in[h]} f_i \leq (p-1)(2h-p+1) + (h-p+1)(t+1-p).$$

Over p, this value is maximized when $p = t+1$, yielding $\sum_{i\in[h]} f_i \leq t(2h-t)$. Since $h \leq \delta - 1$, we then have $\sum_{i\in[h]} f_i \leq 2(\delta-1)t - t^2$.

We now combine bounds (2.4) and (2.5):

$$\frac{n\delta + \sum_{i\in[h]} \tau_i + \sum_{i\in T} \tau_i}{2} \leq (k-h)(2\delta-2) + \sum_{i=1}^{h}(2\delta + \tau_i + f_i - 1).$$

Hence, since $k \leq \delta - 1$,

$$\begin{aligned}
\frac{n\delta}{2} &\leq (2\delta-1)(\delta-1) + \frac{1}{2}\sum_{[h]\setminus T} \tau_i + \sum_{i\in[h]} f_i \\
&\leq (2\delta-1)(\delta-1) + (\delta-1-t)(\delta-3/2) + 2(\delta-1)t - t^2 \\
&\leq 3\delta^2 - \frac{11}{2}\delta + \frac{5}{2} + \left(\delta - \frac{1}{2}\right)t - t^2.
\end{aligned}$$

This bound is maximized when $t = (\delta - \frac{1}{2})/2$. Thus

$$n \leq \frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta},$$

contradicting our choice for the order of G.

It remains to show that this proof provides the framework of a $O(\delta(G)|V(G)|^2)$-time algorithm that generates a rainbow matching of size $\delta(G)$ in a properly edge-colored graph G of order at least $\frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta}$. Given such a graph G, we create a sequence of graphs $\{G_i\}$ as follows, letting $G = G_0$, $\delta = \delta(G)$, and $n = |V(G)|$. First, determine $\delta(G_i)$, $\Delta(G_i)$, and the maximum size of a color class in G_i; this process takes $O(n^2)$-time. If $\Delta(G_i) \leq 3\delta(G_i) - 3$ and the maximum color class has at most $2\delta(G_i) - 2$ edges, then terminate the sequence and set $G_i = G'$. If $\Delta(G_i) > 3\delta(G_i) - 3$, then delete a vertex v of maximum degree and then process the edges of $G_i - v$, iteratively deleting those with two endpoints of degree at least $\delta(G_i)$; the resulting graph is G_{i+1}. If $\Delta(G_i) \leq 3\delta(G_i) - 3$ but a maximum color class C has at least $2\delta(G_i) - 1$ edges, then delete C and then process the edges of $G_i - C$, iteratively deleting those with two endpoints of degree at least $\delta(G_i)$; the resulting graph is G_{i+1}. Note that $\delta(G_{i+1}) = \delta(G_i) - 1$. If this process generates G_δ, we set $G' = G_\delta$ and terminate. Generating the sequence $\{G_i\}$ consists of at most δ steps, each taking $O(n^2)$-time.

Given that $G' = G_i$, the algorithm from the proof of Theorem 2.2 takes $O(\delta n^2)$-time to generate a matching of size $\delta - i$ in G'. The preprocessing step and the process of determining a smallest color class and choosing an edge in that class whose endpoints have minimum degree sum both take $O(n^2)$-time. This process is repeated at most δ times.

A matching of size $\delta - (i + 1)$ in G_{i+1} is easily extended in G_i to a matching of size $\delta - i$ using the vertex of maximum degree or maximum color class. The process of extending the matching takes $O(\delta)$-time. Thus the total run-time of the algorithm generating the rainbow matching of size δ in G is $O(\delta n^2)$. ■

It is worth noting that the analysis of the greedy algorithm used in the proof of Theorem 2.2 could be improved. In particular, the bound $c_{i+1} \geq c_i - 2$ is sharp only if at step i there are an equal number of edges of color i and $i + 1$ and both endpoints of e_i are incident to edges with color $i + 1$. However, since one of the endpoints of e_i has degree at most δ, at most $\delta - 1$ color classes can lose two edges in step i. Since the maximum size of a color class in G is at most $2\delta - 2$, if G has order at least

6δ, then there are at least $3\delta/2$ color classes. Thus, for small values of i, the bound $c_i \leq 2(k - i + 1)$ can likely be improved. However, we doubt that such analysis of this algorithm can be improved to yield a bound on $|V(G)|$ better than 6δ.

3. 2-Factors with a Bounded Number of Odd Components

3.1 Introduction

Throughout this chapter all cycles have an implicit clockwise orientation. For some vertex v on a cycle C we will denote the first, second, and i^{th} predecessor (respectively successor) of v as v^-, v^{--}, and $v^{(-i)}$ (respectively v^+, v^{++}, and $v^{(+i)}$) respectively. Given x and y on C, $C(x, y)$ is the set of vertices $\{x^+, x^{++}, \ldots, y^-\}$ and $C[x, y]$ is the set $\{x, x^+, \ldots, y\}$. We also let xCy (respectively xC^-y) denote the path $xx^+ \ldots y$ (respectively $xx^- \ldots y$). Given a subgraph H of G, for an edge $e = xy$ we define $dist_H(e, v) = \min\{dist_H(x, v), dist_H(y, v)\}$. If there is no path in H from an endpoint of e to v, then we say that $dist_H(e, v) = \infty$. For some $U \subseteq V(G)$, if $U \subseteq N(v)$, then we say that v *dominates* U. For a subgraph H of G and a vertex v, if v dominates $V(H)$, then we will frequently instead say that v dominates H. The *circumference* of a graph G, $c(G)$, is the length of a longest cycle in G.

Recall that when a graph G contains no copies of the claw, $K_{1,3}$, G is claw-free and that we denote by $\langle v; x, y, z\rangle$ the claw with central vertex v and leaves x, y, and z. Likewise, $\langle v; x_1, x_2, ..., x_t\rangle$ denotes a copy of $K_{1,t}$. For the subgraph induced by $U \subseteq V(G)$ we use the notation $G[U]$.

A graph is *hamiltonian* if it contains a spanning cycle, which is a 2-factor with exactly one component. The problem of determining when a graph is hamiltonian is a classical and widely studied problem in graph theory (cf. [34, 35]). Aside from the hamiltonian problem, there are many results throughout the literature that give conditions ensuring the existence of a 2-factor with given properties. Many results of this type are outlined in [34, 35], and there have been numerous developments since the writing of those surveys (see, for instance, [2]) .

Let $odd(\mathcal{F})$ denote the number of odd cycles in a 2-factor $\mathcal{F}$ of a graph G, and let $odd(G)$ denote the minimum of $odd(\mathcal{F})$ over all 2-factors $\mathcal{F}$ of G. If G does not contain a 2-factor, we will set $odd(G) = \infty$. The focus of this chapter is the following.

Problem 3.1 *Given $k \geq 0$, determine conditions that guarantee a graph G has $odd(G) \leq k$.*

We note here that determining if a graph has two disjoint perfect matchings, or equivalently has $odd(G) = 0$ is NP-complete, even in the case where G is a bridgeless cubic graph, which is when G has no cut-edge and is 3-regular. This follows from the fact [48] that it is NP-hard to determine if a cubic graph has a proper edge-coloring with three colors. Such a coloring exists if and only if G contains a pair of disjoint perfect matchings.

In addition to the general literature on the structure of 2-factors in graphs, $odd(G)$ arises when studying several other interesting problems. For instance, results of Huck and Kochol [51], Häggkvist and McGuinness [40], and Huck [50] showed that a cubic bridgeless graph with $odd(G)$ at most four has a family $\mathcal{C}$ of at most five even subgraphs such that each edge of G lies in exactly two members of $\mathcal{C}$. As a consequence, such graphs satisfy the Circuit Double Cover (CDC) Conjecture (cf. [78, 80]). Understanding the CDC for cubic graphs is crucial given that Seymour [78] noted that a smallest counterexample to the CDC Conjecture must be cubic.

Note that $odd(G)$ is a measure of how far G is from having a pair of edge-disjoint perfect matchings, as G contains such a pair if and only if $odd(G) = 0$. Several other related metrics have been considered. Let $B_2(G) = \{(B, B') : B, B' \text{ are edge-disjoint matchings of } G\}$, $P = \{(B, B') \in B_2(G) : |E(B)| + |E(B')| \text{ is maximum}\}$, and H be a maximum matching chosen from the pairs of matchings in P. Then, given a maximum matching M in G, $\mu(G) = |M|/|H|$ measures how close G is to having a pair of disjoint matchings of maximum size. Mkrtchyan, Musoyan, and Tserunyan studied $\mu(G)$ and showed the following.

Theorem 3.2 [68] *For any graph G, $\mu(G) \leq \dfrac{5}{4}$. This bound is tight.*

Another measure of how close G is to having a pair of disjoint perfect matchings is given by the ratio of edges covered by a pair of disjoint matchings to the number of vertices in a graph. This notion was explored by Mkrtchyan, Petrosyan, and Vardanyan, and they showed the following.

Theorem 3.3 [69] *Let G be a cubic graph. Then G contains a pair of disjoint matchings covering at least $\frac{4}{5}|V(G)|$ edges in G.*

The problem of determining when a graph has k edge-disjoint perfect matchings for some $k \geq 2$ has also been studied in several contexts. Hilton [46] and Zhang and Zhu [91] examined the number of disjoint 1-factors in regular graphs, and recently Hou [49] considered the problem for nearly-regular graphs (those graphs with $\Delta(G) - \delta(G) \leq 1$). Hoffman and Rodger [47] give necessary and sufficient conditions for a complete multipartite graph to contain k edge-disjoint 1-factors.

In this chapter, we are generally interested in investigating Problem 3.1 in the context of forbidden subgraphs. In Section 3.2 we examine the stability of $odd(G)$ under the Ryjáček closure for claw-free graphs. In Section 3.3 we specifically consider the case $k = 0$ and examine pairs of forbidden subgraphs that ensure a graph has a pair of disjoint 1-factors. We then present several constructions and open problems related to Problem 3.1 in highly connected claw-free graphs.

3.2 Closure and $odd(G)$ for Claw-Free Graphs

In [73] Ryjáček introduced the following closure concept. In a graph G, a vertex v is *eligible* if $G[N(v)]$ is connected but not complete. The graph formed by completing the neighborhood of a vertex v is the *local completion* (of G) at v. The closure of a graph G, denoted $cl(G)$, is constructed by iteratively performing the local completion operation at eligible vertices until none remain.

This closure is an important tool for the study of cycle structure and hamiltonian-type properties in large part due to the following result from Ryjáček.

Theorem 3.4 [73] *Let G be a claw-free graph. Then*

1. *the closure* $cl(G)$ *is well-defined,*

2. *there is a triangle-free graph* H *such* $cl(G) = L(H)$*, where* $L(H)$ *is the line graph of* H*,*

3. $c(G) = c(cl(G))$.

Given a graph class $\mathscr{C}$, we say that a property $\mathscr{P}$ is *stable* in $\mathscr{C}$ provided that every graph G in $\mathscr{C}$ has property $\mathscr{P}$ if and only if $cl(G)$ has property $\mathscr{P}$. Thus, Theorem 3.4 implies that hamiltonicity (and more generally the property "$c(G) = k$") is stable in the class of claw-free graphs.

Of particular interest here, Ryjáček, Saito, and Schelp showed the following.

Theorem 3.5 [74] *If* G *is a claw-free graph, then* G *has a* 2*-factor with at most* k *cycles if and only if* $cl(G)$ *has a* 2*-factor with at most* k *cycles.*

The main result of this section is the following extension of Theorem 3.5.

Theorem 3.6 [18] *If* G *is a claw-free graph, then* $odd(G) \leq k$ *if and only if* $odd(cl(G)) \leq k$.

Proof: Clearly, if $odd(G) \leq k$, then $odd(cl(G)) \leq k$, so we proceed by demonstrating the converse. Let G be a claw-free graph and let $G = G_0, G_1, \ldots, G_t = cl(G)$ be a sequence of graphs such that G_i is the graph formed by local completion of G_{i-1} at vertex v_i. Suppose to the contrary that $odd(G_t) \leq k$ but $odd(G) > k$, and let $i \geq 1$ be the minimum integer such that G_i has a 2-factor $\mathcal{C}$ with at most k odd cycles and denote v_i as v.

Let $E^i_{\mathcal{C}} = (E(\mathcal{C}) \cap E(G_i)) \setminus E(G_{i-1})$, so that $E^i_{\mathcal{C}}$ is the set of edges of the 2-factor $\mathcal{C}$ in G_i which are not in G_{i-1}, and observe that the endpoints of these edges are all in the neighborhood of v. Choose $\mathcal{C}$ to minimize $|E^i_{\mathcal{C}}|$ and furthermore, among all such choices of $\mathcal{C}$ select an edge $e \in E^i_{\mathcal{C}}$ such that $dist_{\mathcal{C}}(e, v)$ is maximum. For each cycle in

$\mathcal{C}$, let C_w denote the cycle containing a particular vertex w and similarly let C_f denote the cycle containing a particular edge f. Let $e = u_1u_2$, where u_2 is the predecessor of u_1 in C_e, and let P be a shortest path between u_1 and u_2 in $G_{i-1}[N(v)]$. Since G_{i-1} is claw-free, P has order at most four; denote the vertices of P as $u_1 = x_1x_2...x_\ell = u_2$ with $3 \le \ell \le 4$.

The proof proceeds in cases by considering $dist_{\mathcal{C}}(e, v)$. Subcases are then based on the length of P and the location of certain vertices on given cycles in $\mathcal{C}$. In most cases, we draw a contradiction by finding a 2-factor with at most k odd cycles in G_i that either uses fewer edges in $E^i_{\mathcal{C}}$ than $\mathcal{C}$, or violates the pertinent assumption about $dist_{\mathcal{C}}(e, v)$. We begin by illuminating several useful edges and non-edges of G_{i-1}.

Claim 3.7 *If $C_v = C_e$, the following hold:*

1. $u_2v^- \notin E(G_{i-1})$,
2. $u_1v^+ \notin E(G_{i-1})$,
3. $v^-v^+ \notin E(G_{i-1})$,
4. *if* $u_2 \neq v^+$, *then* $u_2v^+ \in E(G_{i-1})$.

Proof: If u_2v^- is an edge, then replacing C_v with $u_2v^-C_v^-u_1vC_vu_2$ results in a 2-factor $\mathcal{C}'$ such that $odd(\mathcal{C}') \le k$ and $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$, a contradiction. Similarly, if $u_1v^+ \in E(G_{i-1})$, replacing C_v with $u_1v^+C_vu_2vC_v^-u_1$ contradicts the minimality of $|E^i_{\mathcal{C}}|$.

Suppose C_v is a triangle, then $v^- = u_1$ and $v^+ = u_2$, so clearly, v^-v^+ is not an edge of G_{i-1}. If C_v is a 4-cycle, then $u_1 = v^-$ or $u_2 = v^+$. However, Parts (1) and (2) imply that $u_2v^- \notin E(G_{i-1})$ and $u_1v^+ \notin E(G_{i-1})$, which is to say $v^-v^+ \notin E(G_{i-1})$. Lastly if C_v has more than four vertices and $v^-v^+ \in E(G_{i-1})$, then replacing C_v with $v^-v^+C_vu_2vu_1C_vv^-$ contradicts the minimality of $|E^i_{\mathcal{C}}|$.

Finally, consider $\langle v; u_1, u_2, v^+\rangle$. By assumption, $u_1u_2 \notin E(G_{i-1})$, and since Part (2) holds $u_1v^+ \notin E(G_{i-1})$, thus since G_{i-1} is claw-free, $u_2v^+ \in E(G_{i-1})$. ■

The following will also be useful throughout the remainder of the proof.

Claim 3.8 *Let $f = w_1w_2 \in \mathcal{C}$ such that $C_e \neq C_f$. If $u_1w_1 \in E(G_{i-1})$ and $u_2w_2 \in E(G_{i-1})$, then there is a cycle C such that $V(C) = V(C_e) \cup V(C_f)$ where $e \notin C$.*

Proof: Specifically assume that w_1 is the predecessor of w_2 (without loss of generality) on C_f, and replace C_e and C_f with $w_1u_1C_{u_1}u_2w_2C_{w_2}w_1$, resulting in a 2-factor $\mathcal{C}'$ such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. ■

Suppose $C_v \neq C_e$, and consider v^+, the successor of v on C_v. The claw $\langle v; v^+, u_1, u_2\rangle$ in G_{i-1} implies that without loss of generality v^+u_2 is an edge in G_{i-1}, since $e = u_1u_2 \notin E(G_{i-1})$. However, by Claim 3.8, we can replace C_e and C_v with $u_2v^+C_vvu_1C_eu_2$, contradicting the minimality of $|E^i_{\mathcal{C}}|$. Thus $C_v = C_e$, so we may apply Claim 3.7 as needed going forward.

We now proceed by considering $dist_{\mathcal{C}}(e, v)$.

Case 1: $dist_{\mathcal{C}}(e, v) \geq 3$.

By Claim 3.7, $u_2v^- \notin E(G_{i-1})$, $u_1v^+ \notin E(G_{i-1})$, and $u_2v^+ \in E(G_{i-1})$. Considering the claw $\langle v; u_1, u_2, v^-\rangle$ we see that $u_1v^- \in E(G_{i-1})$. We can now replace C_v with $C_v^1 = u_1C_vv^-u_1$ and $C_v^2 = vC_vu_2v$. If C_v is odd, this results in a 2-factor $\mathcal{C}'$ with $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. If C_v is even, then C_v^1 and C_v^2 have the same parity. If both are even, the minimality of $|E^i_{\mathcal{C}}|$ is contradicted. Otherwise, replace C_v with $u_1C_vvu_1$ and $v^+C_vu_2v^+$, to contradict the minimality of $|E^i_{\mathcal{C}}|$.

Case 2: $dist_{\mathcal{C}}(e, v) = 2$.

Without loss of generality, assume $e = v^{--}v^{(-3)} = u_1u_2$. Note that since $dist_{\mathcal{C}}(e, v) = 2$, Claim 3.7 implies $v^+v^{(-3)} \in E(G_{i-1})$.

We also claim that if $x_2 \neq v^-$, then $C_{x_2} \neq C_v$. Indeed, suppose otherwise and note that if x_2^-v or x_2^+v is an edge of G_{i-1}, then replacing C_v with either

$x_2^- vv^- v^{--} x_2 C_v v^{(-3)} v^+ C_v x_2^-$ or $x_2^+ vv^- v^{--} x_2 C_v^- v^+ v^{(-3)} C_v^- x_2^+$ results in a 2-factor $\mathcal{C}'$ such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. Also, this implies that $x_2^- \neq v^+$ and $x_2^+ \neq v^{(-3)}$ since v cannot be adjacent to x_2^+ or x_2^-. Thus, since $\langle x_2; v, x_2^-, x_2^+ \rangle$ is not induced, $x_2^- x_2^+ \in E(G_{i-1})$. However, exchanging C_v with $vv^- v^{--} x_2 v$ and $v^+ C_v x_2^- x_2^+ C_v v^{(-3)} v^+$ contradicts the minimality of $|E^i_{\mathcal{C}}|$. Therefore $x_2 \notin V(C_v)$ when $x_2 \neq v^-$.

Case 2.1: $\ell = 3$, *so that in particular* $P = v^{--} x_2 v^{(-3)} = u_1 x_2 u_2$.

By Claim 3.7, $u_2 = v^{(-3)}$ is not adjacent to v^-, so $x_2 \neq v^-$, which implies that $C_{x_2} \neq C_v$.

Next, $\langle x_2; x_2^+, v^{(-3)}, v^{--} \rangle$ cannot be induced in G_{i-1}, so consider each of $x_2^+ v^{(-3)}$ and $x_2^+ v^{--}$. If $x_2^+ v^{(-3)} \in E(G_{i-1})$ or $x_2^+ v^{--} \in E(G_{i-1})$, then replacing C_v and C_{x_2} with either $x_2^+ C_{x_2} x_2 v^{--} C_v v^{(-3)} x_2^+$ or $x_2^+ v^{--} C_v v^{(-3)} x_2 C_{x_2}^- x_2^+$ results in a 2-factor $\mathcal{C}'$ with at most k odd cycles such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. Thus, since $\langle x_2; x_2^+, v^{(-3)}, v^{--} \rangle$ is not induced in G_{i-1}, P contains two internal vertices.

Case 2.2: $C_{x_2} = C_v$.

At the start of Case 2, we showed that if $x_2 \neq v^-$, then $C_v \neq C_{x_2}$, so we can assume $x_2 = v^-$ and $\ell = 4$. By Claim 3.7, $x_3 \neq v^+$, since $v^- v^+ \notin E(G_{i-1})$.

We now claim that x_3 is not on C_v. Indeed, assume otherwise and consider $\langle x_3; v^{(-3)}, v^-, x_3^- \rangle$. If $v^{(-3)} x_3^- \in E(G_{i-1})$, we may replace C_v with $v^{(-3)} x_3^- C_v^- vv^{--} v^- x_3$ $C_v v^{(-3)}$ resulting in a 2-factor $\mathcal{C}'$ with $odd(\mathcal{C}') \leq k$ such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. If $v^- x_3^- \in E(G_{i-1})$, replacing C_v with $v^- x_3^- C_v^- v^+ v^{(-3)} C_v^- x_3 vv^{--} v^-$ similarly contradicts the minimality of $|E^i_{\mathcal{C}}|$. However, from Claim 3.7, $v^- v^{(-3)} \notin E(G_{i-1})$. Thus x_3 is not on C_v, so $C_{x_3} \neq C_v$ as claimed.

Consider the claw $\langle x_3; x_3^+, v^-, v^{(-3)} \rangle$. Since $v^- v^{(-3)}$ is not an edge, one of $x_3^+ v^-$ or $x_3^+ v^{(-3)}$ is an edge in G_{i-1}. If $x_3^+ v^- \in E(G_{i-1})$, replace C_v and C_{x_3} with $vv^{--} v^- x_3^+ C_{x_3} x_3 v^{(-3)} C_v^- v$. Otherwise, if $x_3^+ v^{(-3)} \in E(G_{i-1})$, replace C_v and C_{x_3} with $vv^{--} v^- x_3 C_{x_3}^- x_3^+ v^{(-3)} C_v^- v$. Both scenarios result in a 2-factor $\mathcal{C}'$ with at most k odd cycles such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$, contradicting the minimality of $|E^i_{\mathcal{C}}|$.

Case 2.3: $C_{x_3} = C_v$.

In this case, we first show that $x_3 \neq v^+$. Assume otherwise and consider $\langle x_2; v^{--}, v^+, x_2^- \rangle$. If $v^{--}x_2^- \in E(G_{i-1})$, then replacing C_v and C_{x_2} with $x_2v^+C_vv^{(-3)}v$ $v^-v^{--}x_2^-C_{x_2}^-x_2$ contradicts the minimality of $|E_{\mathcal{C}}^i|$. If $v^+x_2^- \in E(G_{i-1})$, then by replacing C_v and C_{x_2} with $v^+x_2^-C_{x_2}^-x_2v^{--}v^-vv^{(-3)}C_v^-v^+$ in $\mathcal{C}$ we contradict the minimality of $|E_{\mathcal{C}}^i|$. As Claim 3.7 shows, $v^{--}v^+ \notin E(G_{i-1})$, implying that $\langle x_2; v^{--}, v^+, x_2^- \rangle$ is induced, a contradiction, thus $x_3 \neq v^+$.

Now that $x_3 \neq v^+$, we use $\langle v; v^-, v^{(-3)}, x_2 \rangle$ to show that $v^-x_2 \in E(G_{i-1})$ and $v^-x_2^- \notin E(G_{i-1})$. From here, $\langle x_2; v^-, x_2^-, x_3 \rangle$ is used to complete this subcase, as $x_3x_2^-$ allows us to use $\langle x_3; x_3^-, x_3^+, x_2^- \rangle$ to show that $x_3^+x_3^- \in E(G_{i-1})$ which can be used to contradict $|E_{\mathcal{C}}^i|$ and x_3v^- allows us to contradict $x_2 \notin C_v$.

By Claim 3.7, $v^-v^{(-3)} \notin E(G_{i-1})$. If $x_2v^{(-3)} \in E(G_{i-1})$, then $v^{--}x_2v^{(-3)}$ contradicts the minimality of P. Thus, since $\langle v; v^-, v^{(-3)}, x_2 \rangle$ is not induced, $x_2v^- \in E(G_{i-1})$. Also, if $v^-x_2^- \in E_{\mathcal{C}}^i$, then replacing C_v and C_{x_2} with $v^-x_2^-C_{x_2}^-x_2vC_vv^-$ results in a 2-factor $\mathcal{C}'$ with $odd(\mathcal{C}') \leq k$ such that $dist_{\mathcal{C}'}(e, v) > dist_{\mathcal{C}}(e, v)$, a contradiction to the maximality of $dist_{\mathcal{C}}(e, v)$.

Since $\langle x_2; v^-, x_2^-, x_3 \rangle$ is not induced, either $x_3x_2^-$ or v^-x_3 is an edge in G_{i-1}. First, assume that $x_3x_2^- \in E(G_{i-1})$. If $x_3^+x_2^-$ or $x_3^-x_2^-$ is an edge in G_{i-1}, then either $x_3^+x_2^-C_{x_2}^-x_2v^{--}C_vx_3v^{(-3)}C_v^-x_3^+$ or $x_3^-C_v^-v^+v^{(-3)}C_v^-x_3vv^-v^{--}x_2C_{x_2}x_2^-x_3^-$ can replace C_v and C_{x_2}, resulting in a 2-factor $\mathcal{C}'$ with at most k odd cycles such that $|E_{\mathcal{C}'}^i| < |E_{\mathcal{C}}^i|$. Since $\langle x_3; x_3^+, x_3^-, x_2^- \rangle$ is not induced, $x_3^-x_3^+ \in E(G_{i-1})$. But then replacing C_v and C_{x_2} with $x_2^-C_{x_2}^-x_2v^{--}C_vx_3^-x_3^+C_vv^{(-3)}x_3x_2^-$ contradicts the minimality of $|E_{\mathcal{C}}^i|$. This means that $v^-x_3 \in E(G_{i-1})$. Since we could have chosen $P = v^{--}v^-x_3v^{(-3)}$, from Case 2.2, the minimality of $|E_{\mathcal{C}}^i|$ is contradicted, completing Case 2.3.

Case 2.4: $C_{x_3} = C_{x_2}$.

By Claim 3.8, $x_3 \notin \{x_2^+, x_2^-\}$. Suppose C_{x_2} is a 4-cycle, and consider the claw $\langle x_2; x_2^+, x_2^-, v^{--} \rangle$. If $x_2^+x_2^- \in E(G_{i-1})$, then $v^{--}C_vv^{(-3)}x_3x_2^-x_2^+x_2v^{--}$ may replace C_v

and C_{x_2}, resulting in a 2-factor $\mathcal{C}'$ with at most k odd cycles such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$, a contradiction. If $x_2^+v^{--}$ (respectively $x_2^-v^{--}$) is an edge in G_{i-1}, then replace C_v and C_{x_2} with $v^{--}C_vv^{(-3)}x_3C_{x_2}x_2^+v^{--}$ (respectively $v^{--}C_vv^{(-3)}x_3C^-_{x_2}x_2^-v^{--}$) to contradict the minimality of $|E^i_{\mathcal{C}}|$. Thus, we can assume C_{x_2} is not a 4-cycle.

If both $x_2^-x_2^+$ and $x_3^-x_3^+$ are edges in G_{i-1}, then replacing C_v and C_{x_2} with $v^{--}C_vv^{(-3)}x_3x_2v^{--}$ and $x_2^+C_{x_2}x_3^-x_3^+C_{x_2}x_2^-x_2^+$ yields a 2-factor $\mathcal{C}'$ with $odd(\mathcal{C}') \leq k$ and $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$.

First assume $x_2^-x_2^+ \notin E(G_{i-1})$. Since $\langle x_2; x_2^+, x_2^-, v\rangle$ is not induced, one of x_2^-v or x_2^+v is an edge of G_{i-1}. Assume without loss of generality that $x_2^-v \in E(G_{i-1})$. Consider the claw $\langle v; x_2^-, v^-, v^+\rangle$. By Claim 3.7, $v^-v^+ \notin E(G_{i-1})$. If $v^+x_2^- \in E(G_{i-1})$, then replacing C_v and C_{x_2} with $vv^{(-3)}C_v^-v^+x_2^-C^-_{x_2}x_2v^{--}v^-v$ results in a 2-factor $\mathcal{C}'$ with $odd(\mathcal{C}') \leq k$ such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. If $v^-x_2^- \in E(G_{i-1})$, then replace C_v and C_{x_2} with $vC_vv^{(-3)}v$ and $x_2C_{x_2}x_2^-v^-v^{--}x_2$ to again contradict the minimality of $|E^i_{\mathcal{C}}|$. However G_{i-1} is claw-free, implying that $x_2^-x_2^+ \in E(G_{i-1})$, so $x_3^-x_3^+ \notin E(G_{i-1})$.

Since $x_3^+x_3^- \notin E(G_{i-1})$ and $\langle x_3; x_3^+, x_3^-, v\rangle$ is not induced, one of x_3^+v or x_3^-v is an edge in G_{i-1}. Without loss of generality, assume that $x_3^+v \in E(G_{i-1})$. Again, by Claim 3.7 $v^-v^+ \notin E(G_{i-1})$. Thus, since $\langle v; x_3^+, v^-, v^+\rangle$ is not induced, either $x_3^+v^- \in E(G_{i-1})$ or $x_3^+v^+ \in E(G_{i-1})$. If $x_3^+v^- \in E(G_{i-1})$, then $v^-x_3^+C_{x_2}x^3v^{(-3)}C_v^-vv^{--}v^-$ can replace C_v and C_{x_2} to contradict the minimality of $|E^i_{\mathcal{C}}|$. Otherwise, if $x_3^+v^+ \in E(G_{i-1})$, replacing C_v and C_{x_2} with $x_2v^{--}v^-vx_2$ and $v^+C_vv^{(-3)}x_3C^-_{x_2}x_2^+x_2^-C^-_{x_2}x_3^+v^+$ contradicts the minimality of $|E^i_{\mathcal{C}}|$. This contradiction completes Case 2.4.

Case 2.5: *C_{x_3}, C_{x_2}, and C_v are distinct.*

In this case, we begin by using $\langle x_2; x_3, x_2^+, v^{--}\rangle$ to show that $x_3x_2^+ \in E(G_{i-1})$. This edge allows us to consider $\langle x_3; x_3^-, x_2^+, v^{(-3)}\rangle$. Since this claw is not induced in G_{i-1}, we are able to use edges in G_{i-1} to contradict the minimality of $|E^i_{\mathcal{C}}|$.

Since $\langle x_2; x_3, x_2^+, v^{--}\rangle$ cannot be induced, x_3v^{--}, $x_2^+v^{--}$, or $x_3x_2^+$ is an edge of G_{i-1}. Note that $x_3v^{--} \notin E(G_{i-1})$, as otherwise $v^{--}x_3v^{(-3)}$ would contradict the minimality of P. Suppose $x_2^+v^{--} \in E(G_{i-1})$, and consider $\langle v; x_2, v^-, v^{(-3)}\rangle$. If $x_2v^{(-3)} \in E(G_{i-1})$, then $v^{--}x_2v^{(-3)}$ contradicts the minimality of P. By Claim 3.7 $v^-v^{(-3)} \notin E(G_{i-1})$. This means $x_2v^- \in E(G_{i-1})$. However, now replace C_v and C_{x_2} with $v^{--}x_2^+C_{x_2}x_2v^-v^{--}$ and $vC_vv^{(-3)}v$ to contradict the minimality of $|E^i_{\mathcal{C}}|$. Therefore $x_3x_2^+ \in E(G_{i-1})$.

Consider $\langle x_3; x_3^-, x_2^+, v^{(-3)}\rangle$. This cannot be induced, so at least one of $x_3^-x_2^+$, $x_2^+v^{(-3)}$, or $x_3^-v^{(-3)}$ must be an edge of G_{i-1}. When $x_3^-x_2^+ \in E(G_{i-1})$, replace C_v, C_{x_2}, and C_{x_3} with $x_2^+x_3^-C^-_{x_3}x_3v^{(-3)}C^-_vv^{--}x_2C^-_{x_2}x_2^+$, resulting in a 2-factor $\mathcal{C}'$ such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. If $x_2^+v^{(-3)} \in E(G_{i-1})$, replace C_v and C_{x_2} with $x_2^+v^{(-3)}C^-_vv^{--}x_2C^-_{x_2}x_2^+$ to contradict the minimality of $|E^i_{\mathcal{C}}|$. Finally, the edge $x_3^-v^{(-3)}$ allows us to replace C_v, C_{x_2}, and C_{x_3} with $x_3^-v^{(-3)}C^-_vv^{--}x_2C^-_{x_2}x_2^+x_3C_{x_3}x_3^-$, again contradicting the minimality of $|E^i_{\mathcal{C}}|$. This contradiction completes the proof of Case 2.5, which also completes Case 2.

Case 3: $dist_{\mathcal{C}}(e, v) = 1$.

Without loss of generality, assume $e = v^-v^{--}$. By Claim 3.7, $v^-v^+ \notin E(G_{i-1})$, so $x_2 \neq v^+$.

Case 3.1: $C_{x_2} = C_v$.

If $x_2 = v^{++}$, then replacing C_v with $v^-x_2C_vv^{--}v^+vv^-$ results in a 2-factor $\mathcal{C}'$ with at most k odd cycles such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$, contradicting the minimality of $|E^i_{\mathcal{C}}|$. Thus $dist_{C_v}(v, x_2) \geq 3$.

Next we show that either $\langle x_2; x_2^-x_2^+, v^-\rangle$ is induced or there is a 2-factor in G_{i-1} which contradicts the minimality of $|E^i_{\mathcal{C}}|$. If $x_2^-x_2^+ \in E(G_{i-1})$, then replacing C_v with $vx_2v^-C^-_vx_2^+x_2^-C^-_vv$ results in a 2-factor $\mathcal{C}'$ such that $dist_{\mathcal{C}'}(e, v) > dist_{\mathcal{C}}(e, v)$. If $x_2^-v^-$ or $x_2^+v^-$ is an edge of G_{i-1} then replacing C_v with $vC_vx_2^-v^-x_2C_vv^{--}v$ or $vC_vx_2v^-x_2^+C_vv^{--}v$ contradicts the minimality of $|E^i_{\mathcal{C}}|$. Thus $\langle x_2; x_2^-x_2^+, v^-\rangle$ is induced, a contradiction. Therefore $x_2 \notin V(C_v)$.

Case 3.2: $\ell = 3$, *so in particular* $P = v^-x_2v^{--} = u_1x_2u_2$.

By Claim 3.8, $\langle x_2; x_2^+, v^{--}, v^- \rangle$ is induced in G_{i-1}, so P must have two internal vertices.

Case 3.3: $C_{x_3} = C_v$.

First we show that $x_3 \neq v^+$. Indeed assume otherwise, and consider $\left\langle x_2; x_2^-, v^+, v^- \right\rangle$. If $v^+x_2^- \in E(G_{i-1})$, then we can replace C_v and C_{x_2} with $v^+x_2^-C_{x_2}^-x_2 v^-vv^{--}C_v^-v^+$ to contradict the minimality of $|E_{\mathcal{C}}^i|$. If $v^-x_2^- \in E(G_{i-1})$ then replacing C_v and C_{x_2} with $v^-x_2^-C_{x_2}^-x_2v^+C_vv^{--}vv^-$ contradicts the minimality of $|E_{\mathcal{C}}^i|$. By Claim 3.7 $v^-v^+ \notin E(G_{i-1})$, so $\left\langle x_2; x_2^-, v^+, v^- \right\rangle$ is induced in G_{i-1}.

We now use $\langle x_2; v^-, x_3, x_2^- \rangle$ to show that $x_2^-x_3 \in E(G_{i-1})$. If $x_2^-v^- \in E(G_{i-1})$, then replace C_v and C_{x_2} with $vx_2C_{x_2}x_2^-v^-v^{--}C_v^-v$ to contradict the maximality of $dist_{\mathcal{C}}(e, v)$. If $x_3v^- \in E(G_{i-1})$, then $v^-x_3v^{--}$ contradicts the minimality of P. Thus, $x_2^-x_3 \in E(G_{i-1})$.

Finally, consider $\langle x_3; x_3^-, v^{--}, x_2^- \rangle$. If $x_2^-v^{--} \in E(G_{i-1})$ replacing C_v and C_{x_2} with $x_2^-C_{x_2}^-x_2v^-C_vv^{--}x_2^-$ results in a 2-factor $\mathcal{C}'$ with $odd(\mathcal{C}') \leq k$ such that $|E_{\mathcal{C}'}^i| < |E_{\mathcal{C}}^i|$. If $x_3^-v^{--} \in E(G_{i-1})$, then replacing C_v and C_{x_2} with $v^{--}C_v{}^-x_3x_2^-C_{x_2}^-x_2v^-C_vx_3^-v^{--}$ similarly contradicts the minimality of $|E_{\mathcal{C}}^i|$. Finally if $x_2^-x_3^- \in E(G_{i-1})$, then $vC_vx_3^-x_2^-C_{x_2}^-x_2v^-C_v^-x_3v$ can replace C_v and C_{x_2} to contradict the maximality of $dist_{\mathcal{C}}(e, v)$. Since $\langle x_3; x_3^-, v^{--}, x_2^- \rangle$ cannot be induced, $x_3 \notin C_v$.

Case 3.4: $C_{x_3} = C_{x_2}$.

First, considering $\langle x_2; x_2^-, x_2^+, v^- \rangle$, we show that $x_2^-x_2^+ \in E(G_{i-1})$. By Claim 3.8, $x_3 \neq x_2^-$ and $x_3 \neq x_2^+$. Without loss of generality, $v^-x_2^+ \notin E(G_{i-1})$ (respectively $v^-x_2^-$), as otherwise, replacing C_v and C_{x_2} with $v^-x_2^+C_{x_2}x_2vC_vv^-$ results in a 2-factor such that $C_v = C_{x_2}$, which was considered in Case 3.1, and so $x_2^-x_2^+ \in E(G_{i-1})$.

Now consider the claw $\langle x_3; x_3^+, x_2, v^{--} \rangle$. By the minimality of P, $x_2v^{--} \notin E(G_{i-1})$. If $v^{--}x_3^+ \in E(G_{i-1})$, then replacing C_v and C_{x_2} with $v^{--}x_3^+C_{x_2}x_2^-x_2^+C_{x_2}x_3 x_2v^-C_vv^{--}$ results in a 2-factor $\mathcal{C}'$ with at most k odd cycles such that $|E_{\mathcal{C}'}^i| < |E_{\mathcal{C}}^i|$. Otherwise, if $x_2x_3^+ \in E(G_{i-1})$, then replace C_v and C_{x_2} with $x_2x_3^+C_{x_2}x_2^-x_2^+C_{x_2}x_3v^{--}$

$C_v^- v^- x_2$ to contradict the minimality of $|E^i_{\mathcal{C}}|$. This contradiction completes the proof of Case 3.4.

Case 3.5: *C_{x_3}, C_{x_2} and C_v are distinct.*

First, consider the claw $\langle x_2; x_2^+, v^-, x_3\rangle$. As in Case 3.4, $v^- x_2^+ \notin E(G_{i-1})$ as otherwise, replacing C_v and C_{x_2} with $v^- x_2^+ C_{x_2} x_2 v C_v v^-$ results in a 2-factor such that $C_v = C_{x_2}$, considered in Case 3.1. By the minimality of P, $v^- x_3 \notin E(G_{i-1})$. So we have that $x_3 x_2^+ \in E(G_{i-1})$.

Next, we use $\langle x_3; x_2, x_3^+, v^{--}\rangle$ to show that $x_3^+ v^{--} \in E(G_{i-1})$. By the minimality of P, $x_2 v^{--} \notin E(G_{i-1})$. If $x_3^+ x_2 \in E(G_{i-1})$, replacing C_{x_2} and C_{x_3} with $x_2 x_3^+ C_{x_3} x_3 x_2^+ C_{x_2} x_2$ yields a 2-factor such that $C_{x_2} = C_{x_3}$, which is Case 3.4. Thus, $x_3^+ v^{--} \in E(G_{i-1})$; however, we can now replace C_v, C_{x_2}, and C_{x_3} with $v^{--} x_3^+ C_{x_3} x_3 x_2^+ C_{x_2} x_2 v^- C_v v^{--}$, resulting in a 2-factor $\mathcal{C}'$ such that $|E^i_{\mathcal{C}'}| < |E^i_{\mathcal{C}}|$. This contradiction completes the proof of Case 3.5, and the proof of Theorem 3.6. ■

3.3 Disjoint 1-factors

As noted in the introduction, a 2-factor with no odd components is equivalent to a pair of disjoint 1-factors. Going forward we will refer to such a 2-factor as an *even 2-factor*. This section specifically considers Problem 3.1 when $k = 0$.

Problem 3.9 *Determine conditions in a graph G which guarantee that G has a pair of disjoint perfect matchings.*

In light of this problem, we have the following immediate corollary of Theorem 3.6 in the case $k = 0$.

Corollary 3.10 [18] *Let G be a claw-free graph. Then G contains a pair of disjoint perfect matchings if and only if $cl(G)$ contains a pair of disjoint perfect matchings.*

3.3.1 Forbidden Subgraph Conditions

It is clear that if a graph G has an even 2-factor, then G has a 2-factor, so we will consider conditions that ensure a graph has a 2-factor. In [31] Faudree, Faudree, and Ryjáček characterized the pairs of forbidden subgraphs which guarantee that a large enough 2-connected graph has a 2-factor. In the results that follow, the *generalized net* $N(i, j, k)$ is the graph obtained by associating one endpoint of each of the paths P_{i+1}, P_{j+1} and P_{k+1} with distinct vertices of a triangle. Following convention, we will let $B(i, j)$ and Z_i denote the generalized nets $N(i, j, 0)$ and $N(i, 0, 0)$, respectively.

Theorem 3.11 [31] *Let X and Y be connected graphs with at least three edges, and let G be a 2-connected graph of order $n \geq 10$. Then, G being $\{X, Y\}$-free implies that G has a 2-factor if and only if, up to the order of the pairs, $X = K_{1,3}$ and Y is a subgraph of either P_7, Z_4, $B(4, 1)$, or $N(3, 1, 1)$, or $X = K_{1,4}$ and $Y = P_4$.*

We begin by considering $\{K_{1,4}, P_4\}$-free graphs. Let $\mathcal{E}$ be the family of graphs composed of two odd cliques and an edge wz joined to two vertices x and y where x may or may not be adjacent to y (see Figure 3.1). Every graph G in $\mathcal{E}$ is $\{K_{1,4}, P_4\}$-free and does not contain a pair of disjoint 1-factors, as the edge wz is in every perfect matching of G. Our next result demonstrates that graphs in the family $\mathcal{E}$ are the only 2-connected $\{K_{1,4}, P_4\}$-free graphs with no even 2-factor. We require the following lemma from Faudree, Faudree, and Ryjáček.

Lemma 3.12 [31] *If G is a 2-connected $\{K_{1,4}, P_4\}$-free graph of order at least nine, then G has a 2-factor with at most two components.*

Theorem 3.13 [18] *Any 2-connected $\{K_{1,4}, P_4\}$-free graph of even order at least ten contains an even 2-factor or is a member of the family $\mathcal{E}$.*

Proof: Let G be a 2-connected $\{K_{1,4}, P_4\}$-free graph of even order $n \geq 10$ that does not contain an even 2-factor. Thus every 2-factor has at least two components and by Lemma 3.12 there is a 2-factor $\mathcal{F}$ with exactly two components, call them

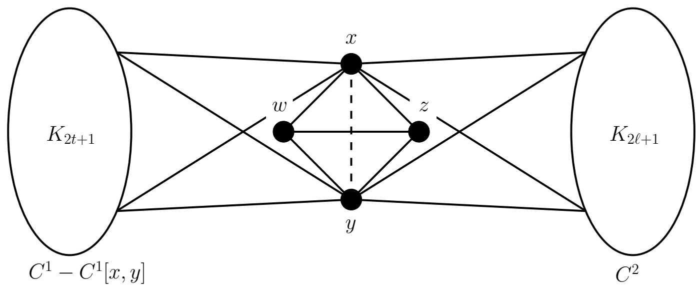

Figure 3.1: The family $\mathcal{E}$ of 2-connected $\{K_{1,4}, P_4\}$-free graphs that do not contain disjoint 1-factors.

C^1 and C^2. Since G is 2-connected there are at least two disjoint edges, e_1 and e_2, between C^1 and C^2. Let $e_1 = xx'$ and $e_2 = yy'$, where $x, y \in C^1$ and $x', y' \in C^2$. If $xy \in E(C^1)$ and $x'y' \in E(C^2)$, then C^1 and C^2 can be combined into a single cycle $xC^1yy'C^{2-}x'x$. Since G is P_4-free, $G[\{x^-, x, x', x'^+\}]$ is not an induced P_4. If $x^-x'^+$ is an edge, then $xC^1x^-x'^+C^2x'x$ is an even 2-factor in G, so without loss of generality, $xx'^+ \in E(G)$. Similarly, since $G[\{x^-, x, x'^+, x'^{++}\}]$ is not an induced P_4, $xx'^{++} \in E(G)$. A similar argument shows that there is an induced P_4 for each edge from x to C^2 unless x dominates C^2. Now, if $xy \in E(C^1)$, then replace C^1 and C^2 with $yC^1xy'^+C^2y'y$, so $y^- \neq x$. For the same reason that x dominates C^2, y must also dominate C^2. (Note that if y' dominated C^1 instead, then we could combine C^1 and C^2.)

Let D be the set of vertices in C^1 that dominate C^2. There can be no two edges $f_1 = uv \in E(C^1)$ and $f_2 = u'v' \in E(C^2)$ such that $uu' \in E(G)$ and $vv' \in E(G)$ lest C^1 and C^2 be combined, and there are at least two vertices in D, so $|V(C^1)| \geq 4$ which implies that $|V(C^1)| \geq 5$ since $|V(C^1)|$ is odd. If for some vertex $v \in D$, the edge v^-v^+ is in G, then C^1 can be shortened by using the edge v^-v^+ to skip v, and C^2 can be extended by including v, forming an even 2-factor. Thus no such edge exists. Since for any pair of vertices v_1 and v_2 in C^2, the graph $\langle x; x^-, x^+, v_1, v_2\rangle$ is not induced, the edge v_1v_2 exists. Thus $V(C^2)$ forms a clique.

Suppose e_1 and e_2 were chosen such that $C^1(x, y) \cap D = \emptyset$. Let v be any vertex in $C^1(x, y^-)$ such that xv is an edge of G. Then the edge xv^+ is in G since $G[\{x', x, v, v^+\}]$ is not an induced P_4. Now x dominates $C^1(x, y)$ and similarly y dominates $C^1(x, y)$. Let i be the smallest integer such that $x^{(+i)} \in C^1(x, y)$ does not dominate $C^1(x, y)$, and let j be the smallest integer such that $y^{(-j)}$ is not adjacent to $x^{(+i)}$. Consider $\langle x; x', x^-, x^{(+i)}, y^{(-j)}\rangle$. If either $x^-x^{(+i)}$ or $x^-y^{(-j)}$ is an edge, then $x^-x^{(+i)}C^1y^-x^{(+i-1)}C^{1-}xx'C^2x'^-yC^1x^-$ or $x^-y^{(-j)}C^1y^-x^{(+i)}C^1y^{(-j-1)}x^{(+i-1)}C^{1-}xx'C^2$ $x'^-yC^1x^-$ is a hamiltonian cycle. If $x'x^- \in E(G)$, $x'x^-C^{1-}xx'^+C^{2-}x$ is a hamiltonian cycle, and similarly if $x'x^{+i} \in E(G)$ or $x'y^{-j} \in E(G)$, then $x'x^{+i}C^{1-}x^+x^{+i+1}C^{1-}xx'^+$ $C^{2-}x'$ or $x'y^{-j}C^{1-}x^+y^{-j+1}C^{1-}xx'^+C^{2-}x'$ is a hamiltonian cycle. Consequently, no such i exists and $G[C^1(x, y)]$ forms a clique. If $|D| = 2$, then this argument can be repeated to show that $G[C^1(y, x)]$ is also a clique.

Suppose $|D| \geq 3$ and let $z \in D$ such that $C^1(x, z) \cap D = \{y\}$. As in the case that $|D| = 2$, $G[C^1(y, z)]$ forms a clique.

If there is an edge between $C^1(x, y)$ and $C^1(y, z)$, then we may rearrange C^1 such that y^-y^+ is an edge and remove y from C^1 and add it to C^2, and therefore no such edge exists. If $x^-y^+ \in E(G)$, then $C^1[x, y]$ can be pulled into C^2 to form $\widehat{C}^1 = y^+C^1x^-y^+$ and $\widehat{C}^2 = xC^1yx'^-C^{2-}x'x$. Consider the path $G[\{x^{--}, x^-, x, x^+\}]$. From before, x^-x^+ is not an edge, and $x^{--}x^+$ allows us to replace $\widehat{C}^1$ and $\widehat{C}^2$ with $x^{--}x^+\widehat{C}^{2-}xx^-\widehat{C}^1x^{--}$, so xx^{--} must be an edge. Iterating this argument, we get that x dominates $\widehat{C}^1$. However, since z dominates C^2, $zx' \in E(G)$ and the cycle $zx'\widehat{C}^2xz^+\widehat{C}^1z$ spans G and so is an even 2-factor. Thus, if $x^-y^+ \in E(G)$, then $|D| = 2$.

Now if $xy \in E(G)$, then $G[\{x^-, x, y, y^+\}]$ is an induced P_4, otherwise consider $G[\{x, y^-, y, y^+\}]$. If $xy^+ \in E(G)$, then $\langle x; x^-, x^+, y^+, x'\rangle$ is induced. Thus, $G[\{x, y^-, y, y^+\}]$ is an induced P_4 unless $|D| = 2$.

As shown above, $|C^1(x, y)| > 0$. Since C^1 is an odd cycle, either $|C^1(x, y)|$ or $|C^1(y, x)|$ is even, so without loss of generality, assume that $|C^1(x, y)|$ is even.

If $|C^1(x,y)| \geq 4$, then $C^1[y,x)$, $C^1(x,y)$, and $C^2 + x$ form an even 2-factor. If $|C^1(x,y)| = 2$, then G is a member of $\mathcal{E}$. ∎

Turning our attention to connected, but not necessarily 2-connected, graphs, Fujisawa and Saito showed the following.

Theorem 3.14 [33] *Let F_1 and F_2 be connected graphs of order at least three. Then there exists a positive integer n_0 such that every connected $\{F_1, F_2\}$-free graph of order at least n_0 and minimum degree at least two has a 2-factor if and only if without loss of generality F_1 is a subgraph of $K_{1,3}$ and F_2 is a subgraph of Z_2.*

We give the following related result for disjoint 1-factors.

Theorem 3.15 [18] *Every connected $\{K_{1,3}, Z_2\}$-free graph of even order with minimum degree $\delta(G) = \delta \geq 3$ contains an even 2-factor with at most two components.*

Proof: Let G be a connected $\{K_{1,3}, Z_2\}$-free graph of even order. Gould and Jacobson [36] showed that every 2-connected $\{K_{1,3}, Z_2\}$-free graph is hamiltonian, so we need only consider the case where G has a cut vertex x. Since G is $K_{1,3}$-free, x must lie in exactly two blocks, so let B_1 and B_2 be the blocks containing x.

Since G is claw-free, the neighborhood of any vertex is connected or two disjoint cliques, so $G[N_{B_1}(x)]$ and $G[N_{B_2}(x)]$ are cliques. Further, since $d(x) \geq 3$, without loss of generality, $|N_{B_1}(x)| \geq 2$. Let $v_1, v_2 \in N_{B_1}(x)$ and $u \in N_{B_2}(x)$. Suppose that $N_{B_2}(x) \neq V(B_2)$, and let $w \in V(B_2)\backslash N(x)$ such that $uw \in E(G)$. We know that there is such a w since G is connected. However, $G[\{x, v_1, v_2, u, w\}]$ is an induced Z_2. Thus $N_{B_2}(x) = V(B_2)$, and similarly $N_{B_1}(x) = V(B_1)$.

Finally, suppose there is some third block, B_3, which necessarily cannot contain x. Without loss of generality, let $v \in B_2 \cap B_3$ be a cut vertex of G. Let $u \in N_{B_3}(v)$ and x be the only cut vertex of B_1. This means that since $\delta(G) \geq 3$, $|V(B_1)| \geq 4$ and x is adjacent to v. Now, let $x_1, x_2 \in V(B_1)$ such that $x_1 \neq x$ and $x_2 \neq x$. Then $G[\{x_1, x_2, x, v, u\}]$ is an induced Z_2, a contradiction.

Thus G consists of blocks B_1 and B_2, both of which are complete. Since G has even order, one block has even order and the other odd order. Without loss of generality, let B_1 have odd order. Then $B_1 - \{x\}$ and B_2 are even 2-connected $\{K_{1,3}, Z_2\}$-free graphs so are hamiltonian, and this yields an even 2-factor of G. ∎

Considering either a clique of any size with a pendant edge or the graph obtained by identifying a vertex in a clique of any size and a vertex of K_3, the minimum degree condition in Theorem 3.15 cannot be weakened in order to still guarantee an even 2-factor in the graph.

3.3.2 Minimum Degree and Connectivity Conditions

There are a number of results that give minimum degree and connectivity conditions implying a claw-free graph has a 2-factor. Egawa and Ota [27] and Choudum and Paulraj [13] independently showed that a claw-free graph with minimum degree at least four has a 2-factor. Subsequently, Yoshimoto [90] and Aldred, Egawa, Fujisawa, Ota and Saito [3] demonstrated that $\delta(G) \geq 3$ suffices when G is 2-connected and claw-free.

We show next that there are no such conditions that ensure a 3-connected claw-free graph of even order contains an even 2-factor. Let $\mathcal{P}$ be the Petersen graph, and define $\mathbb{P}_\delta$ as the graph obtained by replacing each vertex of $\mathcal{P}$ with a clique of order $\delta + 1$. The three incident edges at one vertex v of $\mathcal{P}$ are now incident with three different vertices in the clique replacing v in $\mathbb{P}_\delta$. (See Figure 3.2 for $\mathbb{P}_4$.)

Theorem 3.16 [18] *For even $\delta \geq 4$, $\mathbb{P}_\delta$ is a 3-connected $K_{1,3}$-free graph with minimum degree δ and no even 2-factor.*

Proof: It is clear that $\mathbb{P}_\delta$ is 3-connected since $\mathcal{P}$ is 3-connected. The neighborhood of any vertex is either a clique or two disjoint cliques, thus $\mathbb{P}_\delta$ is $K_{1,3}$-free.

Consider a perfect matching M in $\mathbb{P}_\delta$, and a perfect matching M' that is disjoint from M. Let $F_{\mathcal{P}} \subset E(\mathbb{P}_\delta)$ be the set of edges that were originally edges of $\mathcal{P}$ (these are the edges that are not completely contained in a single clique of order $\delta+1$). Each

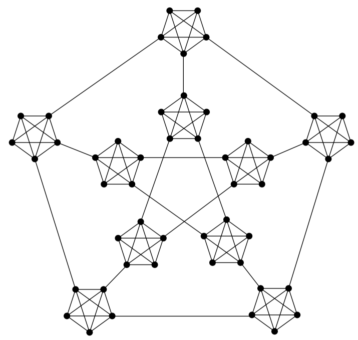

Figure 3.2: The graph $\mathbb{P}_4$.

clique is odd, so M must contain $B \subseteq F_{\mathcal{P}}$ where each clique contains one or three vertices of $V(B)$. If $V(B)$ has three vertices in a single clique C, then there is no M', since $\mathbb{P}_\delta - M$ has C as an odd component. Therefore, $V(B)$ contains exactly one vertex in each clique, and corresponds to a perfect matching in $\mathcal{P}$. However, any perfect matching in $\mathcal{P}$ leaves two components, each isomorphic to C_5. The corresponding components in $\mathbb{P}_\delta$ are copies of C_5 with vertices replaced by odd cliques. These two components are odd (of order at least $5(\delta + 1)$), and so M' cannot exist. Thus, $\mathbb{P}_\delta$ does not contain an even 2-factor. ■

4. Extremal Theorems for Degree Sequence Packing

4.1 Introduction

There are a number of necessary and sufficient conditions for a sequence to be graphic, including the seminal Havel-Hakimi Algorithm [42, 45] and the Erdős-Gallai Criteria [28]. However, a given graphic sequence may have a large family of nonisomorphic realizations, and as such considerable attention has been given to the study of when a graphic sequence has a realization with a given property. Such problems can be divided into two broad classes, described as "forcible" problems and "potential" problems in [70]. Given a graph property $\mathscr{P}$, we say that a graphic sequence π is *forcibly* $\mathscr{P}$*-graphic* if every realization of π has property $\mathscr{P}$, and that π is *potentially* $\mathscr{P}$*-graphic* if at least one realization of π has property $\mathscr{P}$.

Results on forcible degree sequences are often stated as traditional problems in structural or extremal graph theory, where a necessary and/or sufficient condition is given in terms of the degrees of the vertices (or equivalently the number of edges) of a given graph. For instance, minimum degree thresholds for the existence of certain graph structures, such as the threshold for hamiltonicity in Dirac's Theorem [23], can be thought of as forcible theorems. Two older, but exceptionally thorough surveys on forcible and potential problems are due to Hakimi and Schmeichel [43] and Rao [71] and a more recent survey on forcible "Chvátal-Type" theorems (in the spirit of [15]) is due to Bauer et al. [5].

A number of degree sequence analogues to classical problems in extremal graph theory appear throughout the literature, including potentially graphic sequence variants of Hadwiger's Conjecture [26, 72], the Erdős-Sós Conjecture [63], graph Ramsey numbers [10] and the Turán problem (c.f. [32]). In this chapter, we consider an extension of the classical graph packing literature to degree sequences. In particular, we prove a potentially $\mathscr{P}$-graphic analogue to a widely-studied graph packing conjecture of Bollobás and Eldridge [7] and, independently, Catlin [12], which implies a graphic sequence version of the Sauer-Spencer graph packing theorem [76]. We conclude by using similar techniques to prove a pair of related results that have applications to discrete imaging science.

4.1.1 Graph Packing

Two n-vertex graphs G_1 and G_2 *pack* if G_1 is a subgraph of $\overline{G_2}$, or alternatively if G_1 and G_2 can be expressed as edge-disjoint subgraphs of K_n. Graph packing has received a great deal of attention with interesting results and challenging open problems appearing throughout the literature ([56], [88] and [89] are detailed and useful surveys).

In 1978, Sauer and Spencer proved the following classical theorem.

Theorem 4.1 [76] *Let G_1 and G_2 be graphs of order n with maximum degrees Δ_1 and Δ_2 respectively. If*

$$\Delta_1 \Delta_2 < \frac{n}{2},$$

then G_1 and G_2 pack.

Likely the most notable open conjecture in graph packing is due to Bollobás and Eldridge [7] and, independently, Catlin [12].

Conjecture 4.2 [7, 12] *Let G_1 and G_2 be n-vertex graphs with maximum degrees Δ_1 and Δ_2, respectively. If*

$$(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$$

then G_1 and G_2 pack.

We note that, if true, Conjecture 4.2 implies Theorem 4.1. The Bollobás-Eldridge-Catlin conjecture has been settled in several cases, including when $\Delta_1 \leq 2$ by Aigner and Brandt [1] and Alon and Fisher [4]. The case when $\Delta_1 = 3$ was shown by Csaba, Shokoufandeh, and Szemerédi [17] for large n utilizing the regularity lemma. For $\Delta_1, \Delta_2 \geq 300$, Kaul, Kostochka and Yu [55] showed that $(\Delta_1 + 1)(\Delta_2 + 1) \leq 0.6n + 1$

implies that the two graphs pack, which improves the Sauer-Spencer theorem, and is a partial solution to Conjecture 4.2. Other partial results were obtained by Corrádi and Hajnal [16] and Hajnal and Szemerédi [41].

4.1.2 Packing Graphic Sequences

The notion of packing graphic sequences was investigated in [11], where the following key definition appears. If π_1 and π_2 are (not necessarily monotone) graphic sequences, with $\pi_1 = (d_1^{(1)}, \ldots, d_n^{(1)})$ and $\pi_2 = (d_1^{(2)}, \ldots, d_n^{(2)})$, then π_1 and π_2 *pack* if there exist edge-disjoint graphs G_1 and G_2, both with vertex set $\{v_1, \ldots, v_n\}$, such that

$$d_{G_1}(v_i) = d_i^{(1)} \quad \text{and} \quad d_{G_2}(v_i) = d_i^{(2)}.$$

It is critical to note here that the order of the terms in π_1 and π_2 is fixed, so that the statement "π_1 and π_2 pack" is not equivalent to "π_1 and π_2 have realizations that pack". This framework allows for some interesting distinctions between packing graphs and packing graphic sequences. On the other hand, when a theorem certifies that a pair of graphs pack, it generally does not give insight into *how* they pack. Here, by fixing the ordering of π_1 and π_2, we provide insight into how a pair of graphs with these degree sequences might feasibly pack, if in fact they do.

Given a (not necessarily graphic) sequence π, let $\Delta(\pi)$ and $\delta(\pi)$ denote the maximum and minimum terms in π, respectively. Further, given two sequences π_1 and π_2 of the same length, let $\pi_1 + \pi_2$ denote the "vector sum" of π_1 and π_2. One of the main results from [11] is the following.

Theorem 4.3 *Let π_1 and π_2 be n-term graphic sequences with $\Delta = \Delta(\pi_1 + \pi_2)$ and $\delta = \delta(\pi_1 + \pi_2)$. If*

$$\Delta \leq \sqrt{2\delta n} - (\delta - 1),$$

then π_1 and π_2 pack, except that strict inequality is required when $\delta = 1$. This result is sharp for all n and δ.

As was noted in [11], this theorem can be viewed as an "additive" analogue to the Sauer-Spencer theorem, since $\Delta_1 + \Delta_2 < \sqrt{2n}$ implies that $\Delta_1\Delta_2 < \frac{n}{2}$. We modify and strengthen the techniques introduced in the proof of Theorem 4.3 to obtain our main results here.

4.1.3 Statement of Main Results

Throughout the statement and proof of the following results, given graphic sequences π_1 and π_2 we let $\Delta_i = \Delta(\pi_i)$ and $\delta_i = \delta(\pi_i)$ for $i \in \{1, 2\}$. The main result of this chapter is as follows.

Theorem 4.4 [19] *Let π_1 and π_2 be graphic sequences with $\Delta_2 \geq \Delta_1$ and $\delta_1 \geq 1$. If*

$$\begin{cases} (\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \delta_1 n + 1 & \textit{when } \Delta_2 + 2 \geq \Delta_1 + \delta_1 \\ \frac{(\Delta_2+1+\Delta_1+\delta_1)^2}{4} \leq \delta_1 n + 1 & \textit{when } \Delta_2 + 2 < \Delta_1 + \delta_1, \end{cases}$$

then π_1 and π_2 pack.

Theorem 4.4 holds regardless of the orderings of π_1 and π_2 (although these orderings are fixed). Given this, we cannot assume that $\delta(\pi_1) = \delta(\pi_2) = 0$, as it would be possible to order π_1 and π_2 so that the zero terms correspond, which would impact the relative strength of the hypothesis. It seems feasible that the conditions that $\Delta_1 \leq \Delta_2$ and $\delta(\pi_1) \geq 1$ could be replaced by the weaker hypothesis that $\delta(\pi_1 + \pi_2) \geq 1$, although we are unable to obtain such a result at this time.

A simple calculation, which we give below, demonstrates that Theorem 4.4 implies the following direct analogue to the Bollobás-Eldridge-Catlin conjecture.

Corollary 4.5 [19] *Let π_1 and π_2 be graphic sequences with $\Delta_2 \geq \Delta_1$ and $\delta_1 \geq 1$. If $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$, then π_1 and π_2 pack. This result is best possible.*

Much as the Bollobás-Eldridge-Catlin conjecture implies the Sauer-Spencer theorem, we also obtain the following.

Corollary 4.6 [19] *Let π_1 and π_2 be graphic sequences with $\Delta_2 \geq \Delta_1$ and $\delta_1 \geq 1$. If $\Delta_1\Delta_2 < \frac{n}{2}$, then π_1 and π_2 pack. This result is best possible.*

4.2 Sharpness

In [54], Kaul and Kostochka characterized the sharpness examples for Theorem 4.1. Specifically, graphs G_1 and G_2 satisfying $\Delta_1\Delta_2 = \frac{n}{2}$ pack, unless n is even, G_1 is a matching of size $\frac{n}{2}$, and G_2 is either $K_{\frac{n}{2},\frac{n}{2}}$ or any graph that contains $K_{\frac{n}{2}+1}$ as a component.

In a similar manner, to see that Corollaries 4.5 and 4.6 are sharp, let n be even and consider $\pi_1 = (1^n)$ and $\pi_2 = (\frac{n}{2}^{\frac{n+2}{2}}, 0^{\frac{n-2}{2}})$. These sequences are uniquely realized as a perfect matching and $K_{\frac{n}{2}+1} \cup \left(\frac{n}{2} - 1\right) K_1$, which do not pack, regardless of the orderings of π_1 and π_2. We show the following.

Theorem 4.7 [19] *Theorem 4.4 is strictly stronger than Corollary 4.5 unless $\delta_1 = 1$. Further, Corollary 4.5 is strictly stronger than Corollary 4.6 unless $\Delta_1 = \delta_1 = 1$.*

Proof: We first note that when $\delta_1 = 1$, the conditions for Theorem 4.4 and Corollary 4.5 are equivalent, so we suppose that $\delta_1 > 1$.

If $\Delta_2 + 2 \geq \Delta_1 + \delta_1$, then since $\delta_1 - 1 > 0$ and $\Delta_1 > 1$, we have

$$\delta_1 - 1 < (\delta_1 - 1)\Delta_1\Delta_2 + (\delta_1 - 1)\Delta_1.$$

This yields

$$\Delta_1\Delta_2 + \delta_1\Delta_2 + \Delta_1 + \delta_1 - 1 < \delta_1\Delta_1\Delta_2 + \delta_1\Delta_2 + \delta_1\Delta_1,$$

or

$$\frac{(\Delta_2 + 1)(\Delta_1 + \delta_1) - 1}{\delta_1} < (\Delta_2 + 1)(\Delta_1 + 1) - 1 \leq n.$$

If $\Delta_2 + 2 < \Delta_1 + \delta_1$ then we have that $\delta_1 \geq \Delta_2 + 3 - \Delta_1$. Also, $\delta_1 \leq \Delta_1$ implies

$\delta_1 + \Delta_1 \leq 2\Delta_1$. Using these two inequalities, we have the following

$$\frac{(\Delta_2 + 1 + \Delta_1 + \delta_1)^2}{4\delta_1} - \frac{1}{\delta_1} \leq \frac{(\Delta_2 + 1 + 2\Delta_1)^2}{4(\Delta_2 + 3 - \Delta_1)} - \frac{1}{\Delta_2 + 3 - \Delta_1}.$$

It suffices to show that

$$\frac{(\Delta_2 + 1 + 2\Delta_1)^2}{4(\Delta_2 + 3 - \Delta_1)} - \frac{1}{\Delta_2 + 3 - \Delta_1} < \Delta_1\Delta_2 + \Delta_1 + \Delta_2 + 1. \tag{4.1}$$

Inequality (4.1) simplifies to

$$4\Delta_1^2\Delta_2 + 8\Delta_1^2 < 4\Delta_1\Delta_2^2 + 3\Delta_2^2 + 14\Delta_2 + 8\Delta_1\Delta_2 + 4\Delta_1 + 15,$$

which is always true.

Note that if $\Delta_1 = 1$, then the conditions for Corollary 4.5 and Corollary 4.6 are the same. We now show that if $\Delta_1 > 1$, then Corollary 4.5 is strictly stronger than Corollary 4.6. The condition in Corollary 4.5 is $\Delta_1\Delta_2 + \Delta_1 + \Delta_2 \leq n$ and the condition in Corollary 4.6 is $2\Delta_1\Delta_2 + 1 \leq n$. Since $\Delta_1\Delta_2 + \Delta_1 + \Delta_2 < 2\Delta_1\Delta_2 + 1$ when $\Delta_1 > 1$, the assertion holds. ■

Kundu's k-factor Theorem [60], proved independently by Lovász for $k = 1$ [66], states that a graphic sequence $\pi = (d_1, \ldots, d_n)$ has a realization containing a k-factor if and only if $\pi' = (d_1 - k, \ldots, d_n - k)$ is also graphic. This together with Theorem 4.7 allows us to partially characterize the sharpness of Corollary 4.5 and completely characterize the sharpness of Corollary 4.6. The latter characterization is analogous to the characterization for graph packing from [54].

Theorem 4.8 [19] *Let π_1 and π_2 be graphic sequences with $\Delta_2 \geq \Delta_1$ and $\delta_1 \geq 1$.*

(a) If

$$(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 2 \quad \text{and} \quad \delta_1 \neq 1,$$

then π_1 and π_2 pack.

(b) If

$$\Delta_1\Delta_2 = \frac{n}{2},$$

then π_1 and π_2 pack unless $\Delta_1 = 1$ and $\pi_1 + \pi_2$ is not graphic.

4.3 Proofs of Theorem 4.4, Corollary 4.5, and Corollary 4.6

Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be graphs. We say a vertex pair (x, y) is a *bad pair for* (G_1, G_2) or a (G_1, G_2)*-bad pair* if $xy \in E_1 \cap E_2$. Let $b(G_1, G_2)$ denote the number of (G_1, G_2)-bad pairs. We begin by proving Theorem 4.4.

Proof: (of Theorem 4.4) Let π_1 and π_2 be graphic sequences that do not pack. Choose $G_1 = G(\pi_1)$ and $G_2 = G(\pi_2)$ to have the fewest bad pairs among all realizations of π_1 and π_2 and let $G = G_1 \cup G_2$. For a given (G_1, G_2)-bad pair (x, y) we define $I(x, y) = V - (N_G(x) \cup N_G(y))$. Among all choices of G_1 and G_2 that minimize $b(G_1, G_2)$, choose G_1, G_2 and a bad pair (x, y) such that the size of $I = I(x, y)$ is maximum. For $i \in \{1, 2\}$, let $Q_i(y)$ be $N_{G_i}(y) - N_G[x]$ and define $Q_i(x)$ similarly. If either $Q_1(x)$ or $Q_1(y)$ is nonempty, assume without loss of generality that $|Q_1(x)| \leq |Q_1(y)|$. Otherwise, if both $Q_1(x)$ and $Q_1(y)$ are empty, then assume without loss of generality that $|Q_2(x)| \leq |Q_2(y)|$.

Throughout the proof we will make use of the following sets. First, let $\overline{Y} = V(G) - N_G[y]$. Define A to be a subset of $N_{G_1}(\overline{Y})$ such that every vertex of A has at least two neighbors in G_1 in $\overline{Y}$. Finally, let $B = N_{G_1}(\overline{Y}) - A$ and $R = A \cup \{v \in N_G[y] : A \subseteq N_G(v)\}$.

We prove Theorem 4.4 by counting the number of edges in G_1 between R and $V(G) - R$ to reach a contradiction. In order to gain the desired count, we first show particular edge structures in I, $\overline{Y}$, and $N_{G_1}(\overline{Y})$. We then show that A is not empty and further that R is a vertex cover of G_1.

As it will be useful in various instances, we next show that if the condition $(\Delta_2+1)(\Delta_1+\delta_1) \leq \delta_1 n+1$ is contradicted, then the condition $\frac{1}{4}(\Delta_2+1+\Delta_1+\delta_1)^2 \leq \delta_1 n + 1$ is also contradicted.

Claim 4.9 $(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \frac{1}{4}(\Delta_2 + 1 + \Delta_1 + \delta_1)^2$.

Proof: Let $\Delta_1 + \delta_1 = \Delta_2 + 1 + t$, where t is an integer. We have that

$$(\Delta_2 + 1)^2 + t(\Delta_2 + 1) \leq (\Delta_2 + 1)^2 + t(\Delta_2 + 1) + \frac{t^2}{4}.$$

Factoring each side yields

$$(\Delta_2 + 1)(\Delta_2 + 1 + t) \leq \frac{1}{4}\left((2\Delta_2 + 2)^2 + 2t(2\Delta_2 + 2) + t^2\right),$$

which is the same as

$$(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \frac{1}{4}(2\Delta_2 + 2 + t)^2 = \frac{1}{4}(\Delta_2 + 1 + \Delta_1 + \delta_1)^2.$$

■

Claim 4.10 *If u and v are vertices in G such that xu and yv are not in $E(G)$, then uv is not in $E(G)$.*

Proof: Assume otherwise, and without loss of generality let uv be an edge of G_1. We may then exchange the edges xy and uv with the non-edges xu and yv in G_1 to create another realization of π_1. Since xu and yv are not in G, this reduces the number of bad pairs, a contradiction. ■

Claim 4.10 immediately implies that I is an independent set in G.

Several times throughout the proof we use the observation that $|N_G[y]| \leq \Delta_1+\Delta_2$. This bound follows from our assumption that (x, y) is a bad pair and thus x is a neighbor of y in both G_1 and G_2. Next we show that $N_G[y] \neq V(G)$.

Claim 4.11 $\overline{Y} \neq \emptyset$.

Proof: Toward contradiction, suppose that $N_G[y] = V(G)$. Thus $|N_G[y]| = n$, and therefore $\Delta_1 + \Delta_2 \geq n$. By assumption, $(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \delta_1 n + 1$, which implies that

$$(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \delta_1(\Delta_1 + \Delta_2) + 1.$$

Expanding and rearranging, this yields

$$0 \leq \Delta_1(\delta_1 - 1 - \Delta_2) - \delta_1 + 1.$$

However, $\delta_1 - 1 - \Delta_2 < 0$ and $-\delta_1 + 1 \leq 0$, a contradiction.

Consequently, $N_G[y] \neq V(G)$. ■

Claim 4.12 *$\overline{Y}$ is independent in G_1.*

Proof: Otherwise, suppose there are vertices u and v in $\overline{Y}$ that form an edge in G_1. By Claim 4.10, both u and v must be adjacent to x. If there is some vertex $z \in Q_1(y)$, then removing the edges uv, xy, and yz from G_1 and adding the non-edges yu, yv, and xz to G_1 would create another realization of G_1' of π_1 such that $b(G_1', G_2) < b(G_1, G_2)$. If $Q_1(y)$ is empty, then since $|Q_1(x)| \leq |Q_1(y)|$, we have that $Q_1(x)$ is also empty, and therefore since we have assumed $|Q_2(x)| \leq |Q_2(y)|$ and $u \in Q_2(x)$, there is some z in $Q_2(y)$. We then exchange the edges yz and xu in G_2 and the edges uv and xy in G_1, for the non-edges yu and xz in G_2 and the non-edges xu and yv in G_1 to again create realizations of π_1 and π_2 with fewer than $b(G_1, G_2)$ bad pairs. Thus, $\overline{Y}$ is independent in G_1. ■

Claim 4.13 *$N_{G_1}(\overline{Y}) \cup \{x, y\}$ is a clique in G.*

Proof: Let $u \in \overline{Y}$ and $w \in N_{G_1}(u)$. By Claim 4.12, $w \notin \overline{Y}$ and therefore $w \in N_G[y]$. If $w \neq x$, then since $uy \notin E(G)$, by Claim 4.10, $wx \in E(G)$. Thus,

$N_{G_1}(u) \subseteq N_G[x] \cap N_G(y)$.

Consequently, suppose $w, w' \in N_{G_1}(\overline{Y})$ are such that $ww' \notin E(G)$. Let $u \in N_{G_1}(w) \cap \overline{Y}$ and $u' \in N_{G_1}(w') \cap \overline{Y}$ (u and u' need not be distinct). Note that without loss of generality $x \neq w$ since $xw' \in E(G)$. If $u \in I$, then replacing the edges uw, $u'w'$, and xy in G_1 with the non-edges xu, yu', and ww' contradicts the minimality of $b(G_1, G_2)$. Thus $u \notin I$, and likewise $u' \notin I$.

Next, assume there is some $z \in Q_1(y)$. By Claim 4.10, $uz \notin E(G)$. Remove the edges wu, $w'u'$, and yz from G_1 and add the edges ww', yu', and zu to create a realization G_1' of π_1 with $b(G_1', G_2) = b(G_1, G_2)$. However, neither x nor y are adjacent to vertices in $\{z\} \cup I(x, y)$, which contradicts the maximality of I.

It remains to consider the case where $Q_1(y) = \emptyset$. Similar to the proof of Claim 4.12, since $Q_1(y)$ is empty, $Q_1(x)$ is empty, therefore $u, u' \in Q_2(x)$ and there must be a vertex z in $Q_2(y)$. Also note that since $u, u' \in Q_2(x)$ the edges xu and xu' are in G_2. Exchanging the edges wu, $w'u'$, and xy in G_1 with ux and the non-edges $u'y$ and ww' creates another realization G_1' of π_1 such that (u, x) is a (G_1', G_2)-bad pair and $b(G_1', G_2) = b(G_1, G_2)$. However, by Claim 4.10 u is not adjacent to vertices in $\{z\} \cup I(x, y)$, and x is not adjacent to vertices in $\{z\} \cup I(x, y)$. Therefore $I(u, x) > I(x, y)$. Hence, $N_{G_1}(\overline{Y}) \cup \{x, y\}$ is a clique in G. ■

Claim 4.14 $A \neq \emptyset$.

Proof: For sake of contradiction, suppose A is empty, and therefore $N_{G_1}(\overline{Y}) = B$. Since $\overline{Y}$ is independent in G_1 we have that $\delta_1|\overline{Y}| \leq |B|$. Thus,

$$n = |\overline{Y}| + |N_G[y]| \leq \frac{|B|}{\delta_1} + \Delta_1 + \Delta_2.$$

We proceed by showing that $|B| \leq \Delta_1 + \Delta_2 - 2$, which establishes the desired contradiction. By the definition of $\overline{Y}$, y is not adjacent to vertices in $\overline{Y}$, and therefore $y \notin B$. If $x \notin B$, then $|B| \leq |N_G(y)| - |\{x\}| \leq \Delta_1 + \Delta_2 - 2$. If $x \in B$, then since x has a neighbor in $\overline{Y}$, $Q_1(x) \neq \emptyset$. By assumption $|Q_1(x)| \leq |Q_1(y)|$, thus there is some vertex z in $N_G[y]$ not adjacent to x. Now we have that $|B| \leq |N_G[y] - \{y, z\}| \leq \Delta_1 + \Delta_2 - 2$. Inserting this upper bound of $|B|$ into the above inequality we have that

$$\delta_1 n + 1 \leq (\delta_1 + 1)(\Delta_1 + \Delta_2) - 1.$$

By Claim 4.9,

$$(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \frac{(\Delta_2 + 1 + \Delta_1 + \delta_1)^2}{4},$$

so that the hypothesis of the theorem yields

$$(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq (\delta_1 + 1)(\Delta_1 + \Delta_2) - 1,$$

which simplifies to

$$\Delta_2 \Delta_1 + \delta_1 < \delta_1 \Delta_1 + \Delta_2.$$

Combining the Δ_2 terms and the δ_1 terms, we have

$$\Delta_2(\Delta_1 - 1) < \delta_1(\Delta_1 - 1),$$

which is a contradiction when $\Delta_1 = 1$. Otherwise, it contradicts the assumption that $\delta_1 \leq \Delta_2$. ■

By Claim 4.14, $A \neq \emptyset$ and by Claim 4.13, $N_{G_1}(\overline{Y}) \cup \{x, y\}$ is a clique in G and therefore $N_{G_1}(\overline{Y}) \cup \{x, y\} \subseteq R$.

Claim 4.15 *Every edge of G_1 is incident with R.*

Proof: Towards contradiction let zz' be an edge of G_1 not incident with R. By Claim 4.12 we know that z and z' must be in $N_G[y] - R$, so there exist vertices w and w' (not necessarily distinct) in A which are not adjacent to z and z' (respectively). Also, we have distinct vertices u and u' in $\overline{Y}$ such that wu and $w'u'$ are edges in G_1.

We can remove the edges zz', uw, and $u'w'$ from G_1 and add the non-edges wz, $w'z'$, and uu' to form a realization G_1' of π_1. It is possible that, via this edge-exchange, (u, u') is a bad pair of (G_1', G_2), implying that $b(G_1', G_2) = b(G_1, G_2)+1$. However, the sets $Q_1(x)$, $Q_2(x)$, $Q_1(y)$, and $Q_2(y)$ are not affected by these exchanges. Now, $\overline{Y}$ is no longer independent in G_1 and (x, y) is still a bad pair. As in the proof of Claim 4.12 we now exchange edges to obtain a realization G_1'' of π_1 such that (x, y) and (u, u') are no longer bad pairs and no other bad pairs are created. Thus, $b(G_1'', G_2) < b(G_1, G_2)$, a contradiction.

Therefore R is a vertex cover of G_1, as desired. ∎

We conclude the proof by finding lower and upper bounds on the number of edges in G_1 between R and $V - R$, which we denote by $e_1(R, V - R)$. The necessary lower bound follows easily from the assertion that $V - R$ is independent in G_1

$$\delta_1(n - |R|) \leq e_1(R, V - R).$$

While $\Delta_1|R|$ is a straightforward upper bound for $e_1(R, V - R)$, we require a stronger bound to obtain the desired result.

Suppose $|R| \leq \Delta_2+1$. Since $\{x, y\} \subseteq R$, in G_1 both x and y have at most $\Delta_1 - 1$ neighbors in $V - R$. The remaining vertices of R each have at most Δ_1 neighbors in $V - R$. Thus, $e_1(R, V - R)$ is bounded above by

$$(|R| - 2)\Delta_1 + 2(\Delta_1 - 1) = |R|\Delta_1 - 2.$$

Combining the upper and lower bounds on $e_1(R, V - R)$ yields

$$\delta_1(n - |R|) \leq |R|\Delta_1 - 2,$$

which is the same as

$$\delta_1 n + 1 < |R|(\Delta_1 + \delta_1).$$

By our assumption on $|R|$ we have the following contradiction,

$$\delta_1 n + 1 < (\Delta_2 + 1)(\Delta_1 + \delta_1).$$

Now assume that $|R| = \Delta_2 + 1 + t$, where t is a positive integer. Notice that $|N_G[y]| \leq \Delta_1 + \Delta_2$ implies that

$$|N_G[y] - R| \leq \Delta_1 + \Delta_2 - (\Delta_2 + 1 + t) = \Delta_1 - t - 1.$$

As y has no neighbors in $\overline{Y}$, y has at most $\Delta_1 - t - 1$ neighbors of G_1 in $V - R$. If there is another vertex $w \in R - N_{G_1}(\overline{Y})$, then w also has no neighbors in $\overline{Y}$ and thus has at most $\Delta_1 - t - 1$ neighbors of G_1 in $V - R$. If $R - N_{G_1}(\overline{Y}) = \{y\}$, then R is a clique. In this case, x has at most $\Delta_1 + \Delta_2 - |R|$ neighbors of G_1 in $V - R$. As $\Delta_1 + \Delta_2 - |R| = \Delta_1 - t - 1$, we have that there are at least two vertices in R with at most $\Delta_1 - t - 1$ neighbors of G_1 in $V - R$.

Each of the remaining vertices of R have at most $\Delta_1 - t$ neighbors of G_1 in $V - R$. In particular, if $v \in B$, then v has one neighbor of G_1 to $\overline{Y}$ and at most $\Delta_1 - t - 1$ neighbors of G_1 to $N_G[y] - R$. If $v \in A$, then v is adjacent to every vertex of A, and therefore has at most $\Delta_1 + \Delta_2 - |R| + 1$ neighbors of G_1 to $V - R$, which is $\Delta_1 - t$.

Therefore, we have that

$$e_1(R, V - R) \leq 2(\Delta_1 - t - 1) + (|R| - 2)(\Delta_1 - t) = |R|(\Delta_1 - t) - 2.$$

Combining this with the lower bound of $e_1(R, V - R)$, we have

$$\delta_1 n + 1 < |R|(\Delta_1 + \delta_1 - t).$$

Since $\Delta_2 + 1 + t = |R|$, we expand the right side to obtain

$$\delta_1 n + 1 < (\Delta_2 + 1)(\Delta_1 + \delta_1) - t(\Delta_2 + 1 - (\Delta_1 + \delta_1)) - t^2.$$

If $\Delta_2 + 2 \geq \Delta_1 + \delta_1$, then we contradict our claim that $(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \delta_1 n + 1$. Otherwise, $\Delta_2 + 2 < \Delta_1 + \delta_1$. In this case, the right side is maximized when $t = \frac{1}{2}(-\Delta_2 - 1 + \Delta_1 + \delta_1)$, which yields

$$\delta_1 n + 1 < \frac{(\Delta_2 + 1 + \Delta_1 + \delta_1)^2}{4}.$$

This contradiction completes the proof. ■

We next prove Corollary 4.5.

Proof: By assumption $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$. If $\delta_1 = 1$, then neccessarily $\Delta_2 + 2 \geq \Delta_1 + \delta_1$. Also, we have that $(\Delta_2 + 1)(\Delta_1 + \delta_1) = (\Delta_2 + 1)(\Delta_1 + 1)$ and $\delta_1 n + 1 = n + 1$. Therefore $(\Delta_2 + 1)(\Delta_1 + \delta_1) \leq \delta_1 n + 1$.

If $\delta_1 > 1$, we consider two cases. First suppose $\Delta_2 + 2 \leq \Delta_1 + \delta_1$. Since $\delta_1 - 1 > 0$ and $\Delta_1 > 1$, we have

$$\delta_1 - 1 < (\delta_1 - 1)\Delta_1\Delta_2 + (\delta_1 - 1)\Delta_1.$$

This yields

$$\Delta_1\Delta_2 + \delta_1\Delta_2 + \Delta_1 + \delta_1 - 1 < \delta_1\Delta_1\Delta_2 + \delta_1\Delta_2 + \delta_1\Delta_1,$$

which is equivalent to

$$\frac{(\Delta_1 + \delta_1)(\Delta_2 + 1) - 1}{\delta_1} < (\Delta_2 + 1)(\Delta_1 + 1) - 1.$$

By assumption this is at most n, completing this case, as now we have

$$(\Delta_1 + \delta_1)(\Delta_2 + 1) \leq n\delta_1 + 1.$$

Now, suppose $\delta_1 > 1$ and $\Delta_2 + 2 > \Delta_1 + \delta_1$. Since $\Delta_2 \geq \Delta_1$, we have

$$0 < 4\Delta_1\Delta_2(\Delta_2 - \Delta_1) + 8\Delta_1(\Delta_2 - \Delta_1) + 8\Delta_1 + 3\Delta_2^2 + 10\Delta_2 + 3.$$

With some manipulation, we have that

$$\frac{(\Delta_2 + 1 + 2\Delta_1)^2}{4(\Delta_2 + 3 - \Delta_1)} - \frac{1}{\Delta_2 + 3 - \Delta_1} < (\Delta_2 + 1)(\Delta_1 + 1) - 1.$$

By assumption, $(\Delta_2 + 1)(\Delta_1 + 1) \leq n + 1$. Noting that since $\Delta_2 + 2 < \Delta_1 + \delta_1$ we have that $\Delta_2 + 3 - \Delta_1 \leq \delta_1$, and that $\delta_1 + \Delta_1 \leq 2\Delta_1$, we have the following

$$\frac{(\Delta_2 + 1 + \Delta_1 + \delta_1)^2}{4} \leq n\delta_1 + 1,$$

as desired. ∎

Finally, we give the straightforward proof that Corollary 4.5 implies Corollary 4.6.

Proof: For $\Delta_1 > 1$, since $\Delta_1 \geq \Delta_2$, we have that $\Delta_1 + \Delta_2 \leq 2\Delta_2$ and $2\Delta_2 \leq \Delta_1\Delta_2$, thus

$$\Delta_1\Delta_2 + \Delta_1 + \Delta_2 \leq 2\Delta_1\Delta_2.$$

By assumption, $2\Delta_1\Delta_2 < n$, therefore $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$. If, instead, $\Delta_1 = 1$, then $2\Delta_2 < n$ or $2\Delta_2 + 1 \leq n$, and since $(\Delta_1 + 1)(\Delta_2 + 1) - 1 = 2\Delta_2 + 1$, we have that $(\Delta_1 + 1)(\Delta_2 + 1) \leq n + 1$ as desired. ∎

4.4 Discrete Tomography

Tomography is the process of imaging through sectioning, for example constructing a three dimensional image from a series of 2-dimensional cross-sections or projections. Of interest here is *discrete tomography*, which uses low-dimensional projections to reconstruct discrete objects, such as the atomic structure of crystalline lattices and other polyatomic structures.

4.4.1 The k-color Tomography Problem

Numerous papers (c.f. [24, 25, 37, 38]) study the *k-color Tomography Problem*, in which the goal is to color the entries of an $m \times n$ matrix using k colors so that each row and column receives a prescribed number of entries of each color. The colors represent different types of atoms appearing in a crystal, and the number of times an atom appears in a given row or column is generally obtained using high resolution transmission electron microscopes [57, 77]. This is precisely the problem of packing the degree sequences of k bipartite graphs with partite sets of size m and n.

4.4.2 A Sauer-Spencer-type Theorem for the Discrete Tomography Problem

A sequence $\pi = (a_1, \ldots, a_r; b_1, \ldots, b_s)$ is *bigraphic* if there is a bipartite graph G such that $\pi = \pi(G)$ with partite sets X and Y, and the degrees of the vertices in X and Y are $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$, respectively. Two bigraphic sequences, $\pi_1 = (a_1^{(1)}, \ldots, a_r^{(1)}; b_1^{(1)}, \ldots, b_s^{(1)})$ and $\pi_2 = (a_1^{(2)}, \ldots, a_r^{(2)}; b_1^{(2)}, \ldots, b_s^{(2)})$ pack if there exist edge-disjoint bipartite graphs G_1 and G_2, both with partite sets $X = \{x_1, \ldots, x_r\}$ and $Y = \{y_1, \ldots, y_s\}$, such that for $j \in \{1, 2\}$,

$$d_{G_j}(x_i) = a_i^{(j)}$$

for $1 \leq i \leq r$, and

$$d_{G_j}(y_i) = b_i^{(j)}$$

for $1 \leq i \leq s$.

The following is a tomographic analogue to Corollary 4.6.

Theorem 4.16 [19] *Let π_1 and π_2 be bigraphic sequences with parts of sizes r and s, and $\Delta_i = \Delta(\pi_i)$ and $\delta_i = \delta(\pi_i)$ for $i \in \{1, 2\}$, such that $\Delta_1 \leq \Delta_2$ and $\delta_1 \geq 1$. If*

$$\Delta_1\Delta_2 \leq \frac{(r+s)}{4}$$

then π_1 and π_2 pack.

The other main result of this section, which takes δ_1 into account, improves on Theorem 4.16 when $\delta_1 \geq 3$.

Theorem 4.17 [19] *Let π_1 and π_2 be bigraphic sequences with parts of sizes r and s, and $\Delta_i = \Delta(\pi_i)$ and $\delta_i = \delta(\pi_i)$ for $i \in \{1, 2\}$, such that $\Delta_1 \leq \Delta_2$ and $\delta_1 \geq 1$. If*

$$\Delta_1\Delta_2 \leq \delta_1\frac{(r+s)}{8}$$

then π_1 and π_2 pack.

As before, we say a vertex pair (x, y) is a *bad pair for* (G_1, G_2) or a (G_1, G_2)-*bad pair* if $xy \in E(G_1) \cap E(G_2)$.

Let π_1 and π_2 be bigraphic sequences that do not pack, choose $G_1 = G(\pi_1)$ and $G_2 = G(\pi_2)$ to have the fewest bad pairs among all realizations of π_1 and π_2 and let $G = G_1 \cup G_2$. Fix a (G_1, G_2)-bad pair (x, y) and let X and Y be the partite sets of G, where $x \in X$ and $y \in Y$. Let $I_X = X - N_G(y)$ and $I_Y = Y - N_G(x)$. We now have the following lemmas, the first of which is analogous to Claim 4.10.

Lemma 4.18 *The set $I_X \cup I_Y$ is independent.*

Proof: Suppose otherwise, so in particular let $z \in I_X$ and $z' \in I_Y$ such that $zz' \in E(G)$. Exchanging the edges xy and zz' with the non-edges zy and $z'x$ decreases the number of (G_1, G_2)-bad pairs, contradicting the choice of G_1 and G_2. ■

Lemma 4.19 *The subgraph of G induced by $N_{G_1}(I_Y) \cup N_{G_1}(I_X) \cup \{x, y\}$ is a complete bipartite graph.*

Proof: First, note that by Lemma 4.18 and the definition of I_Y, x is adjacent to every vertex in $N_{G_1}(I_X)$ and likewise, y is adjacent to every vertex in $N_{G_1}(I_Y)$. Suppose then that there is some $w \in N_{G_1}(I_Y)$ and $w' \in N_{G_1}(I_X)$ such that ww' is not an edge in G. Now we have that there is some $z' \in I_Y$ and $z \in I_X$ such that wz' and $w'z$ are edges in G_1. Exchanging the edges $w'z$, wz', and xy (all in G_1) with the non-edges ww', xz', and yz decreases the number of bad pairs in G, a contradiction. ■

We are now ready to prove Theorems 4.16 and 4.17.

Proof: (of Theorem 4.16) Observe first that each vertex in $N_{G_1}(I_X)$ (respectively $N_{G_1}(I_Y)$) can have at most Δ_1 neighbors in I_X (resp. I_Y) so that

$$|I_X| + |I_Y| \leq \Delta_1(|N_{G_1}(I_X)| + |N_{G_1}(I_Y)|).$$

We further have that

$$n - (|N_G(x)| + |N_G(y)|) \leq |I_X| + |I_Y|,$$

and

$$|N_{G_1}(I_Y)| + |N_{G_1}(I_X)| \leq |N_G(x)| + |N_G(y)| - 2 \leq 2(\Delta_1 + \Delta_2) - 4.$$

Taken together, these yield that

$$n - (2(\Delta_1 + \Delta_2) - 2) \leq \Delta_1(2(\Delta_1 + \Delta_2) - 4),$$

so

$$\frac{n}{2} \leq (\Delta_1 + 1)(\Delta_1 + \Delta_2) - 2\Delta_1 - 1.$$

As $\Delta_1\Delta_2 \leq \frac{n}{4}$, it follows that

$$2\Delta_1\Delta_2 \leq \Delta_1^2 + \Delta_1\Delta_2 + \Delta_1 + \Delta_2 - 2\Delta_1 - 1,$$

so that

$$\Delta_1\Delta_2 - \Delta_2 \leq \Delta_1^2 - \Delta_1 - 1.$$

However, then

$$\Delta_2 \leq \Delta_1 - \frac{1}{\Delta_1 - 1},$$

a contradiction, since $\Delta_1 \leq \Delta_2$. ■

Proof: (of Theorem 4.17) By Lemma 4.18, I_X and I_Y are independent, so every vertex in $I_X \cup I_Y$ must have at least δ_1 neighbors in $N_{G_1}(I_Y) \cup N_{G_1}(I_X)$. Also, as in Theorem 4.16, each vertex in $N_{G_1}(I_Y) \cup N_{G_1}(I_X)$ has at most Δ_1 neighbors in $I_X \cup I_Y$. Therefore,

$$\delta_1(|I_X| + |I_Y|) \leq \Delta_1(|N_{G_1}(I_Y)| + |N_{G_1}(I_X)|)$$

so that

$$|I_X| + |I_Y| \leq \frac{\Delta_1}{\delta_1}\left(|N_{G_1}(I_Y)| + |N_{G_1}(I_X)|\right).$$

Again, we have that

$$|N_{G_1}(I_Y)| + |N_{G_1}(I_X)| \leq |N_G(x)| + |N_G(y)| - 2 \leq 2(\Delta_1 + \Delta_2 - 2).$$

Let $r + s = n$, so that

$$|I_X| + |I_Y| = n - (|N_G(x)| + |N_G(y)|).$$

Combining the above equations yields

$$n - 2(\Delta_1 + \Delta_2) + 2 \leq 2\frac{\Delta_1}{\delta_1}(\Delta_1 + \Delta_2 - 2).$$

By isolating Δ_2,

$$\frac{\delta_1 n}{2(\Delta_1 + \delta_1)} - \Delta_1 + \frac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1} \leq \Delta_2.$$

Notice that $\Delta_1 + \delta_1 \leq 2\Delta_1$, so we have

$$\frac{\delta_1 n}{4\Delta_1} \leq \Delta_2 + \Delta_1 - \frac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1}.$$

By assumption, $\Delta_2 \leq \dfrac{\delta_1 n}{8\Delta_1}$, so

$$2\Delta_2 \leq 2\left(\frac{\delta_1 n}{8\Delta_1}\right) \leq \Delta_2 + \Delta_1 - \frac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1},$$

which implies

$$\Delta_2 \leq \Delta_1 - \frac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1}.$$

Since $\dfrac{2\Delta_1 + \delta_1}{\Delta_1 + \delta_1} > 0$, and $\Delta_2 \geq \Delta_1$, we arrive at a contradiction, completing the proof. ■

5. Future Work and Open Problems

5.1 Rainbow Matchings in Edge-Colored Graphs

Recently, Theorems 2.1 and 2.2 have been improved by Lo and Tan [64] and by Gyárfás and Sárközy [39]. In fact, the extension to edge-colored graphs, but not necessarily properly edge-colored graphs, has been considered. In this case, we consider the *minimum color degree*, $\hat{\delta}(G)$, which is the minimum number of distinct colors at any vertex in a graph G. The bounds in this realm had been on the order of $(\hat{\delta})^2$ (see [58]) until the following result.

Theorem 5.1 [64] *Let G be an edge-colored graph with minimum color degree $\hat{\delta}$. If $n \geq 4\hat{\delta} - 4$ for $\hat{\delta} \geq 4$ or $n \geq 4\hat{\delta} - 3$ for $\hat{\delta} \geq 3$, then G contains a rainbow matchings of size $\hat{\delta}$.*

It is worth noting however, that the proof for Theorem 5.1 uses extremal techniques to show existence, and the relevance of an algorithm remains, especially in light of the following complexity results for rainbow matchings.

Theorem 5.2 [52] *Given an edge-colored graph G and integer k, it is NP-complete to determine if G contains a rainbow matching with at least k edges, even when G is an edge-colored bipartite graph.*

Theorem 5.3 [61] *Given an edge-colored graph G, determining the size of the largest rainbow matching is APX-complete, even when G is*

1. *an edge-colored complete graph,*
2. *a properly edge-colored path,*
3. *a properly edge-colored P_8-free tree in which every color is used at most twice,*
4. *a properly edge-colored P_5-free linear forest in which every color is used at most twice,*

5. *a properly edge-colored P_4-free bipartite graph in which every color is used at most twice.*

Given these results, our algorithm, which runs in $O(\delta|V(G)|^2)$-time, is quite powerful.

5.2 Open Problems and Conjectures Regarding 2-factors with a Bounded Number of Components

Theorem 3.16 implies that no minimum degree condition on 1-, 2-, and 3-connected claw-free graphs will ensure the existence of an even 2-factor. For 4-connected claw-free graphs, we look to the well-known conjecture of Matthews and Sumner [67].

Conjecture 5.4 (The Matthews Sumner Conjecture) *If G is a 4-connected claw-free graph, then G is hamiltonian.*

This conjecture has several equivalences, all of which remain open. Every line graph is claw-free, and thus Thomassen conjectured the following.

Conjecture 5.5 [81] *Every 4-connected line graph is hamiltonian.*

A closed trail is *dominating* provided every edge has an endpoint on the trail. A *snark* is a 2-edge-connected, 3-regular graph that is not 3-edge-colorable with girth at least 5 and no non-trivial 3-edge cut. The following conjecture, seemingly unrelated to Conjectures 5.4 and 5.5, is also equivalent to these conjectures, as shown by Broersma et al. [8].

Conjecture 5.6 [8] *Every snark contains a dominating cycle.*

A hamiltonian cycle in a graph of even order is an even 2-factor. Thus in a similar vein as the above, we make the following conjecture.

Conjecture 5.7 [18] *If G is a 4-connected, claw-free graph of even order, then G has an even 2-factor if and only if G is hamiltonian.*

We also conjecture the following extension of Theorem 3.4, and in particular Corollary 3.10.

Conjecture 5.8 [18] *If G is a k-connected, claw-free graph of even order at least $2k$, then G has k disjoint 1-factors if and only if $cl(G)$ has k disjoint 1-factors.*

For $k \geq 3$, the condition that G is k-connected in Conjecture 5.8 is necessary. To see this, consider the graph G obtained by connecting a vertex v to $k-1$ vertices in an odd clique of order at least $2k-1$. The closure of G is complete, but v does not have sufficient degree in G to be in k disjoint 1-factors.

5.3 Degree Sequence and Mixed Packing Problems

Mixed packing refers to a mixture of graph packing and sequence packing where we consider a fixed graph G and a graphic sequence π. In general, G *packs* with π if there is some realization G' of π such that G and G' pack. In this case, vertex permutations of G are allowed in order to pack G and G'. We have the following conjecture.

Conjecture 5.9 [20] *Let G be an n-vertex graph and $\pi = (d_1, d_2, \ldots, d_n)$ be an n-term graphic sequence with Δ_π the maximum term of π and Δ_G the maximum degree in G. If*

$$(\Delta_G + 1)(\Delta_\pi + 1) \leq n + 1,$$

then G and π pack.

Conjecture 5.9 is a mixed packing analogue to the Bollobás-Eldridge-Catlin graph packing conjecture, and is stronger than Corollary 4.5 since we can use 2-switches in both sequences when packing two sequences, but no longer have this flexibility with mixed packing.

In the event that we have a fixed graph G and graphic sequence π such that the vertices of G are fixed with respect to π, meaning that $V(G) = \{v_1, \ldots, v_n\}$ and

$d_\pi(v_i) = d_i$, then G *φ-packs* with π. This is a more restrictive version of mixed packing, and so determining conditions such that G and π φ-pack would also be interesting.

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