INSTABILITY IN MULTICOLORING AND MULTICLIQUE SEQUENCES

by

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ABSTRACT

The m-chromatic number $\chi_m(G)$ of a graph G is the fewest colors needed so each node has m colors and no color appears on adjacent nodes. Stahl has shown that for any graph G, there is a p and k_0 such that for the multicoloring sequence $\chi_1(G), \chi_2(G), \chi_3(G), \ldots$, we have $\chi_{k+p}(G) = \chi_k(G) + \chi_p(G)$ for all $k > k_0$. Stahl defined as "chromatically unstable" all graphs for which $k_0 > 0$, and found the first known such graph, Grötzch's graph, for which $k_0 = 1$. Here we give several other graphs which are chromatically unstable, including two for which $k_0 = 3$. Additionally, we examine the multiclique sequences of these graphs, finding that most are "cliquishly unstable" as well.

This abstract accurately represents the content of the candidate's thesis. I recommend its publication.

Signed ____________________

David C. Fisher

DEDICATION

To John Marriott

and

To Joseph and Jeanne Sutorik

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1 Introduction

A *coloring* of a graph G is a labeling of the nodes with $1, 2, \ldots, m$, so that adjacent nodes have different labels. Alternatively, it is easy to see but tedious to explain that a coloring can be thought of as a collection of maximal independent sets such that each node is in at least one set. The *coloring number* $\chi(G)$ is the fewest number of maximal independent sets in a coloring of G. Similarly, the *k-coloring number* $\chi_k(G)$ is the fewest maximal independent sets so that each node is in at least k of the sets. The k-coloring number may be found through the solution of an integer program. Let K be the *node maximal independence matrix* of G, where the rows of K represent the nodes of G and the columns represent all maximal independent sets of G. If node i is in independent set j, then $k_{ij} = 1$; otherwise $k_{ij} = 0$. Then $\chi_k(G)$ is the value of the integer program:

$$\text{Minimize } \mathbf{1}^T\mathbf{x} \text{ subject to } K\mathbf{x} \geq k\mathbf{1}, \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{x} \text{ integer,} \tag{1}$$

where $\mathbf{1}$ and $\mathbf{0}$ are vectors of all ones and all zeros, respectively, and $\mathbf{x}$ is a vector such that x_j is the integer weight on independent set j. Since $\chi(G) = \chi_1(G)$, this method also finds the value of $\chi(G)$ by setting $k = 1$.

Fractional coloring is a generalization of coloring in that the weights on the maximal independent sets need not be integer. The *fractional coloring number*

$\chi_f(G)$ is the value of the linear program:

$$\text{Minimize } \mathbf{1}^T\mathbf{x} \text{ subject to } K\mathbf{x} \geq \mathbf{1}, \text{ and } \mathbf{x} \geq \mathbf{0}. \tag{2}$$

The problem of finding the chromatic number is dual to that of finding a maximum clique. A *clique* of G is a complete subgraph of G. An alternative definition is that a clique is a set of nodes such that each maximal independent set of G contains at most one node of the set. The *clique number* $\omega(G)$ is the maximum number of nodes in any clique of G. Expanding the idea, the *k-clique number* $\omega_k(G)$ is the largest sum of integer weights on the nodes so that the sum of weights in any maximal independent set is at most k. The value of the following integer program, where $\mathbf{y}$ is a vector such that y_i is a weight on node i, is $\omega_k(G)$:

$$\text{Maximize } \mathbf{1}^T\mathbf{y} \text{ subject to } K^T\mathbf{y} \leq k\mathbf{1}, \mathbf{y} \geq \mathbf{0} \text{ and } \mathbf{y} \text{ integer.} \tag{3}$$

The dual to the fractional coloring linear program (2) is the following:

$$\text{Maximize } \mathbf{1}^T\mathbf{y} \text{ subject to } K^T\mathbf{y} \leq \mathbf{1}, \text{ and } \mathbf{y} \geq \mathbf{0}. \tag{4}$$

The value of (4) is the *fractional clique number*, $\omega_f(G)$, that is the largest sum of (not necessarily integer) weights which may be placed on the nodes of G so that for each maximal independent set, the sum of weights on that set is at most one. Since (4) is the linear programming dual of (2), we have:

$$\omega(G) \leq \omega_f(G) = \chi_f(G) \leq \chi(G). \tag{5}$$

The fractional clique number and the k-clique number are related in that a maximal k-clique may be transformed into a fractional clique by dividing the weight on each node by k. Since this fractional clique may not be maximal,

$$\omega_f(G) \geq \frac{\omega_k(G)}{k},$$

which implies

$$\omega_k(G) \leq \lfloor k\omega_f(G) \rfloor \tag{6}$$

By similar reasoning,

$$\chi_f(G) \leq \frac{\chi_k(G)}{k},$$

implying

$$\chi_k(G) \geq \lceil k\chi_f(G) \rceil \tag{7}$$

A graph construction which will be utilized heavily in this report is that of Mycielski. Let G be a graph with nodes $v_1, v_2, \ldots, v_m$. Then the *Mycielskian* $\mu(G)$ of G is the graph with nodes $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, z$ with x_i adjacent to x_j and y_j if and only if v_i is adjacent to v_j in G; with y_i adjacent to z for all i; and no other edges. For example, the Mycieskian of C_5 is illustrated in Figure 1.

Larsen, Propp, and Ullman [**4**] proved the following useful results for the Mycielskian of any graph G:

$$\omega(\mu(G)) = \omega(G) \tag{8}$$

$$\chi(\mu(G)) = \chi(G) + 1 \tag{9}$$

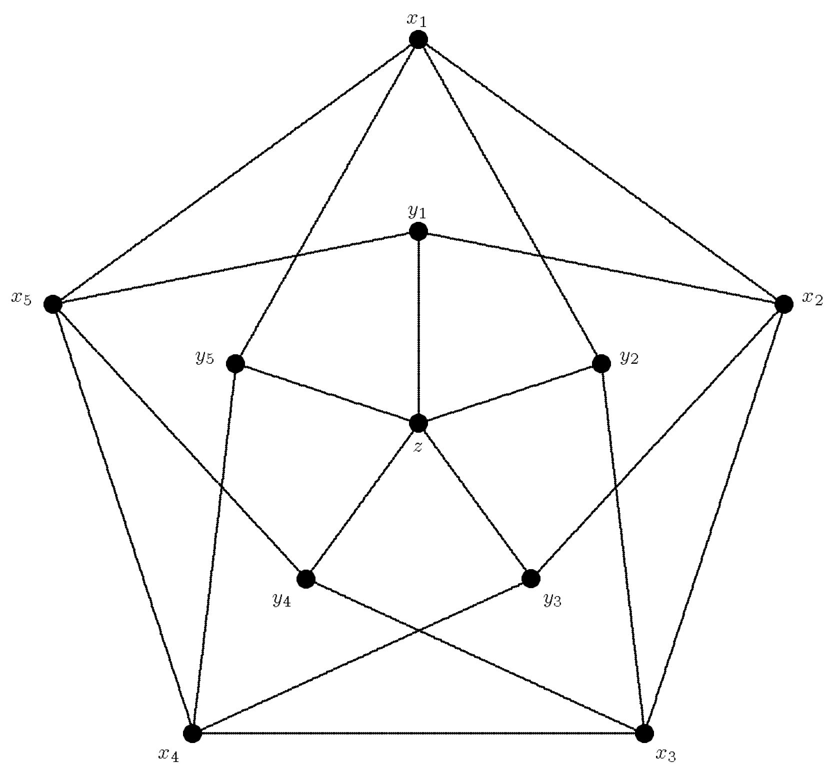

Figure 1: The Mycielskian of C_5.

$$\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)} \tag{10}$$

These results will be used extensively in later proofs, where we expand upon several results by Stahl.

Stahl [**5**] showed that if a subadditive integer sequence (subadditive meaning $f_{l+m} \leq f_l + f_m$) has the property that $\frac{f_l}{l} \geq \frac{f_p}{p}$ for some p and all l, then there exists a k_0 such that $f_{k+p} = f_k + f_p$ for all $k > k_0$. Since an $(l+m)$-coloring may be formed from a minimal l-coloring and a minimal m-coloring, $\chi_{l+m}(G) \leq \chi_l(G) + \chi_m(G)$, and therefore the *multicoloring sequence* $\chi_1(G), \chi_2(G), \chi_3(G), \ldots$ is subadditive. If p is the least common denominator of a minimal fractional coloring, then p times that fractional coloring is a p-coloring, so by (7), $\frac{\chi_k(G)}{k} \geq \frac{\chi_p(G)}{p}$ for some p and all k. Thus

$$\chi_{k+p}(G) = \chi_k(G) + \chi_p(G) \text{ for all } k \geq k_0, \tag{11}$$

where p is the least common denominator of a minimal fractional coloring. Therefore if k_0 is known, we can find $\chi_k(G)$ for all k in finite time.

For most common graphs, $k_0 = 0$. Stahl [**6**] termed graphs for which $k_0 > 0$ *chromatically unstable*, and published the first known example of such a graph. The graph is known as Grötzch's graph, or alternatively, as $\mu(C_5)$. The *chromatic stability number* k_0 of Grötzch's graph is 1.

Here we prove that several in the series of Mycielskians of odd cycles are chromatically unstable, including two where the stability number is 3. Additionally, we examine the multiclique sequences of this series of graphs, and find that several are *cliquishly unstable*, meaning that $k_0 > 0$ for the multiclique sequence $\omega_1(G), \omega_2(G), \omega_3(G), \ldots$. The highest clique stability number we have found computationally is 18.

2 Instability in Multicoloring Sequences

Stahl [**6**] has shown that the Mycielskian of C_5 has the property of chromatic instability. We will show that the Mycielskians of C_7, C_9, C_{11}, and C_{13} have this property as well.

2.1 Cycles

For completeness, we first show that all cycles are chromatically stable.

Theorem 2.1. *For n even, $\chi_k(C_n) = 2k$.*

Proof: For a k-coloring of an even cycle, any two adjacent vertices require k distinct colors each. Therefore, $\chi_k(C_n) \geq 2k$. But since an even cycle can be k-colored by alternating between two sets of k colors as the cycle is traversed, $\chi_k(C_n) \leq 2k$. Then for even cycles, $\chi_k(C_n) = 2k$ for all k, and therefore even cycles are chromatically stable. $\square$

Lemma 2.1. *$\chi_f(C_n) = \frac{2n}{n-1}$ for n odd.*

Proof: An odd cycle contains n maximal independent sets of size $\frac{n-1}{2}$. So placing a weight of $\frac{2}{n-1}$ on each node gives a fractional clique of weight $\frac{2n}{n-1}$. Therefore, $\chi_f(C_n) \geq \frac{2n}{n-1}$. Additionally, since we can form a fractional coloring of an odd cycle by placing a weight of $\frac{2}{n-1}$ on each of the n maximal independent sets,

$\chi_f(C_n) \leq \frac{2n}{n-1}$. $\square$

The next theorem, showing odd cycles to be chromatically stable, is from Stahl [**5**], although our proof is somewhat simpler.

Theorem 2.2. *For n odd,* $\chi_k(C_n) = \left\lceil \frac{2nk}{n-1} \right\rceil$.

Proof: Since $\chi_k(G) \geq \lceil k\chi_f(G) \rceil$, by Lemma 2.1, $\chi_k(C_n) \geq \left\lceil \frac{2nk}{n-1} \right\rceil$. To show $\chi_k(G) \leq \lceil k\chi_f(G) \rceil$, we first note that $\left\lceil \frac{2nk}{n-1} \right\rceil = 2k+1$ when $k < \frac{n}{2}$. Suppose $k < \frac{n}{2}$. Then for n odd, we may k-color C_n with $2k+1$ colors as follows. Number the vertices sequentially around the cycle, beginning with 1. Use numbers to represent colors, and for each vertex $i = 1, \ldots, n$, color vertex i with a set of colors as follows:

$$\begin{cases} 1 + \{k(i-1), k(i-1)+1, \ldots, ik-1\}(\text{mod}(2k+1)) & \text{if } i \leq 2k+1 \\ 1 \text{ to } k & \text{if } i > 2k+1 \text{ and even} \\ k+1 \text{ to } 2k & \text{if } i > 2k+1 \text{ and odd} \end{cases}$$

For example, to 4-color C_{15} with nine colors, we would follow the above procedure to produce the result illustrated in Figure 2.

We have shown that $\chi_k(G) = \left\lceil \frac{2nk}{n-1} \right\rceil$ for $k < \frac{n}{2}$. Now if $k > \frac{n}{2}$, then:

$$\chi_k(C_n) \leq \chi_{\frac{n-1}{2}}(C_n) + \chi_{k-(\frac{n-1}{2})}(C_n) = \left\lceil \frac{2n(\frac{n-1}{2})}{n-1} \right\rceil + \left\lceil \frac{2n(k-(\frac{n-1}{2}))}{n-1} \right\rceil = \left\lceil \frac{2nk}{n-1} \right\rceil.$$

Therefore, $\chi_k(C_n) = \left\lceil \frac{2nk}{n-1} \right\rceil$ for n odd. $\square$

2.2 Mycielskians of Cycles

We now prove results regarding the chromatic stability of Mycieskians of cycles.

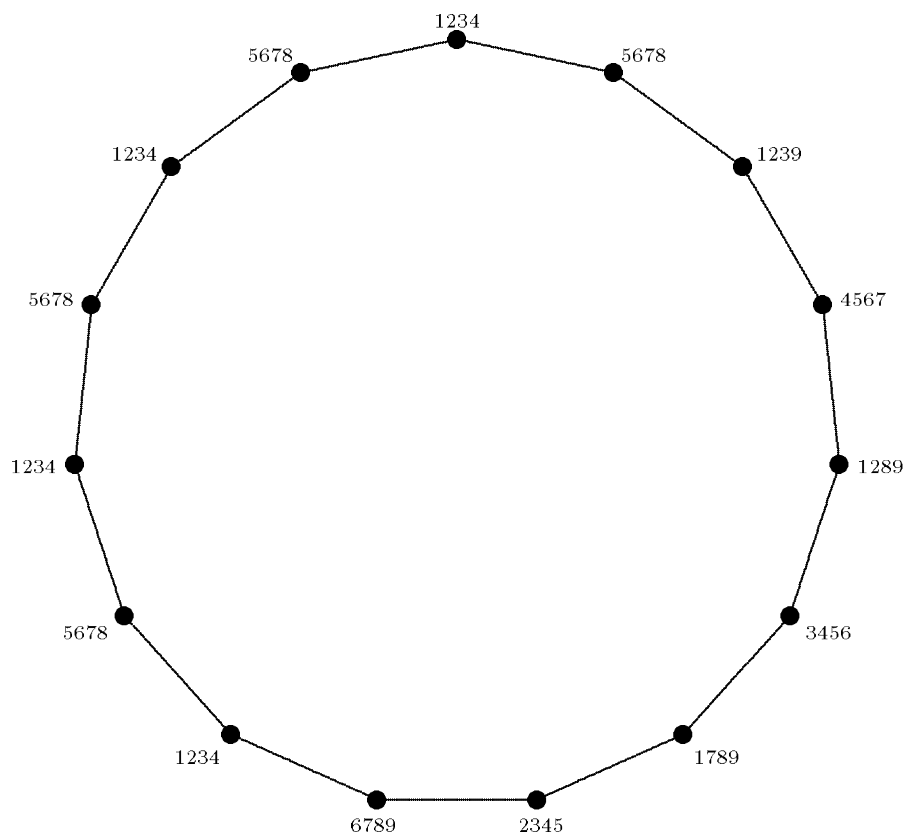

Figure 2: A 4-coloring of C_{15} with nine colors.

2.2.1 Mycielskians of Even Cycles

As with even cycles, the Mycielskians of even cycles have the property of chromatic stability.

Theorem 2.3. *For n even, $\chi_k(\mu(G)) = \left\lceil \frac{5k}{2} \right\rceil$.*

Proof: Each $\mu(C_n)$, n even, contains an induced 5-cycle. By Theorem 2.2, a minimum k-coloring of a 5-cycle requires $\left\lceil \frac{5k}{2} \right\rceil$ colors, therefore $\chi_k(\mu(C_n)) \geq \left\lceil \frac{5k}{2} \right\rceil$. To show $\chi_k(\mu(C_n)) \leq \left\lceil \frac{5k}{2} \right\rceil$, we use a proof by induction. For n even, $\mu(C_n) - \{z\}$ is a bipartite graph, and hence, may be 1-colored with two colors. Coloring the center with a third color shows that $\chi_1(\mu(C_n)) \leq \left\lceil \frac{5}{2} \right\rceil$. Additionally, $\mu(C_n) - \{z\}$ may be 2-colored with four colors by coloring one partite set with the colors 1 and 2, and the other with colors 3 and 4. Now replace 1 and 3 on the inner nodes by 5. Then the center may be colored by 1 and 3. Therefore, $\chi_2(\mu(C_n)) \leq \left\lceil \frac{5(2)}{2} \right\rceil$. For $k > 2$, assume $\chi_{k-2}(\mu(C_n)) \leq \left\lceil \frac{5(k-2)}{2} \right\rceil$. Then

$$\chi_k(\mu(C_n)) \leq \chi_{k-2}(\mu(C_n)) + \chi_2(\mu(C_n)) \leq \left\lceil \frac{5(k-2)}{2} \right\rceil + 5 = \left\lceil \frac{5k}{2} \right\rceil.$$

Therefore, the Mycielskian of an even cycle is chromatically stable. □

2.2.2 Mycielskians of Odd Cycles

Stahl [**6**] showed that $\mu(C_n)$ is chromatically unstable when $n = 5$. We will show the same property holds when $n = 7, 9, 11,$ and 13. Of special importance is the fact that the chromatic stability number rises to three for $\mu(C_{11})$. For completeness, we begin by showing that $\mu(C_3)$ is chromatically stable.

Theorem 2.4. $\chi_k(\mu(C_3)) = \left\lceil \frac{10k}{3} \right\rceil$ for $k = 1, 2, 3, \ldots$

Proof: Since $\chi_f(C_3) = 3$ and $\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}$, we have $\chi_f(\mu(C_3)) = \frac{10}{3}$. Then $\chi_k(\mu(C_3)) \geq \lceil k\chi_f(C_3) \rceil = \left\lceil \frac{10k}{3} \right\rceil$.

To show $\chi_k(\mu(C_3)) \leq \left\lceil \frac{10k}{3} \right\rceil$, we first illustrate that the inequality holds for $k = 1, 2,$ and 3 by showing the appropriate k-colorings in Figure 3.

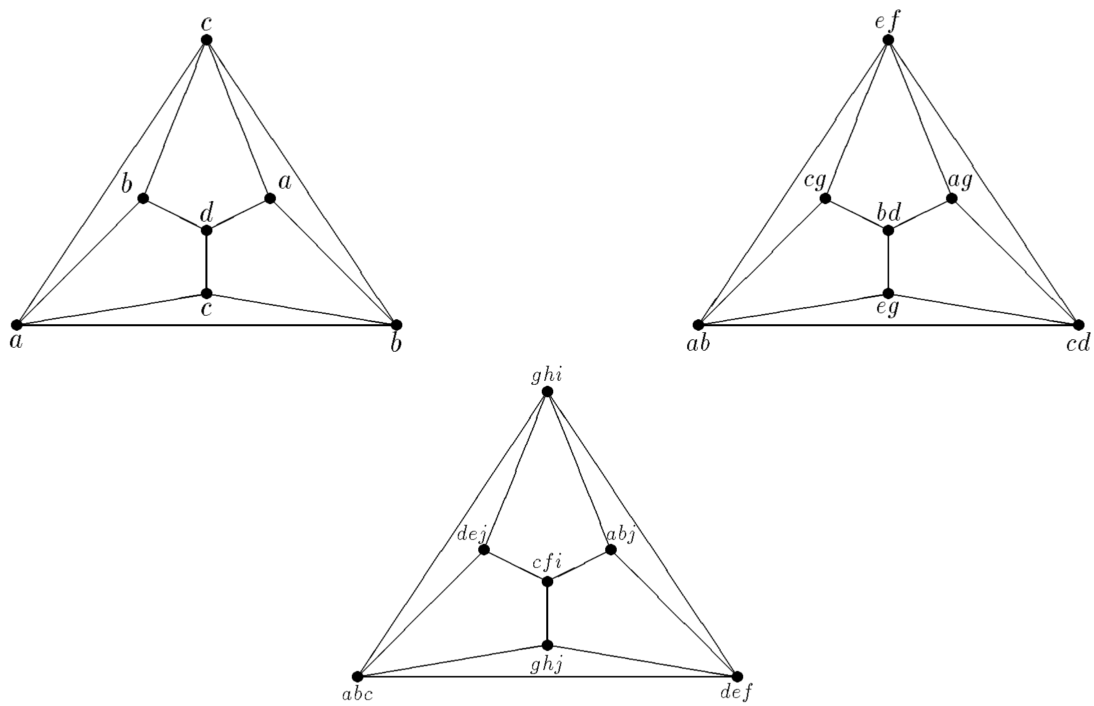

Figure 3: 1-, 2-, and 3-colorings of $\mu(C_3)$.

Using induction, for $k > 3$, assume $\chi_{k-3}(\mu(C_3)) \leq \left\lceil \frac{10(k-3)}{3} \right\rceil$. Then

$$\chi_k(\mu(C_3)) \leq \chi_{k-3}(\mu(C_3)) + \chi_3(\mu(C_3)) \leq \left\lceil \frac{10(k-3)}{3} \right\rceil + 10 = \left\lceil \frac{10k}{3} \right\rceil.$$

Therefore, $\chi_k(\mu(C_3)) = \left\lceil \frac{10k}{3} \right\rceil$ for $k > 1$. $\square$

We now present examples of graphs which are chromatically unstable, beginning with a proof of Stahl's result for $\mu(C_5)$. We will use the following lemma.

Lemma 2.2. $\chi_1(\mu(C_n)) = 4$ *for n odd.*

Proof: Since the chromatic number of any odd cycle is 3, the result follows from (9). □

Theorem 2.5.

$$\chi_k(\mu(C_5)) = \begin{cases} 4 & \text{for } k = 1 \\ \left\lceil \frac{29k}{10} \right\rceil & \text{otherwise} \end{cases}$$

Proof: Since $\chi_f(C_5) = \frac{5}{2}$ and $\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}$, we have $\chi_f(\mu(C_5)) = \frac{29}{10}$. Then $\chi_k(\mu(C_5)) \geq \lceil k\chi_f(C_5) \rceil = \left\lceil \frac{29k}{10} \right\rceil$. This lower bound holds for all $k > 1$. However, for $k = 1$, Lemma 2.2 shows $\chi_1(\mu(C_5)) = 4$.

We now have only to show that $\chi_k(\mu(C_5)) \leq \left\lceil \frac{29k}{10} \right\rceil$ for $k > 1$. This is done with a proof by induction. For $k = 2$, 3, 10, and 11, Table 1 illustrates that the inequality holds.

Furthermore, since $\chi_{k+l}(G) \leq \chi_k(G) + \chi_l(G)$:

$$\begin{array}{rcrcrcl}
\chi_4(\mu(C_5)) & \leq & \chi_2(\mu(C_5)) & + & \chi_2(\mu(C_5)) & = & 12 \\
\chi_5(\mu(C_5)) & \leq & \chi_2(\mu(C_5)) & + & \chi_3(\mu(C_5)) & = & 15 \\
\chi_6(\mu(C_5)) & \leq & \chi_3(\mu(C_7)) & + & \chi_3(\mu(C_5)) & = & 18 \\
\chi_7(\mu(C_5)) & \leq & \chi_3(\mu(C_5)) & + & \chi_4(\mu(C_5)) & = & 21 \\
\chi_8(\mu(C_5)) & \leq & \chi_4(\mu(C_5)) & + & \chi_4(\mu(C_5)) & = & 24 \\
\chi_9(\mu(C_5)) & \leq & \chi_4(\mu(C_5)) & + & \chi_5(\mu(C_5)) & = & 27
\end{array}$$

Thus, $\chi_k(\mu(C_5)) \leq \left\lceil \frac{29k}{10} \right\rceil$ for $2 \leq k \leq 11$. For $k > 11$, assume $\chi_{k-10}(\mu(C_5)) \leq \left\lceil \frac{29(k-10)}{10} \right\rceil$. Then

$$\chi_k(\mu(C_5)) \leq \chi_{k-10}(\mu(C_5)) + \chi_{10}(\mu(C_5)) \leq \left\lceil \frac{29(k-10)}{10} \right\rceil + 29 = \left\lceil \frac{29k}{10} \right\rceil.$$

maximal independent sets	k = 2	3	10	11
$\{x_1, x_3, y_1, y_3\}$	1	1	3	3
$\{x_2, x_4, y_2, y_4\}$	0	1	3	3
$\{x_3, x_5, y_3, y_5\}$	0	1	3	4
$\{x_4, x_1, y_4, y_1\}$	1	1	3	3
$\{x_5, x_2, y_5, y_2\}$	1	1	3	3
$\{x_1, x_3, z\}$	0	1	2	3
$\{x_2, x_4, z\}$	1	0	2	2
$\{x_3, x_5, z\}$	1	0	2	1
$\{x_4, x_1, z\}$	0	1	2	2
$\{x_5, x_2, z\}$	0	1	2	3
$\{x_4, y_1, y_2, y_4\}$	0	0	0	1
$\{y_1, y_2, y_3, y_4, y_5\}$	1	1	4	4
$\chi_k(\mu(C_5))$	6	9	29	32

Table 1: The weight on each independent set in a k-coloring of $\mu(C_5)$.

Therefore, $\chi_k(\mu(C_5)) = \left\lceil \frac{29k}{10} \right\rceil$ for $k > 1$. $\square$

Theorem 2.6.

$$\chi_k(\mu(C_7)) = \begin{cases} 4 & \text{for } k = 1 \\ \left\lceil \frac{58k}{21} \right\rceil & \text{otherwise} \end{cases}$$

Proof: Since $\chi_f(C_7) = \frac{7}{3}$, and $\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}$, we have $\chi_f(\mu(C_7)) = \frac{58}{21}$. Then $\chi_k(\mu(C_7)) \geq \lceil k\chi_f(\mu(C_7)) \rceil = \left\lceil \frac{58k}{21} \right\rceil$. This lower bound holds for all $k > 1$. However, for $k = 1$, Lemma 2.2 shows that $\chi_1(\mu(C_7)) = 4$.

We now show that $\chi_k(\mu(C_7)) \leq \left\lceil \frac{58k}{21} \right\rceil$ for $k > 1$. We will perform a proof by induction. For $k = 2, 3, 5, 6, 9, 13, 17$, and 21, Table 2 illustrates that the inequality holds.

Furthermore, since $\chi_{k+l}(G) \leq \chi_k(G) + \chi_l(G)$:

maximal independent sets	k = 2	3	5	6	9	13	17	21
$\{x_1, x_3, x_5, y_1, y_3, y_5\}$	0	0	1	1	2	2	3	4
$\{x_2, x_4, x_6, y_2, y_4, y_6\}$	0	0	1	1	1	2	3	4
$\{x_3, x_5, x_7, y_3, y_5, y_7\}$	0	1	1	1	1	3	3	4
$\{x_4, x_6, x_1, y_4, y_6, y_1\}$	0	1	1	1	2	4	4	4
$\{x_5, x_7, x_2, y_5, y_7, y_2\}$	0	1	1	1	2	2	3	4
$\{x_6, x_1, x_3, y_6, y_1, y_3\}$	1	0	1	1	2	2	3	4
$\{x_7, x_2, x_4, y_7, y_2, y_4\}$	1	0	1	1	2	2	3	4
$\{x_1, x_3, x_5, z\}$	1	1	1	1	1	3	3	3
$\{x_2, x_4, x_6, z\}$	1	1	1	1	2	2	3	3
$\{x_3, x_5, x_7, z\}$	0	0	1	1	2	0	2	3
$\{x_4, x_6, x_1, z\}$	0	0	1	1	1	0	1	3
$\{x_5, x_7, x_2, z\}$	0	0	0	1	1	2	2	3
$\{x_6, x_1, x_3, z\}$	0	1	0	1	1	3	3	3
$\{x_7, x_2, x_4, z\}$	0	1	1	1	1	3	3	3
$\{x_5, x_7, y_2, y_3, y_5, y_7\}$	1	0	0	0	0	1	1	0
$\{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$	1	2	2	3	4	5	7	9
$\chi_k(\mu(C_7))$	6	9	14	17	25	36	47	58

Table 2: The weight on each independent set in a k-coloring of $\mu(C_7)$.

$$\begin{array}{llllll}
\chi_4(\mu(C_7)) & \leq & \chi_2(\mu(C_7)) & + & \chi_2(\mu(C_7)) & = & 12 \\
\chi_7(\mu(C_7)) & \leq & \chi_2(\mu(C_7)) & + & \chi_5(\mu(C_7)) & = & 20 \\
\chi_8(\mu(C_7)) & \leq & \chi_2(\mu(C_7)) & + & \chi_6(\mu(C_7)) & = & 23 \\
\chi_{10}(\mu(C_7)) & \leq & \chi_5(\mu(C_7)) & + & \chi_5(\mu(C_7)) & = & 28 \\
\chi_{11}(\mu(C_7)) & \leq & \chi_5(\mu(C_7)) & + & \chi_6(\mu(C_7)) & = & 31 \\
\chi_{12}(\mu(C_7)) & \leq & \chi_6(\mu(C_7)) & + & \chi_6(\mu(C_7)) & = & 34 \\
\chi_{14}(\mu(C_7)) & \leq & \chi_5(\mu(C_7)) & + & \chi_9(\mu(C_7)) & = & 39 \\
\chi_{15}(\mu(C_7)) & \leq & \chi_6(\mu(C_7)) & + & \chi_9(\mu(C_7)) & = & 42 \\
\chi_{16}(\mu(C_7)) & \leq & \chi_7(\mu(C_7)) & + & \chi_9(\mu(C_7)) & = & 45 \\
\chi_{18}(\mu(C_7)) & \leq & \chi_9(\mu(C_7)) & + & \chi_9(\mu(C_7)) & = & 50 \\
\chi_{19}(\mu(C_7)) & \leq & \chi_9(\mu(C_7)) & + & \chi_{10}(\mu(C_7)) & = & 53 \\
\chi_{20}(\mu(C_7)) & \leq & \chi_{10}(\mu(C_7)) & + & \chi_{10}(\mu(C_7)) & = & 56 \\
\chi_{22}(\mu(C_7)) & \leq & \chi_9(\mu(C_7)) & + & \chi_{13}(\mu(C_7)) & = & 61
\end{array}$$

Thus far, we have $\chi_k(\mu(C_7)) \leq \left\lceil \frac{58k}{21} \right\rceil$ for $2 \leq k \leq 22$. For $k > 22$, assume $\chi_{k-21}(\mu(C_7)) \leq \left\lceil \frac{58(k-21)}{21} \right\rceil$. Then

$$\chi_k(\mu(C_7)) \leq \chi_{k-21}(\mu(C_7)) + \chi_{21}(\mu(C_7)) \leq \left\lceil \frac{58(k-21)}{21} \right\rceil + 58 = \left\lceil \frac{58k}{21} \right\rceil.$$

Therefore, $\chi_k(\mu(C_7)) = \left\lceil \frac{58k}{21} \right\rceil$ for $k > 1$. □

Theorem 2.7.

$$\chi_k(\mu(C_9)) = \begin{cases} 4 & \text{for } k = 1 \\ \left\lceil \frac{97k}{36} \right\rceil & \text{otherwise} \end{cases}$$

Proof: In a fashion similar to the previous theorem, we find that $\chi_k(\mu(C_9)) \geq \left\lceil \frac{97k}{36} \right\rceil$. Since $\chi_f(C_9) = \frac{9}{4}$, $\chi_f(\mu(C_9)) = \frac{97}{36}$. Then $\chi_k(\mu(C_9)) \geq \lceil k\chi_f(\mu(C_9)) \rceil = \left\lceil \frac{97k}{36} \right\rceil$. But $\chi_1(\mu(C_9))$ does not meet its lower bound of 3, since Lemma 2.2 shows $\chi_1(\mu(C_9)) = 4$.

To show $\chi_k(\mu(C_9)) \leq \left\lceil \frac{97k}{36} \right\rceil$ for $k > 1$, we perform a proof by induction. For $k = 2, 3, 4, 5, 7, 10, 23$, and 36, Table 3 illustrates that the inequality holds.

Furthermore, since $\chi_{k+l}(G) \leq \chi_k(G) + \chi_l(G)$, we have:

maximal independent sets	k = 2	3	4	5	7	10	23	36
$\{x_1, x_3, x_5, x_7, y_1, y_3, y_5, y_7\}$	1	0	1	1	1	2	3	5
$\{x_2, x_4, x_6, x_8, y_2, y_4, y_6, y_8\}$	0	0	0	1	1	1	4	5
$\{x_3, x_5, x_7, x_9, y_3, y_5, y_7, y_9\}$	0	0	0	1	1	1	3	5
$\{x_4, x_6, x_8, x_1, y_4, y_6, y_8, y_1\}$	0	0	0	0	1	1	3	5
$\{x_5, x_7, x_9, x_2, y_5, y_7, y_9, y_2\}$	0	1	0	0	1	1	3	5
$\{x_6, x_8, x_1, x_3, y_6, y_8, y_1, y_3\}$	1	1	1	1	1	1	3	5
$\{x_7, x_9, x_2, x_4, y_7, y_9, y_2, y_4\}$	1	1	1	1	1	1	3	5
$\{x_8, x_1, x_3, x_5, y_8, y_1, y_3, y_5\}$	0	1	1	1	1	2	3	5
$\{x_9, x_2, x_4, x_6, y_9, y_2, y_4, y_6\}$	0	1	1	1	1	2	3	5
$\{x_1, x_3, x_5, x_7, z\}$	0	2	0	1	1	0	2	4
$\{x_2, x_4, x_6, x_8, z\}$	0	1	1	1	1	1	1	4
$\{x_3, x_5, x_7, x_9, z\}$	0	0	1	0	1	2	2	4
$\{x_4, x_6, x_8, x_1, z\}$	0	0	1	0	1	2	3	4
$\{x_5, x_7, x_9, x_2, z\}$	0	0	1	1	1	2	3	4
$\{x_6, x_8, x_1, x_3, z\}$	0	0	0	1	1	2	3	4
$\{x_7, x_9, x_2, x_4, z\}$	0	0	0	0	0	1	3	4
$\{x_8, x_1, x_3, x_5, z\}$	1	0	0	0	0	0	3	4
$\{x_9, x_2, x_4, x_6, z\}$	1	0	0	1	1	0	3	4
$\{x_3, x_5, x_7, y_1, y_3, y_5, y_7, y_9\}$	0	0	0	0	0	0	1	0
$\{x_2, x_4, y_2, y_4, y_6, y_7, y_8, y_9\}$	0	0	0	0	0	1	0	0
$\{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9\}$	1	1	2	2	3	4	10	16
$\chi_k(\mu(C_9))$	6	9	11	14	19	27	62	97

Table 3: The weight on each independent set in a k-coloring of $\mu(C_9)$.

For $k = 6$, 8, 9, and 11:

$$\chi_k(\mu(C_9)) \leq \chi_4(\mu(C_9)) + \chi_{k-4}(\mu(C_9)) \leq 11 + \left\lceil \frac{97(k-4)}{36} \right\rceil = \left\lceil \frac{97k}{36} \right\rceil.$$

For $k = 12, 13, \ldots, 22, 24, 25, \ldots, 35$, and 37:

$$\chi_k(\mu(C_9)) \leq \chi_{10}(\mu(C_9)) + \chi_{k-10}(\mu(C_9)) \leq 27 + \left\lceil \frac{97(k-10)}{36} \right\rceil = \left\lceil \frac{97k}{36} \right\rceil.$$

Thus far, we have $\chi_k(\mu(C_9)) \leq \left\lceil \frac{97}{36} \right\rceil$ for $2 \leq k \leq 37$. For $k > 37$, assume $\chi_{k-36}(\mu(C_9)) \leq \left\lceil \frac{97(k-36)}{36} \right\rceil$. Then

$$\chi_k(\mu(C_9)) \leq \chi_{k-36}(\mu(C_9)) + \chi_{36}(\mu(C_9)) \leq \left\lceil \frac{97(k-36)}{36} \right\rceil + 97 = \left\lceil \frac{97k}{36} \right\rceil.$$

Therefore, $\chi_k(\mu(C_9)) \leq \left\lceil \frac{97}{36} \right\rceil$ for all $k > 37$, and the proof is complete. □

We continue to proofs of the chromatic instability of $\mu(C_{11})$ and $\mu(C_{13})$. Note that the chromatic stability number has increased from 1 to 3.

Lemma 2.3. $\chi_3(\mu(C_{11})) = 9$.

Proof: We first show that $\chi_3(\mu(C_{11})) \geq 9$. Suppose not. Then we are able to find eight independent sets of $\mu(C_{11})$ such that each node is included in at least three sets. To color all twenty-three nodes with three colors each, we need at least $23 \cdot 3 = 69$ node-colors total.

Label the nodes as before, with the outer nodes labeled x_1 through x_{11}, the inner nodes y_1 through y_{11}, and the center z. The largest independent set which includes z is the type of set shown in Figure 4. We refer to this type of set as a

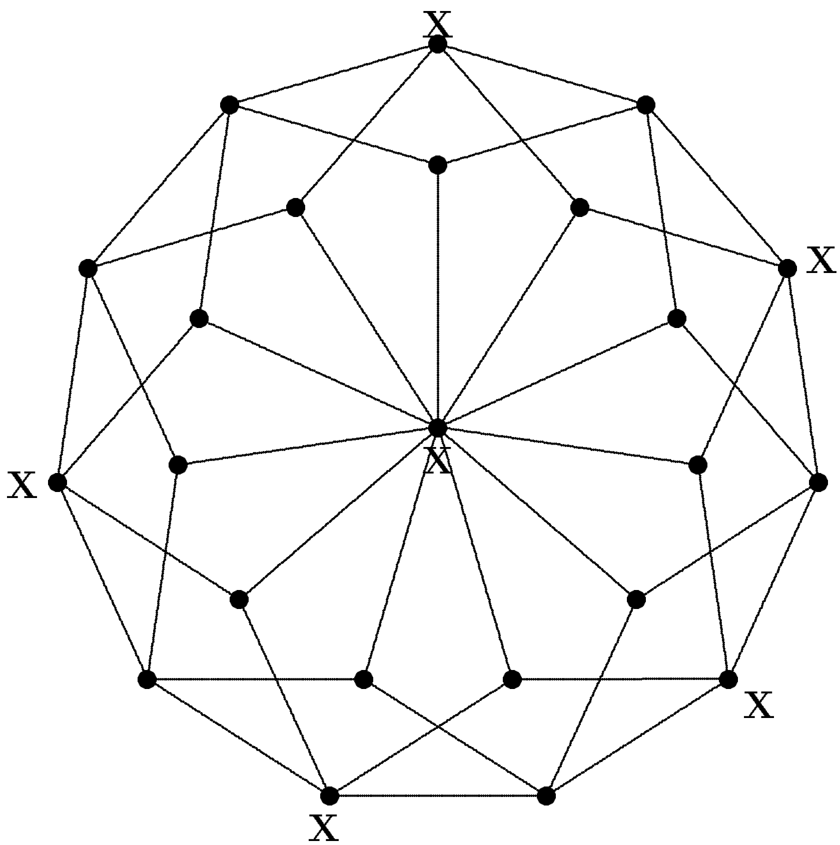

Figure 4: One of 11 Type Z independent sets of $\mu(C_{11})$.

Type Z independent set and notice that through rotation, there are eleven such sets.

Three Type Z independent sets are required to 3-color z, providing at most $3 \cdot 6 = 18$ node-colors. We now require at most five additional independent sets which together provide at least 51 node-colors.

The unique largest independent set of $\mu(C_{11})$ is the one consisting of the eleven inner nodes. Call this set a Type Y set, illustrated in Figure 5.

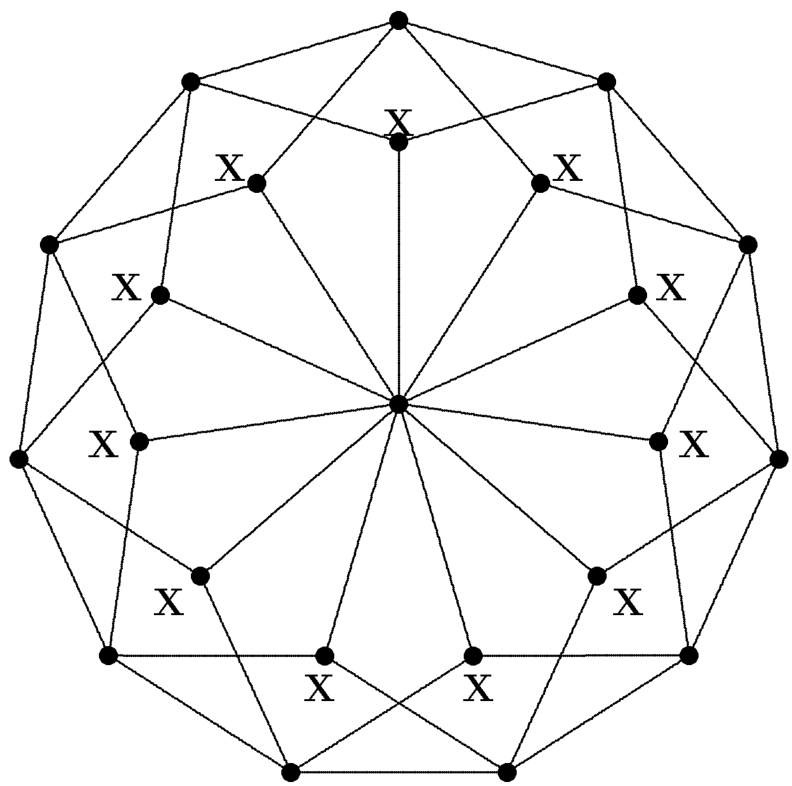

Figure 5: Type Y independent set of $\mu(C_{11})$.

At least one Type Y set must be used to provide 51 node-colors using five independent sets. However, if more than one Type Y set is used, there are three or fewer independent sets with which to color the outer nodes. Since any independent set contains at most five outer nodes, these three independent sets combined with the three independent sets of Type Z provide at most $3 \cdot 5 + 3 \cdot 5 = 30$ node-colors on the outer nodes. But since the outer nodes require at least $3 \cdot 11 = 33$ node-colors, we must use exactly one of the Type Y independent sets.

We have used four independent sets, providing at most $18 + 11 = 29$ node-colors. So we must find four more independent sets which together provide at least 40 node-colors. Since we can no longer use a Type Y set, the only type of independent set containing eleven nodes, each of the four independent sets must contain ten nodes, for a total of exactly 69 node-colors. In other words, each node must be included in exactly three independent sets.

Each Type Z independent set provides five node-colors on the outer nodes. Therefore, we need exactly $33 - (3 \cdot 5) = 18$ more node-colors on the outer nodes. There are two cases which satisfy this criteria. See Figures 6 through 8 for illustrations of the following types of independent sets.

Case 1: Three Type X_5, one Type X_3. One independent set of Type X_3 colors seven inner nodes. These nodes have been previously colored by an independent set of Type Y, so each requires inclusion in exactly one more independent set, a set of Type X_5. Consider the four sequential inner nodes of the Type X_3 set. We require two distinct independent sets of Type X_5 to complete the 3-coloring of these four nodes. But then the last independent set of Type X_5 cannot be used without adding another color to at least one of the four sequential inner nodes, an impossibility since each must have exactly three colors.

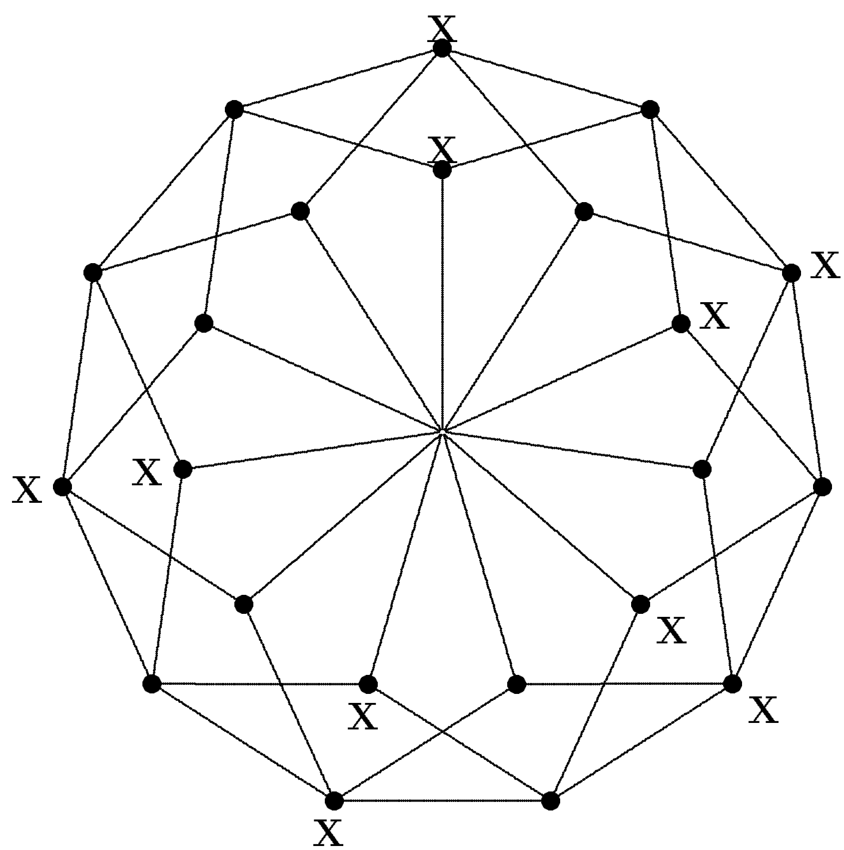

Figure 6: One of 11 Type X_5 independent sets of $\mu(C_{11})$.

Case 2: Two Type X_5, two Type X_4. Any choice of the two independent sets of Type X_4 forces at least one inner node to be colored by both of these

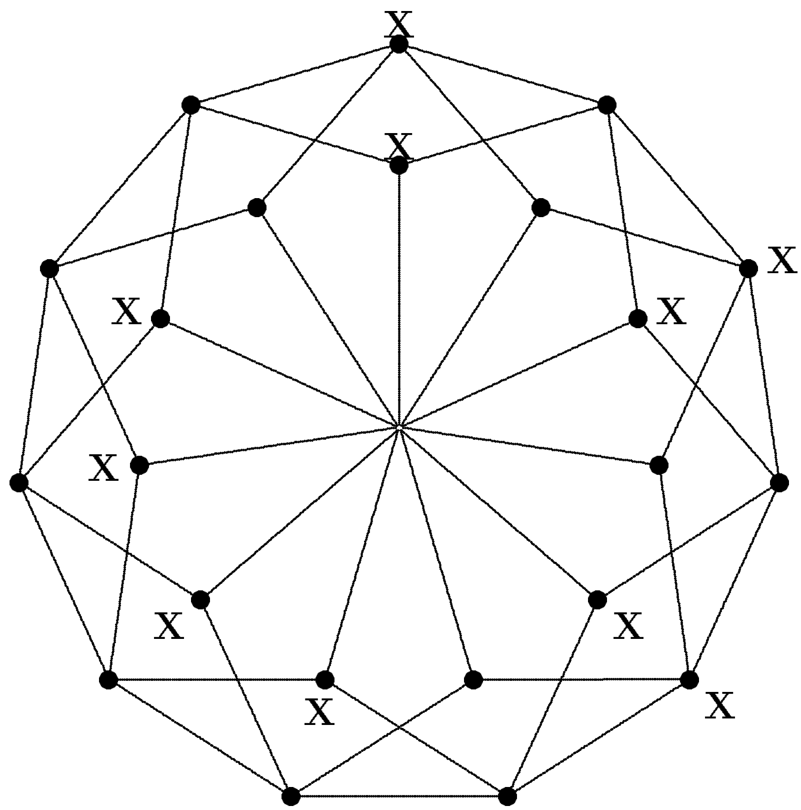

Figure 7: One of 11 Type X_3 independent sets of $\mu(C_{11})$.

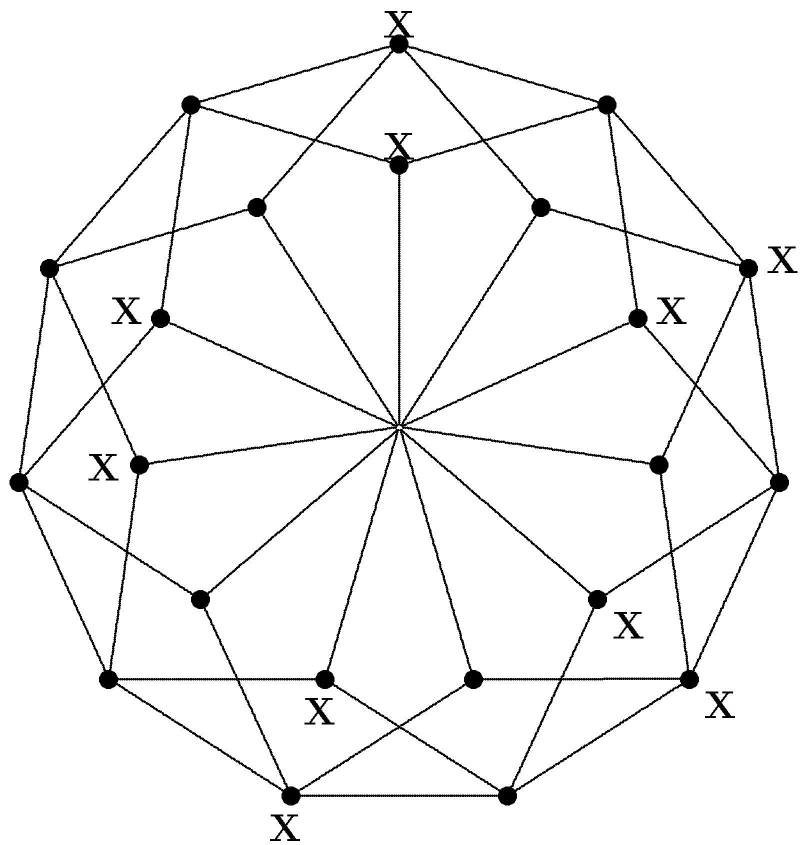

Figure 8: One of 11 Type X_4 independent sets of $\mu(C_{11})$.

independent sets. Call this node y_j. Since y_j was already 1-colored by the independent set of Type Y, it is now 3-colored. Therefore, we cannot include y_j in either of the two Type X_5 sets to be used. Then x_j will not be included in a Type X_5 set either, and therefore must have been colored by three independent sets of Type Z. Consider x_{j+1}, which is adjacent to both x_j and y_j. Since x_j and y_j are colored by six distinct independent sets, we would require three more distinct independent sets to color x_{j+1}, for a total of nine, a contradiction of our hypothesis that $\mu(C_{11})$ can be 3-colored with eight colors.

We have shown that $\chi_3(\mu(C_{11})) \geq 9$. Table 4 illustrates that $\mu(C_{11})$ can be 3-colored with 9 colors; therefore, $\chi_3(\mu(C_{11})) \leq 9$. □

Theorem 2.8.

$$\chi_k(\mu(C_{11})) = \begin{cases} 4 & \text{for } k = 1 \\ 9 & \text{for } k = 3 \\ \left\lceil \frac{146k}{55} \right\rceil & \text{otherwise} \end{cases}$$

Proof: Similar to previous theorems, $\chi_f(C_{11}) = \frac{11}{5}$, which implies $\chi_f(\mu(C_{11})) = \frac{146}{55}$, and $\chi_k(\mu(C_{11})) \geq \left\lceil \frac{146k}{55} \right\rceil$. But for $k = 1$ and 3, this lower bound is not met, since Lemma 2.2 shows $\chi_1(\mu(C_{11})) = 4$ and Lemma 2.3 shows $\chi_3(\mu(C_{11})) = 9$.

To show $\chi_k(\mu(C_{11})) \leq \left\lceil \frac{146k}{55} \right\rceil$ for $k = 2, 4, 5, 6, \ldots$, we perform a proof by induction. For $k = 2, 4, 5, 6, 7, 9, 29, 32$, and 55, Table 4 illustrates that the inequality holds. For $k = 8, 10, 11, \ldots, 28, 30, 31, 33, 34, \ldots, 54, 56$, and 57, we have $\chi_k(\mu(C_{11})) \leq \chi_6(\mu(C_{11})) + \chi_{k-6}(\mu(C_{11})) = \left\lceil \frac{146k}{55} \right\rceil$. For $k = 58$, we have

$\chi_{58}(\mu(C_{11})) \leq \chi_{29}(\mu(C_{11})) + \chi_{29}(\mu(C_{11})) = \left\lceil \frac{146(58)}{55} \right\rceil$.

Thus far, we have $\chi_k(\mu(C_{11})) \leq \left\lceil \frac{146k}{55} \right\rceil$ for $k = 2, 4, 5, \ldots, 58$. For $k > 58$, assume $\chi_{k-55}(\mu(C_{11})) \leq \left\lceil \frac{146(k-55)}{55} \right\rceil$. Then $\chi_k(\mu(C_{11})) \leq \chi_{k-55}(\mu(C_{11})) + \chi_{55}(\mu(C_{11})) = \left\lceil \frac{146k}{55} \right\rceil$ for $k > 58$, and we are done. □

Theorem 2.9.

$$\chi_k(\mu(C_{13})) = \begin{cases} 4 & \text{for } k = 1 \\ 8 \text{ or } 9 & \text{for } k = 3 \\ \left\lceil \frac{205k}{78} \right\rceil & \text{otherwise} \end{cases}$$

Proof: Since $\chi_f(C_{13}) = \frac{13}{6}$, we have $\chi_f(\mu(C_{13})) = \frac{205}{78}$ and $\chi_k(\mu(C_{13})) \geq \left\lceil \frac{205k}{78} \right\rceil$. Lemma 2.2 shows $\chi_1(\mu(C_{13})) = 4$. Computational results have shown that $\chi_3(\mu(C_{13}) = 9$. However, at the time of this paper, a theoretical proof does not exist. Table 5 shows $\chi_3(\mu(C_{13})) \leq 9$.

We prove $\chi_k(\mu(C_{13})) \leq \left\lceil \frac{205k}{78} \right\rceil$ for $k = 2, 4, 5, 6, \ldots$ by induction. For $k = 2, 4, 5, 6, 7, 9, 11, 14, 19, 27, 35$, and 78, Table 5 illustrates that the inequality holds. For $k = 8, 10$, and 12, we have $\chi_k(\mu(C_{13})) \leq \chi_6(\mu(C_{13})) + \chi_{k-6}(\mu(C_{13})) = \left\lceil \frac{205k}{78} \right\rceil$. For $k = 54, 62$, and 70, we have $\chi_k(\mu(C_{13})) \leq \chi_{35}(\mu(C_{13})) + \chi_{k-35}(\mu(C_{13})) = \left\lceil \frac{205k}{78} \right\rceil$. For $k = 13$ through 81 except 14, 19, 27, 35, 54, 62, and 70, we have $\chi_k(\mu(C_{13})) \leq \chi_{11}(\mu(C_{13})) + \chi_{k-11}(\mu(C_{13})) = \left\lceil \frac{205k}{78} \right\rceil$.

For $k > 81$, assume $\chi_{k-78}(\mu(C_{13})) \leq \left\lceil \frac{205(k-78)}{78} \right\rceil$. Then $\chi_k(\mu(C_{13})) \leq \chi_{k-78}(\mu(C_{13})) + \chi_{78}(\mu(C_{13})) = \left\lceil \frac{205k}{78} \right\rceil$ for $k > 81$, and we are done. □

maximal independent sets	k									
	2	3	4	5	6	7	9	29	32	55
$\{x_1, x_3, x_5, x_7, x_9, y_1, y_3, y_5, y_7, y_9\}$	0	0	0	1	0	1	1	3	3	6
$\{x_2, x_4, x_6, x_8, x_{10}, y_2, y_4, y_6, y_8, y_{10}\}$	1	0	1	1	1	1	1	3	3	6
$\{x_3, x_5, x_7, x_9, x_{11}, y_3, y_5, y_7, y_9, y_{11}\}$	0	0	1	1	0	1	1	4	3	6
$\{x_4, x_6, x_8, x_{10}, x_1, y_4, y_6, y_8, y_{10}, y_1\}$	0	1	1	1	0	0	1	3	3	6
$\{x_5, x_7, x_9, x_{11}, x_2, y_5, y_7, y_9, y_{11}, y_2\}$	0	1	0	1	1	0	1	3	3	6
$\{x_6, x_8, x_{10}, x_1, x_3, y_6, y_8, y_{10}, y_1, y_3\}$	0	0	0	0	0	1	1	3	3	6
$\{x_7, x_9, x_{11}, x_2, x_4, y_7, y_9, y_{11}, y_2, y_4\}$	1	0	0	0	0	1	1	3	3	6
$\{x_8, x_{10}, x_1, x_3, x_5, y_8, y_{10}, y_1, y_3, y_5\}$	0	1	0	0	1	1	1	3	3	6
$\{x_9, x_{11}, x_2, x_4, x_6, y_9, y_{11}, y_2, y_4, y_6\}$	0	1	1	0	1	0	1	3	4	6
$\{x_{10}, x_1, x_3, x_5, x_7, y_{10}, y_1, y_3, y_5, y_7\}$	1	1	1	1	1	1	1	3	3	6
$\{x_{11}, x_2, x_4, x_6, x_8, y_{11}, y_2, y_4, y_6, y_8\}$	0	0	0	1	0	1	1	3	5	6
$\{x_1, x_3, x_5, x_7, x_9, z\}$	1	0	1	1	1	0	1	3	2	5
$\{x_2, x_4, x_6, x_8, x_{10}, z\}$	0	0	0	1	0	1	1	2	4	5
$\{x_3, x_5, x_7, x_9, x_{11}, z\}$	0	1	0	1	1	1	1	1	4	5
$\{x_4, x_6, x_8, x_{10}, x_1, z\}$	0	0	0	1	1	1	1	2	4	5
$\{x_5, x_7, x_9, x_{11}, x_2, z\}$	0	0	0	1	0	1	1	3	4	5
$\{x_6, x_8, x_{10}, x_1, x_3, z\}$	0	1	0	0	1	0	0	3	4	5
$\{x_7, x_9, x_{11}, x_2, x_4, z\}$	0	1	1	0	1	1	0	3	4	5
$\{x_8, x_{10}, x_1, x_3, x_5, z\}$	0	0	1	0	0	1	1	3	3	5
$\{x_9, x_{11}, x_2, x_4, x_6, z\}$	0	0	0	0	0	0	1	3	2	5
$\{x_{10}, x_1, x_3, x_5, x_7, z\}$	0	0	0	0	0	0	1	3	1	5
$\{x_{11}, x_2, x_4, x_6, x_8, z\}$	1	0	1	0	1	0	1	3	0	5
$\{x_1, x_3, x_5, x_7, y_1, y_3, y_5, y_7, y_9, y_{10}\}$	0	0	0	0	0	0	0	0	2	0
$\{x_1, x_3, x_5, x_{10}, y_1, y_3, y_5, y_7, y_8, y_{10}\}$	0	0	0	0	0	0	0	0	1	0
$\{x_1, x_6, x_8, x_{10}, y_1, y_3, y_4, y_6, y_8, y_{10}\}$	0	0	0	0	1	0	0	0	0	0
$\{x_2, x_4, x_6, x_{11}, y_2, y_4, y_6, y_8, y_9, y_{11}\}$	0	0	0	0	0	1	0	0	0	0
$\{x_3, x_5, x_7, x_9, y_1, y_3, y_5, y_7, y_9, y_{11}\}$	0	0	0	0	1	0	0	0	0	0
$\{x_4, x_6, x_8, x_{10}, y_1, y_2, y_4, y_6, y_8, y_{10}\}$	0	0	0	0	0	0	0	1	0	0
$\{x_1, x_3, x_6, x_9, z\}$	0	0	0	0	0	1	0	0	0	0
$\{x_2, x_4, x_{11}, y_2, y_4, y_6, y_7, y_8, y_9, y_{11}\}$	0	0	0	0	1	0	0	0	0	0
$\{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}\}$	1	1	2	2	2	3	4	13	14	25
$\chi_k(\mu(C_{11}))$	6	9	11	14	16	19	24	77	85	146

Table 4: The weight on each independent set in a k-coloring of $\mu(C_{11})$.

maximal independent sets	k: 2	3	4	5	6	7	9	11	14	19	27	35	78
$\{x_1, x_3, x_5, x_7, x_9, x_{11}, y_1, y_3, y_5, y_7, y_9, y_{11}\}$	1	1	0	0	0	0	1	1	1	1	3	3	7
$\{x_2, x_4, x_6, x_8, x_{10}, x_{12}, y_2, y_4, y_6, y_8, y_{10}, y_{12}\}$	1	1	0	0	0	0	1	1	2	1	3	3	7
$\{x_3, x_5, x_7, x_9, x_{11}, x_{13}, y_3, y_5, y_7, y_9, y_{11}, y_{13}\}$	1	1	0	0	0	0	1	1	0	1	3	3	7
$\{x_4, x_6, x_8, x_{10}, x_{12}, x_1, y_4, y_6, y_8, y_{10}, y_{12}, y_1\}$	0	1	0	0	0	1	1	1	1	2	2	3	7
$\{x_5, x_7, x_9, x_{11}, x_{13}, x_2, y_5, y_7, y_9, y_{11}, y_{13}, y_2\}$	0	0	0	1	1	1	1	1	1	2	2	4	7
$\{x_6, x_8, x_{10}, x_{12}, x_1, x_3, y_6, y_8, y_{10}, y_{12}, y_1, y_3\}$	0	0	0	1	1	1	1	1	0	2	2	3	7
$\{x_7, x_9, x_{11}, x_{13}, x_2, x_4, y_7, y_9, y_{11}, y_{13}, y_2, y_4\}$	0	0	0	1	1	1	1	1	0	2	2	3	7
$\{x_8, x_{10}, x_{12}, x_1, x_3, x_5, y_8, y_{10}, y_{12}, y_1, y_3, y_5\}$	0	0	1	1	1	1	1	1	0	2	2	3	7
$\{x_9, x_{11}, x_{13}, x_2, x_4, x_6, y_9, y_{11}, y_{13}, y_2, y_4, y_6\}$	0	0	1	1	1	1	1	0	2	2	2	3	7
$\{x_{10}, x_{12}, x_1, x_3, x_5, x_7, y_{10}, y_{12}, y_1, y_3, y_5, y_7\}$	0	0	1	1	1	1	1	1	1	2	2	3	7
$\{x_{11}, x_{13}, x_2, x_4, x_6, x_8, y_{11}, y_{13}, y_2, y_4, y_6, y_8\}$	0	0	1	0	1	1	0	1	2	1	2	3	7
$\{x_{12}, x_1, x_3, x_5, x_7, x_9, y_{12}, y_1, y_3, y_5, y_7, y_9\}$	0	0	1	0	0	1	0	1	1	1	2	3	7
$\{x_{13}, x_2, x_4, x_6, x_8, x_{10}, y_{13}, y_2, y_4, y_6, y_8, y_{10}\}$	0	1	0	0	0	0	1	0	1	1	3	3	7
$\{x_1, x_3, x_5, x_7, x_9, x_{11}, z\}$	0	0	1	1	1	1	1	1	1	3	1	3	6
$\{x_2, x_4, x_6, x_8, x_{10}, x_{12}, z\}$	0	0	1	1	1	1	1	1	0	3	0	3	6
$\{x_3, x_5, x_7, x_9, x_{11}, x_{13}, z\}$	0	0	1	1	1	1	1	0	1	2	0	3	6
$\{x_4, x_6, x_8, x_{10}, x_{12}, x_1, z\}$	1	0	0	1	1	1	1	0	0	0	2	2	6
$\{x_5, x_7, x_9, x_{11}, x_{13}, x_2, z\}$	0	1	0	0	0	0	1	1	0	0	3	1	6
$\{x_6, x_8, x_{10}, x_{12}, x_1, x_3, z\}$	0	0	0	0	0	0	0	1	3	1	3	2	6
$\{x_7, x_9, x_{11}, x_{13}, x_2, x_4, z\}$	0	0	0	0	0	0	0	1	4	1	3	3	6
$\{x_8, x_{10}, x_{12}, x_1, x_3, x_5, z\}$	0	0	0	0	0	0	1	1	2	1	3	3	6
$\{x_9, x_{11}, x_{13}, x_2, x_4, x_6, z\}$	0	0	0	0	0	1	0	1	0	0	3	3	6
$\{x_{10}, x_{12}, x_1, x_3, x_5, x_7, z\}$	0	0	0	0	0	0	0	1	0	0	3	3	6
$\{x_{11}, x_{13}, x_2, x_4, x_6, x_8, z\}$	0	0	0	0	0	0	1	1	0	2	3	3	6
$\{x_{12}, x_1, x_3, x_5, x_7, x_9, z\}$	0	2	0	0	1	1	1	1	1	3	2	3	6
$\{x_{13}, x_2, x_4, x_6, x_8, x_{10}, z\}$	1	0	1	1	1	1	1	1	2	3	1	3	6
$\{x_1, x_3, x_5, x_{12}, y_1, y_3, y_5, y_7, y_8, y_9, y_{10}, y_{12}\}$	0	0	0	0	0	0	0	0	1	0	0	0	0
$\{x_2, x_4, x_6, x_{13}, y_2, y_4, y_6, y_8, y_9, y_{10}, y_{11}, y_{13}\}$	0	0	0	0	0	0	0	1	0	0	0	0	0
$\{x_3, x_5, x_7, x_9, y_1, y_3, y_5, y_7, y_9, y_{11}, y_{12}, y_{13}\}$	0	0	0	0	0	0	0	0	0	0	1	0	0
$\{x_4, x_6, x_8, x_{10}, y_1, y_2, y_4, y_6, y_8, y_{10}, y_{12}, y_{13}\}$	0	0	0	0	0	0	0	1	0	0	0	0	0
$\{x_5, x_7, x_9, x_{11}, y_1, y_2, y_3, y_5, y_7, y_9, y_{11}, y_{13}\}$	0	0	0	0	0	0	0	0	1	0	0	0	0
$\{x_1, x_6, x_8, x_{10}, x_{12}, y_1, y_3, y_4, y_6, y_8, y_{10}, y_{12}\}$	0	0	0	0	0	0	0	0	1	0	0	1	0
$\{x_2, x_4, x_6, x_{11}, x_{13}, y_2, y_4, y_6, y_8, y_9, y_{11}, y_{13}\}$	0	0	0	0	0	0	0	0	0	1	0	0	0
$\{x_4, x_6, x_8, x_{10}, x_{12}, y_1, y_2, y_4, y_6, y_8, y_{10}, y_{12}\}$	0	0	0	0	0	0	0	0	0	0	1	0	0
$\{x_5, x_7, x_9, x_{11}, x_{13}, y_2, y_3, y_5, y_7, y_9, y_{11}, y_{13}\}$	0	0	0	0	0	0	0	0	1	1	0	0	0
$\{x_1, x_{10}, x_{12}, y_1, y_3, y_4, y_5, y_6, y_7, y_8, y_{10}, y_{12}\}$	0	0	0	0	0	0	0	0	1	1	0	0	0
$\{x_3, x_5, x_7, y_1, y_3, y_5, y_7, y_9, y_{10}, y_{11}, y_{12}, y_{13}\}$	0	0	0	0	0	0	0	0	1	0	0	0	0
$\{x_9, x_{11}, x_{13}, y_2, y_3, y_4, y_5, y_6, y_7, y_9, y_{11}, y_{13}\}$	0	0	0	0	0	0	0	1	0	0	0	0	0
$\{x_3, x_5, y_1, y_3, y_5, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}\}$	0	0	0	0	0	0	0	0	1	0	0	0	0
$\{y_1$ through $y_{13}\}$	1	1	2	3	3	3	4	4	4	8	12	16	36
$\chi_k(\mu(C_{13}))$	6	9	11	14	16	19	24	29	37	50	71	92	205

Table 5: The weight on each independent set in a k-coloring of $\mu(C_{13})$.

3 Instability in Multiclique Sequences

Mycielskians of odd cycles have provided interesting results in chromatic instability. These graphs also prove to be useful in discovering instability in multiclique sequences.

3.1 Cycles

We first show that all cycles are cliquishly stable.

Theorem 3.1. *For n even,* $\omega_k(C_n) = 2k$.

Proof: The k-clique number of a graph is the maximum total integer weight which can be placed on the nodes so that each independent set has weight at most k. An even cycle is a bipartite graph; therefore, since we cannot put a weight of more than k on either partite set, $\omega_k \leq 2k$. But by placing a total weight of k on the nodes of each partite set, we find a k-clique, so $\omega_k \geq 2k$. Therefore, every even cycle is cliquishly stable. $\square$

Theorem 3.2. *For n odd,* $\omega_k(C_n) = \left\lfloor \frac{2nk}{n-1} \right\rfloor$.

Proof: Since $\omega_f(C_n) = \chi_f(C_n) = \frac{2n}{n-1}$ for n odd, and $\omega_k \leq \lfloor k\omega_f \rfloor$, we have $\omega_k \leq \left\lfloor \frac{2nk}{n-1} \right\rfloor$. To show the converse, note that $2k = \left\lfloor \frac{2nk}{n-1} \right\rfloor$ when $k < \frac{n}{2}$. If $k < \frac{n}{2}$, then a k-clique of value $2k$ may be formed by placing a weight of k on each of two adjacent nodes in the cycle. Therefore, for $k < \frac{n}{2}, \omega_k(G) \geq \left\lfloor \frac{2nk}{n-1} \right\rfloor$. So if $k > \frac{n}{2}$, then:

$$\omega_k(C_n) \geq \omega_{\frac{n-1}{2}}(C_n) + \omega_{k-(\frac{n-1}{2})}(C_n) = \left\lfloor \frac{2n(\frac{n-1}{2})}{n-1} \right\rfloor + \left\lfloor \frac{2n(k-(\frac{n-1}{2})}{n-1} \right\rfloor = \left\lfloor \frac{2nk}{n-1} \right\rfloor.$$

Therefore, all odd cycles are cliquishly stable. □

3.2 Mycielskians of Cycles

We now proceed to results regarding the cliquish stability of Mycielskians of cycles.

3.2.1 Mycielskians of Even Cycles.

We show that Mycielskians of even cycles are cliquishly unstable.

Theorem 3.3. *For n even, $\mu(C_n) = \left\lfloor \frac{5k}{2} \right\rfloor$.*

Proof: For n even, $\mu(C_n)$ contains an induced 5-cycle. Since $\omega_k(C_5) = \left\lfloor \frac{5k}{2} \right\rfloor$, we have $\omega_k(\mu(C_n)) \geq \left\lfloor \frac{5k}{2} \right\rfloor$. We show that $\omega_k(\mu(C_n)) \leq \left\lfloor \frac{5k}{2} \right\rfloor$ through a proof by induction. For n even, we may find a 1-clique of $\mu(C_n)$ with value 2 by assigning a weight of 1 to each of two adjacent outer nodes. Additionally, we may find a 2-clique of value 5 by assigning a weight of 1 to each of the following: two adjacent outer nodes, say x_i and x_{i+1}, the corresponding inner nodes, y_i and y_{i+1}, and the center node z. Therefore, when $k = 1$ or 2, we have $\omega_k(\mu(C_n)) \leq \left\lfloor \frac{5k}{2} \right\rfloor$. For $k > 2$, assume $\omega_{k-2}(\mu(C_n)) \leq \left\lfloor \frac{5(k-2)}{2} \right\rfloor$. Then

$$\omega_k(\mu(C_n)) \leq \omega_{k-2}(\mu(C_n)) + \omega_2(\mu(C_n)) \leq \left\lfloor \frac{5(k-1)}{2} \right\rfloor + 5 = \left\lfloor \frac{5k}{2} \right\rfloor.$$

Therefore, Mycielskians of even cycles are cliquishly stable. □

3.2.2 Mycielskians of Odd Cycles

We begin by showing that for $n = 3$ and 5, we have that $\mu(C_n)$ is stable in its multiclique sequence. In the following proofs, we let $a_i, b_i,$ and c represent the weights on nodes $x_i, y_i,$ and z respectively. In addition, let $a = a_1 + a_2 + \cdots + a_n$, and $b = b_1 + b_2 + \cdots + b_n$.

Theorem 3.4. $\omega_k(\mu(C_3)) = \left\lfloor \frac{10k}{3} \right\rfloor$.

Proof: Since $\omega_f(\mu(C_3)) = \chi_f(\mu(C_3)) = \frac{10}{3}$, we have $\omega_k(\mu(C_3)) \leq \lfloor k\chi_f(\mu(C_3)) \rfloor = \left\lfloor \frac{10k}{3} \right\rfloor$. We show that $\omega_k(\mu(C_3)) \geq \left\lfloor \frac{10k}{3} \right\rfloor$ for $k = 1, 2,$ and 3 by presenting the appropriate k-cliques in Figure 9.

For $k > 3$ assume $\omega_{k-3}(\mu(C_3)) \geq \left\lfloor \frac{10(k-3)}{3} \right\rfloor$. Then by induction,

$$\omega_k(\mu(C_3)) \geq \omega_{k-3}(\mu(C_3)) + \omega_3(\mu(C_3)) \geq \left\lfloor \frac{10(k-3)}{3} \right\rfloor + 10 = \left\lfloor \frac{10k}{3} \right\rfloor,$$

which concludes our proof. □

Lemma 3.1. *For $k = 9 + 10h$, where $h \geq 0$ and integer,* $\omega_k(\mu(C_5)) \leq \left\lfloor \frac{29k}{10} \right\rfloor - 1$.

Proof: For $k = 9 + 10h$, we can algebraically show that $\left\lfloor \frac{29k}{10} \right\rfloor - 1 = 25 + 29h$. There are two cases.

Case 1: $c \geq 4 + 4h$. We have five independent sets of $\mu(C_5)$ of the form $\{x_1, x_3, z\}$, $\{x_2, x_4, z\}, \ldots, \{x_5, x_2, z\}$. Since $a_1 + a_3 + c \leq 9 + 10h$, $a_2 + a_4 + c \leq 9 + 10h$,

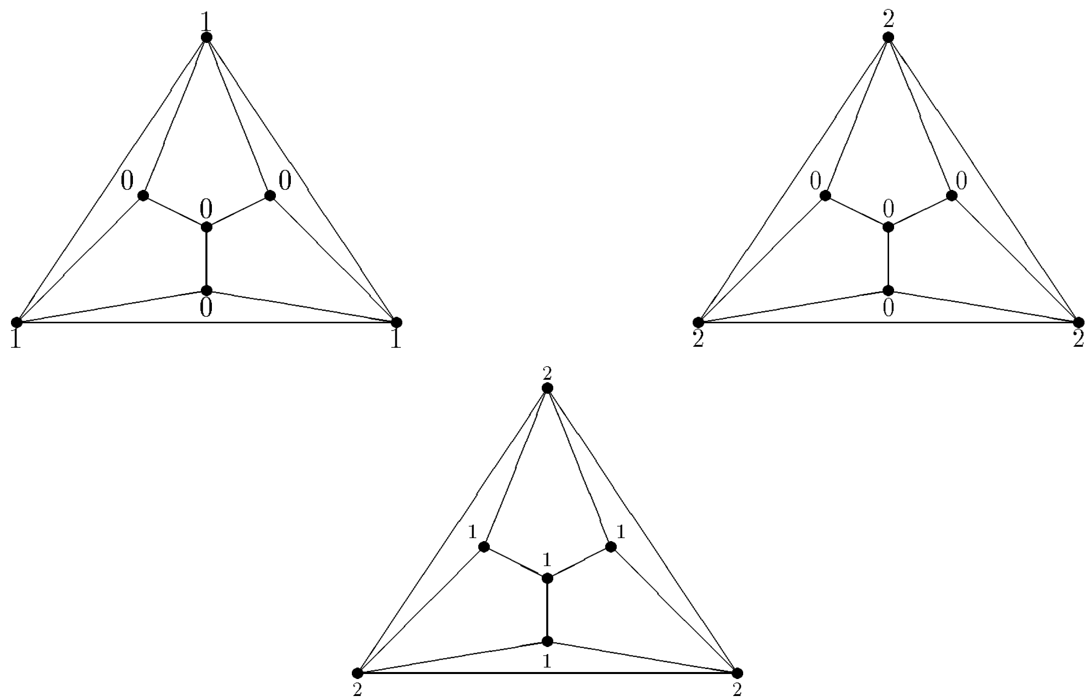

Figure 9: 1-, 2-, and 3-cliques of $\mu(C_3)$.

$\ldots, a_5 + a_2 + c \leq 9 + 10h$, we have that

$$a = \frac{a_1 + a_3}{2} + \frac{a_2 + a_4}{2} + \frac{a_3 + a_5}{2} + \frac{a_4 + a_1}{2} + \frac{a_5 + a_2}{2} \leq 5\left(\frac{9 + 10h - c}{2}\right).$$

Additionally, the independent set $\{y_1, y_2, y_3, y_4, y_5\}$ has weight at most $9 + 10h$. Therefore, summing the weights over all nodes:

$$\omega_{9+10h}(\mu(C_5)) = a + b + c \leq \frac{45 + 50h - 5}{2} + (9 + 10h) + c \leq 25 + 29h.$$

Case 2: $c \leq 3 + 4h$. Now we use independent sets of the form $\{x_1, x_3, y_1, y_3\}$, $\{x_2, x_4, y_2, y_4\}$, $\ldots$, $\{x_5, x_2, y_5, y_2\}$. Since $a_1 + a_3 + b_1 + b_3 \leq 9 + 10h$, $a_2 + a_4 + b_2 + b_4 \leq 9 + 10h$, $\ldots$, $a_5 + a_2 + b_5 + b_2 \leq 9 + 10h$, we have

$$\omega_{9+10h}(\mu(C_5)) = a + b + c = \frac{(a_1 + b_1) + (a_3 + b_3)}{2} + \frac{(a_2 + b_2) + (a_4 + b_4)}{2}$$
$$+ \frac{(a_3 + b_3) + (a_5 + b_5)}{2} + \frac{(a_4 + b_4) + (a_1 + b_1)}{2} + \frac{(a_5 + b_5) + (a_2 + b_2)}{2} + c$$
$$\leq 5\left(\frac{9 + 10h}{2}\right) + c \leq 25 + 29h.$$

Therefore, for $k = 9 + 10h$, we have $\omega_{9+10h}(\mu(C_5)) \leq 25 + 29h = \left\lfloor \frac{29k}{10} \right\rfloor - 1$. $\square$

Theorem 3.5.

$$\omega_k(\mu(C_5)) = \begin{cases} \left\lfloor \frac{29k}{10} \right\rfloor - 1 & \text{for } k = 9 + 10h, h \geq 0 \text{ and integer} \\ \left\lfloor \frac{29k}{10} \right\rfloor & \text{otherwise} \end{cases}$$

Proof: Since $\omega_f(\mu(C_5)) = \chi_f(\mu(C_5)) = \frac{29}{10}$, then $\omega_k(\mu(C_5)) \leq \lfloor k\omega_f(G) \rfloor = \left\lfloor \frac{29k}{10} \right\rfloor$; for $k = 9 + 10h$, Lemma 3.1 shows $\omega_k(\mu(C_5)) \leq \left\lfloor \frac{29k}{10} \right\rfloor - 1$. To prove equality, we use induction. For $k \leq 10$, Table 6 illustrates that equality holds.

						k				
nodes	1	2	3	4	5	6	7	8	9	10
x_1	0	0	1	1	1	1	2	2	2	3
x_2	0	0	0	1	1	2	2	2	2	3
x_3	1	1	1	1	2	2	2	3	3	3
x_4	1	1	1	1	2	2	2	3	3	3
x_5	0	0	1	1	1	2	2	2	2	3
y_1	0	0	0	0	1	2	1	2	2	2
y_2	0	1	1	1	2	1	1	2	2	2
y_3	0	0	1	1	1	1	2	1	2	2
y_4	0	0	1	1	0	1	2	1	2	2
y_5	0	1	0	1	1	1	1	2	1	2
z	0	1	1	2	2	2	3	3	4	4
$\omega_k(\mu(C_5))$	2	5	8	11	14	17	20	23	25	29

Table 6: The weight on each node in a k-clique of $\mu(C_5)$.

For $k > 10$, assume that the result holds for $k-10$. Then since $\omega_k(\mu(C_5)) \geq \omega_{k-10}(\mu(C_5)) + \omega_{10}(\mu(C_5))$, the result follows for $k > 10$. $\Box$

We now present examples of graphs which are cliquishly unstable.

Lemma 3.2. $\omega_3(\mu(C_7)) \leq 7$.

Proof: There are three cases.

Case 1: $c \geq 2$. There are seven independent sets of $\mu(C_7)$ of the form $\{x_1, x_3, x_5, z\}$, $\{x_2, x_4, x_6, z\}, \ldots, \{x_7, x_2, x_4, z\}$. Since $a_1+a_3+a_5+c \leq 3$, $a_2+a_4+a_6+c \leq 3, \ldots, a_7+a_2+a_4+c \leq 3$, we have

$$a = \frac{a_1+a_3+a_5}{3} + \frac{a_2+a_4+a_6}{3} + \cdots + \frac{a_7+a_2+a_4}{3} \leq 7\left(\frac{3-c}{3}\right).$$

The independent set $\{y_1, y_2, \ldots y_7\}$ has weight at most 3, therefore $\omega_3(\mu(C_7)) = a + b + c \leq \frac{21-7c}{3} + 3 + c \leq 7$.

Case 2: $c = 0$. Now we use independent sets of the form $\{x_1, x_3, x_5, y_1, y_3, y_5\}$, $\{x_2, x_4, x_6, y_2, y_4, y_6\}$, $\ldots$, $\{x_7, x_2, x_4, y_7, y_2, y_4\}$. Since $a_1 + a_3 + a_5 + b_1 + b_3 + b_5 \leq 3$, $a_2 + a_4 + a_6 + b_2 + b_4 + b_6 \leq 3$, $\ldots$, $a_7 + a_2 + a_4 + b_7 + b_2 + b_4 \leq 3$, we have

$$\omega_3(\mu(C_7)) = a + b + c = \frac{(a_1 + b_1) + (a_3 + b_3) + (a_5 + b_5)}{3}$$

$$+\frac{(a_2 + b_2) + (a_4 + b_4) + (a_6 + b_6)}{3} + \cdots + \frac{(a_7 + b_7) + (a_2 + b_2) + (a_4 + b_4)}{3} + c$$

$$\leq 7.$$

Case 3: $c = 1$. Each independent set of outer nodes $\{x_1, x_3, x_5\}$, $\{x_2, x_4, x_6\}$, $\ldots$, $\{x_7, x_2, x_4\}$ can have a total weight of at most 2, since including z in each set forms a maximal independent set. So at least one node of each of the above sets has weight 0. Since each node is in three of the above sets, at least three of the outer nodes are weighted 0. We have two cases.

Case a: *No two zero-weighted outer nodes are sequential.* Then there must be exactly three zero-weighted outer nodes. Without loss of generality, suppose $a_1 = a_3 = a_5 = 0$. Since $a_i > 0$ for $i \neq 1, 3, 5$, it follows that $a_7 + a_2 + a_4 + c \geq 4$. But since $\{x_7, x_2, x_4, z\}$ is an independent set, we have $a_7 + a_2 + a_4 + c \leq 3$, a contradiction.

Case b: *Two of the zero-weighted outer nodes are sequential.* Without loss of generality, suppose the two sequential nodes are x_1 and x_2.

Then all other nodes of $\mu(C_7)$ are contained in $S = \{x_3, x_5, x_7, y_3, y_5, y_7\} \cup \{x_4, x_6, y_1, y_2, y_4, y_6\} \cup \{z\}$. But since S is the union of two maximal independent sets and z, the total weight on the nodes of S is at most $3+3+1 = 7$.

Therefore, $\omega_3(\mu(C_7)) \leq 7$. □

Lemma 3.3. *For $k = g + 21h$, where $g = 4$, 8, 20 and h a nonnegative integer,* $\omega_k(\mu(C_7)) \leq \left\lfloor \frac{58k}{21} \right\rfloor - 1$.

Proof: Let q_g represent an integer associated with each g, where $q_4 = 2, q_8 = 4$, and $q_{20} = 9$. For $k = g + 21h$, we can show $\left\lfloor \frac{58k}{21} \right\rfloor - 1 = \left\lfloor \frac{58g}{21} \right\rfloor + 58h - 1$. There are two cases.

Case 1: $c \geq q_g + 9h$. Since $a_1 + a_3 + a_5 + c \leq g + 21h$, $a_2 + a_4 + a_6 + c \leq g + 21h$, $\ldots$, $a_7 + a_2 + a_4 + c \leq g + 21h$, we have

$$a = \frac{a_1 + a_3 + a_5}{3} + \frac{a_2 + a_4 + a_6}{3} + \cdots + \frac{a_7 + a_2 + a_4}{3} \leq 7\left(\frac{g + 21h - c}{3}\right).$$

Additionally, the independent set $\{y_1, y_2, \ldots, y_7\}$ has weight at most $g+21h$, therefore:

$$\omega_{g+21h}(\mu(C_7)) = a+b+c \leq \frac{7g + 147h - 7c}{3} + (g+21h) + c \leq \left\lfloor \frac{58g}{21} \right\rfloor + 58h - 1.$$

Case 2: $c \leq q_g - 1 + 9h$. Since $a_1 + a_3 + a_5 + b_1 + b_3 + b_5 \leq g + 21h$, $a_2 + a_4 + a_6 + b_2 + b_4 + b_6 \leq g + 21h$, $\ldots$, $a_7 + a_2 + a_4 + b_7 + b_2 + b_4 \leq g + 21h$, we have

$$\omega_{g+21h}(\mu(C_7)) = a + b + c = \frac{(a_1 + b_1) + (a_3 + b_3) + (a_5 + b_5)}{3}$$

$$+\frac{(a_2+b_2)+(a_4+b_4)+(a_6+b_6)}{3}+\cdots+\frac{(a_7+b_7)+(a_2+b_2)+(a_4+b_4)}{3}+c$$

$$\leq 7\left(\frac{g+21h}{3}\right)+c \leq \left\lfloor\frac{58g}{21}\right\rfloor+58h-1.$$

Therefore, for $g = 4,\ 8,\ 20$, we have $\omega_k(\mu(C_7)) = \omega_{g+21h}(\mu(C_7)) \leq \left\lfloor\frac{58g}{21}\right\rfloor + 58h - 1 = \left\lfloor\frac{58k}{21}\right\rfloor - 1$. □

Theorem 3.6.

$$\omega_k(\mu(C_7)) = \begin{cases} 7 & \text{for } k=3 \\ \left\lfloor\frac{58k}{21}\right\rfloor - 1 & \text{for } k = g+21h, g = 4,8,20, h \geq 0 \text{ and integer} \\ \left\lfloor\frac{58k}{21}\right\rfloor & \text{otherwise} \end{cases}$$

Proof: Since $\omega_f(\mu(C_7)) = \chi_f(\mu(C_7)) = \frac{58}{21}$, we have $\omega_k(\mu(C_7)) \leq \lfloor k\omega_f(\mu(C_7))\rfloor = \left\lfloor\frac{58k}{21}\right\rfloor$. Lemma 3.2 shows $\omega_3(\mu(C_7)) \leq 7$. By Lemma 3.3, $\omega_k(\mu(C_7)) \leq \left\lfloor\frac{58k}{21}\right\rfloor - 1$ for $k = g+21h$ and the given values of g. We use induction to show equality. For $k = 1,\ 2,\ 5,\ 6,\ 7,\ 10,\ 11,\ 12,\ 15,\ 16$, and 21, Table 7 shows that equality holds. For $k = 3,\ 4,\ 8$, and 9, we have

$$\omega_k(\mu(C_7)) \geq \omega_2(\mu(C_7)) + \omega_{k-2}(\mu(C_7)) = \begin{cases} \left\lfloor\frac{58k}{21}\right\rfloor - 1 & \text{for } k=4,8 \\ \left\lfloor\frac{58k}{21}\right\rfloor & \text{for } k=3,9 \end{cases}$$

For $k = 13,\ 14,\ 17,\ 18,\ 19,\ 20,\ 22,\ 23$, and 24, we have

$$\omega_k(\mu(C_7)) \geq \omega_{12}(\mu(C_7)) + \omega_{k-12}(\mu(C_7)) = \begin{cases} \left\lfloor\frac{58k}{21}\right\rfloor - 1 & \text{for } k=20 \\ \left\lfloor\frac{58k}{21}\right\rfloor & \text{otherwise} \end{cases}$$

For $k > 24$, assume that the result holds for $k-21$. Since $\omega_k(\mu(C_7)) \geq \omega_{k-21}(\mu(C_7)) + \omega_{21}(\mu(C_7))$, the result holds for $k > 24$. □

	k										
nodes	1	2	5	6	7	10	11	12	15	16	21
x_1	1	0	0	1	1	2	2	2	3	3	4
x_2	1	0	1	1	2	2	2	3	2	3	4
x_3	0	0	1	1	2	2	2	3	3	3	4
x_4	0	1	1	1	1	2	2	2	3	3	4
x_5	0	1	1	1	1	2	2	2	3	3	4
x_6	0	0	1	1	1	1	2	2	3	3	4
x_7	0	0	1	1	1	2	2	2	3	3	4
y_1	0	0	1	1	2	1	2	2	2	3	3
y_2	0	0	1	1	1	2	2	1	3	3	3
y_3	0	1	1	1	0	2	1	1	2	2	3
y_4	0	0	1	0	1	1	1	2	2	2	3
y_5	0	0	1	0	1	1	2	2	2	2	3
y_6	0	1	0	1	1	2	2	2	2	2	3
y_7	0	0	0	2	1	1	1	2	2	2	3
z	0	1	2	3	3	4	5	5	6	7	9
$\omega_k(\mu(C_7))$	2	5	13	16	19	27	30	33	41	44	58

Table 7: The weight on each node in a k-clique of $\mu(C_7)$.

For the next two proofs, we will refer to types of maximal independent sets of $\mu(C_9)$ as follows:

$$\begin{array}{ll} \text{Type } X_3: & \{x_1, x_3, x_5, y_1, y_3, y_5, y_7, y_8\}, \{x_2, x_4, x_6, y_2, y_4, y_6, y_8, y_9\}, \ldots, \\ & \{x_9, x_2, x_4, y_9, y_2, y_4, y_6, y_7\} \\ \text{Type } X_4: & \{x_1, x_3, x_5, x_7, y_1, y_3, y_5, y_7\}, \{x_2, x_4, x_6, x_8, y_2, y_4, y_6, y_8\}, \ldots, \\ & \{x_9, x_2, x_4, x_6, y_9, y_2, y_4, y_6\} \\ \text{Type } Y: & \{y_1, y_2, \ldots, y_9\} \\ \text{Type } Z: & \{x_1, x_3, x_5, x_7, z\}, \{x_2, x_4, x_6, x_8, z\}, \ldots, \{x_9, x_2, x_4, x_6, z\} \end{array}$$

Lemma 3.4. $\omega_5(\mu(C_9)) \leq 12$.

Proof: There are three cases.

Case 1: $c \geq 3$. There are nine Type Z independent sets of $\mu(C_9)$. Since $a_1 + a_3 + a_5 + a_7 + c \leq 5$, $a_2 + a_4 + a_6 + a_8 + c \leq 5$, $\ldots$, $a_9 + a_2 + a_4 + a_6 + c \leq 5$, we have

$$a = \frac{a_1 + a_3 + a_5 + a_7}{4} + \frac{a_2 + a_4 + a_6 + a_8}{4} + \cdots + \frac{a_9 + a_2 + a_4 + a_6}{4}$$
$$\leq 9\left(\frac{5-c}{4}\right).$$

The Type Y independent set has weight at most 5, therefore $\omega_5(\mu(C_9)) = a + b + c \leq \frac{45-9c}{4} + 5 + c \leq 12$.

Case 2: $c \leq 1$. Using the Type X_4 independent sets, since $a_1 + a_3 + a_5 + a_7 + b_1 + b_3 + b_5 + b_7 \leq 5$, $a_2 + a_4 + a_6 + a_8 + b_2 + b_4 + b_6 + b_8 \leq 5$, $\ldots$, $a_9 + a_2 + a_4 + a_6 + b_9 + b_2 + b_4 + b_6 \leq 5$, we have $\omega_5(\mu(C_9)) = a + b + c \leq 9\left(\frac{5}{4}\right) + c \leq 12$.

Case 3: $c = 2$. Each Type Z independent set has weight at most 5, and therefore contains at least one outer node of weight 0. Since each x_i is an element of four Type Z sets, there are at least three zero-weighted outer nodes.

Case a: *Two zero-weighted outer nodes are sequential.* For any two sequential outer nodes, there is one Type X_4 and one Type X_3 independent set that together contain all nodes except z and the two sequential zero-weighted outer nodes. Since each independent set is weighted at most 5, for this case, $\omega_5(\mu(C_9)) \leq 12$.

Case b: *No two zero-weighted outer nodes are sequential.* It is easy to show that there exists but one arrangement of three nonsequential zero-weighted outer nodes which does not violate a maximum weight of 5 on all Type Z independent sets; that is, the three zero-weighted outer nodes must be equally spaced around the cycle. All other outer nodes must be at least 1, to avoid falling into Case a. But the non-zero-weighted outer nodes cannot exceed 1, else we could find Type Z independent sets of weight greater than 5.

Each zero-weighted outer node is an element of a Type X_4 independent set which contains three one-weighted outer nodes; therefore, its corresponding inner node has weight at most 2. If any inner node corresponding to a zero-weighted outer node has weight 0, then all other nodes except the center are contained in two Type X_4 independent sets, and then $\omega_5(\mu(C_9)) \leq 5+5+2 = 12$. If one inner node corresponding to a zero-weighted outer node has weight 2, then investigation of the Type X_4 independent sets shows that the other two must have weight 1. Then to exceed a total weight of 12 and to avoid exceeding a maximum weight of 5 on the Type Y independent set, exactly one other inner node must have weight 1; but all such placements result in a set of Type X_3 with weight 6. Therefore, all inner nodes corresponding to zero-weighted outer nodes must have weight 1.

The maximum weight on the Type Y independent set then dictates that the remaining inner nodes have total weight 2, but any placement of the additional weight results in a Type X_4 independent set with weight 6.

Therefore, $\omega_5(\mu(C_9)) \leq 12$. □

Lemma 3.5. *For $k = g+36h$, where $g = 3, 6, 13, 19, 26, 35$, and h a nonnegative integer, $\omega_k(\mu(C_9)) \leq \left\lfloor \frac{97k}{36} \right\rfloor - 1$.*

Proof: Let q_g represent an integer associated with each g, such that $q_3 = 2, q_6 = 3, q_{13} = 6, q_{19} = 9, q_{26} = 12,$ and $q_{35} = 16$. For $k = g + 36h$, it can be shown that $\left\lfloor \frac{97k}{36} \right\rfloor - 1 = \left\lfloor \frac{97g}{36} \right\rfloor + 97h - 1$. There are two cases.

Case 1: $c \geq q_g+16h$. Using the nine Type Z independent sets, since $a_1+a_3+a_5+a_7+c \leq g+36h$, $a_2+a_4+a_6+a_8+c \leq g+36h, \ldots, a_9+a_2+a_4+a_6+c \leq g+36h$, we have

$$a = \frac{a_1 + a_3 + a_5 + a_7}{4} + \frac{a_2 + a_4 + a_6 + a_8}{4} + \cdots + \frac{a_9 + a_2 + a_4 + a_6}{4}$$
$$\leq 9\left(\frac{g + 36h - c}{4}\right).$$

Since the Type Y independent set has weight at most $g + 36h$, we have $\omega_{g+36h}(\mu(C_9)) = a + b + c \leq \frac{9g+324h-9c}{4} + (g + 36h) + c \leq \left\lfloor \frac{97g}{36} \right\rfloor + 97h - 1.$

Case 2: $c \leq q_g - 1 + 16h$. Using the nine Type X_4 independent sets, since $a_1+a_3+a_5+a_7+b_1+b_3+b_5+b_7 \leq g+36h$, $a_2+a_4+a_6+a_8+b_2+b_4+b_6+b_8 \leq g + 36h$, $\ldots$, $a_9 + a_2 + a_4 + a_6 + b_9 + b_2 + b_4 + b_6 \leq g + 36h$, we have $\omega_{g+36h}(\mu(C_9)) = a + b + c \leq \left\lfloor \frac{97g}{36} \right\rfloor + 97h - 1.$

Therefore, for $g = 3, 6, 13, 19, 26, 35$, we have $\omega_k(\mu(C_9)) \leq \left\lfloor \frac{97g}{36} \right\rfloor + 97h - 1 = \left\lfloor \frac{97k}{21} \right\rfloor - 1.$ □

Theorem 3.7.

$$\omega_k(\mu(C_9)) = \begin{cases} 12 & \text{for } k = 5 \\ \left\lfloor \frac{97k}{36} \right\rfloor - 1 & \text{for } k = g + 36h, g = 3, 6, 13, 19, 26, 35, \\ & h \geq 0 \text{ and integer} \\ \left\lfloor \frac{97k}{36} \right\rfloor & \text{otherwise} \end{cases}$$

Proof: Since $\omega_f(\mu(C_9)) = \chi_f(\mu(C_9)) = \frac{97}{36}$, then $\omega_k(\mu(C_9)) \leq \lfloor k\omega_f(\mu(C_9)) \rfloor = \left\lfloor \frac{97k}{36} \right\rfloor$. By Lemma 3.4, we have $\omega_5(\mu(C_9)) \leq 12$, and Lemma 3.5 shows $\omega_k(\mu(C_9)) \leq \left\lfloor \frac{97k}{36} \right\rfloor - 1$ for $k = g + 36h$ and the given values of g. Induction will show equality. For $k = 1, 2, 7, 8, 9, 12, 15, 16, 22, 25, 29$ and 36, Table 8 shows that equality holds. For $k = 3,\ 4,\ 5,\ 6,\ 10,\ 11,\ 13,\ 14,\ 17,\ 18,\ 19,\ 20,\ 23,\ 24,\ 26,\ 27,\ 30,\ 31,\ 33,$ $34,\ 35,\ 38,\ 39,$ and 40, we have

$$\omega_k(\mu(C_9)) \geq \omega_2(\mu(C_9)) + \omega_{k-2}(\mu(C_9)) = \begin{cases} \left\lfloor \frac{97k}{36} \right\rfloor - 1 & \text{for } k = 3, 6, 13, 19, 26, 35, 39 \\ \left\lfloor \frac{97k}{36} \right\rfloor & \text{otherwise} \end{cases}$$

For $k = 21, \omega_k(\mu(C_9)) \geq \omega_9(\mu(C_9)) + \omega_{12}(\mu(C_9)) = \left\lfloor \frac{97(21)}{36} \right\rfloor$. For $k = 28,\ 32,\ 37,$ and 41, $\omega_k(\mu(C_9)) \geq \omega_{16}(\mu(C_9)) + \omega_{k-16}(\mu(C_9)) = \left\lfloor \frac{97k}{36} \right\rfloor$.

	k										
nodes	1	2	7	8	9	12	15	16	25	29	36
x_1	1	1	1	1	2	1	2	3	3	4	5
x_2	0	1	1	1	1	2	2	3	3	4	5
x_3	0	0	1	1	1	2	2	2	3	4	5
x_4	0	0	1	1	1	1	2	2	3	4	5
x_5	0	0	1	1	1	2	2	2	3	4	5
x_6	0	0	1	1	1	2	2	2	3	4	5
x_7	0	0	1	1	1	1	2	2	3	4	5
x_8	0	0	1	1	1	2	2	2	5	4	5
x_9	0	0	0	1	2	2	2	2	5	4	5
y_1	0	1	0	1	0	2	2	1	4	4	4
y_2	0	0	1	0	1	1	1	1	3	3	4
y_3	0	0	1	1	1	1	1	2	3	3	4
y_4	0	0	1	1	1	2	1	2	3	3	4
y_5	0	0	1	0	1	1	1	2	3	3	4
y_6	0	0	1	1	1	1	2	2	3	3	4
y_7	0	0	1	1	1	2	2	2	3	3	4
y_8	0	0	0	1	2	1	2	2	1	3	4
y_9	0	1	1	2	1	1	3	2	2	4	4
z	1	1	3	4	4	5	7	7	11	13	16
$\omega_k(\mu(C_9))$	2	5	18	22	24	32	40	43	67	78	97

Table 8: The weight on each node in a k-clique of $\mu(C_9)$.

For $k > 41$, assume the result holds for $k - 36$. Then since $\omega_k(\mu(C_9)) \geq \omega_{k-36}(\mu(C_9)) + \omega_{36}(\mu(C_9))$, the result holds for $k > 41$. □

In the next two proofs, we refer to the following types of maximal independent sets of $\mu(C_{11})$:

Type X_4 :	$\{x_1, x_3, x_5, x_7, y_1, y_3, y_5, y_7, y_9, y_{10}\}, \{x_2, x_4, x_6, x_8, y_2, y_4, y_6, y_8, y_{10}, y_{11}\}, \ldots, \{x_{11}, x_2, x_4, x_6, y_{11}, y_2, y_4, y_6, y_8, y_9\}$
Type X_5 :	$\{x_1, x_3, x_5, x_7, x_9, y_1, y_3, y_5, y_7, y_9\}, \{x_2, x_4, x_6, x_8, x_{10}, y_2, y_4, y_6, y_8, y_{10}\}, \ldots, \{x_9, x_2, x_4, x_6, x_8, y_9, y_2, y_4, y_6, y_8\}$
Type Y :	$\{y_1, y_2, \ldots, y_{11}\}$
Type Z :	$\{x_1, x_3, x_5, x_7, x_9, z\}, \{x_2, x_4, x_6, x_8, x_{10}, z\}, \ldots, \{x_9, x_2, x_4, x_6, x_8, z\}$

Lemma 3.6. $\omega_5(\mu(C_{11})) \leq 12$.

Proof: There are three cases.

Case 1: $c \geq 3$. There are eleven Type Z independent sets of $\mu(C_{11})$. Since $a_1 + a_3 + a_5 + a_7 + a_9 + c \leq 5$, $a_2 + a_4 + a_6 + a_8 + a_{10} + c \leq 5$, $\ldots$, $a_9 + a_2 + a_4 + a_6 + a_8 + c \leq 5$, we have $a \leq 11\left(\frac{5-c}{5}\right)$. The Type Y independent set has weight at most 5, therefore $\omega_5(\mu(C_{11})) = a + b + c \leq \frac{55-11c}{5} + 5 + c \leq 12$.

Case 2: $c \leq 1$. Using the Type X_5 independent sets, since $a_1 + a_3 + a_5 + a_7 + a_9 + b_1 + b_3 + b_5 + b_7 + b_9 \leq 5$, $a_2 + a_4 + a_6 + a_8 + a_{10} + b_2 + b_4 + b_6 + b_8 + b_{10} \leq 5$, $\ldots$, $a_9 + a_2 + a_4 + a_6 + a_8 + b_9 + b_2 + b_4 + b_6 + b_8 \leq 5$, we have $\omega_5(\mu(C_{11})) = a + b + c \leq 11 + c \leq 12$.

Case 3: $c = 2$. Each Type Z independent set must contain at least two outer nodes with weight 0. Since each pair of outer nodes are contained in at most four Type Z independent sets, and since some duplication may occur, there are at least five zero-weighted outer nodes.

Case a: *Two zero-weighted outer nodes are sequential.* Then all other nodes excluding the center are contained in the union of a Type X_4 and a Type X_5 independent set. Then $\omega_5(\mu(C_{11})) \leq 5 + 5 + 2 = 12$ in this case.

Case b: *No two zero-weighted outer nodes are sequential.* Then the five zero-weighted outer nodes must be spaced at least one apart. But since all other outer nodes have positive weight, there exists a Type Z independent set with weight larger than 5, a contradiction.

Therefore, $\omega_5(\mu(C_{11})) \leq 12$. $\square$

Lemma 3.7. *For $k = g + 55h$, where $g = 8, 14, 17, 23, 26, 34, 43, 49, 52, 54$, and h a nonnegative integer, $\omega_k(\mu(C_{11})) \leq \left\lfloor \frac{146k}{55} \right\rfloor - 1$.*

Proof: Let q_g represent an integer associated with each g, such that $q_8 = 4, q_{14} = 7, q_{17} = 8, q_{23} = 11, q_{26} = 12, q_{34} = 16, q_{43} = 20, q_{49} = 23, q_{52} = 24$ and $q_{54} = 25$. For $k = g + 55h$, we have $\left\lfloor \frac{146k}{55} \right\rfloor - 1 = \left\lfloor \frac{146g}{55} \right\rfloor + 146h - 1$.

Case 1: $c \geq q_g + 25h$. Since each Type Z independent set has weight at most $g + 55h$, we have $a \leq 11\left(\frac{g+55h-c}{5}\right)$. Since the Type Y independent set has weight at most $g + 55h$, we have $\omega_{g+55h}(\mu(C_{11})) = a + b + c \leq \frac{11g+605h-11c}{5} + (g + 55h) + c \leq \left\lfloor \frac{146g}{55} \right\rfloor + 146h - 1$.

Case 2: $c \leq q_g - 1 + 25h$. Since each Type X_5 independent set has weight at most $g + 55h$, and each node is an element of five such sets, we have $\omega_{g+55h}(\mu(C_{11})) = a + b + c \leq 11\left(\frac{g+55h}{5}\right) + c \leq \left\lfloor \frac{58g}{21} \right\rfloor + 58h - 1$.

Therefore, $\omega_k(\mu(C_{11})) \le \left\lfloor \frac{58g}{21} \right\rfloor + 58h - 1 = \left\lfloor \frac{58k}{21} \right\rfloor - 1$. □

Theorem 3.8.

$$\omega_k(\mu(C_{11})) = \begin{cases} 12 & \text{for } k = 5 \\ 17 \text{ or } 18 & \text{for } k = 7 \\ \left\lfloor \frac{146k}{55} \right\rfloor - 1 & \text{for } k = g + 55h, g = 8, 14, 17, 23, 26, 34, 43, 49, 52, 54, \\ & h \ge 0 \text{ and integer} \\ \left\lfloor \frac{146k}{55} \right\rfloor & \text{otherwise} \end{cases}$$

Proof: We have $\omega_k(\mu(C_{11})) \le \lfloor k\omega_f(\mu(C_{11}))\rfloor = \left\lfloor \frac{146k}{55} \right\rfloor$. Lemma 3.6 shows $\omega_5(\mu(C_{11})) \le 12$. Computational results show that $\omega_7(\mu(C_{11})) = 17$, however, a theoretical proof does not yet exist. Lemma 3.7 proves $\omega_{g+55h}(\mu(C_{11})) \le \left\lfloor \frac{146k}{55} \right\rfloor - 1$ for the values of g given above. By induction, we will show equality. For $k = 1$, 2, 9, 10, 11, 16, 19, 20, 25, 28, 37, 46 and 55, Table 9 shows that equality holds. For $k = 3$, 4, 5, 6, 7, 8, 12, 13, 14, 15, 17, 18, 21, 22, 23, 24, 26, 27, 29 and 30, we have

$$\omega_k(\mu(C_{11})) \ge \omega_2(\mu(C_{11})) + \omega_{k-2}(\mu(C_{11})) = \begin{cases} \left\lfloor \frac{146k}{55} \right\rfloor - 1 & \text{for } k = 8, 14, 17, 23, 26 \\ \left\lfloor \frac{146k}{55} \right\rfloor & \text{otherwise} \end{cases}$$

For $k = 31, 32, \ldots, 36, 38, 39, \ldots, 45, 47, 48, \ldots, 54, 56, 57, \ldots, 62$, we have

$$\omega_k(\mu(C_{11})) \ge \omega_{20}(\mu(C_{11})) + \omega_{k-20}(\mu(C_{11})) = \begin{cases} \left\lfloor \frac{146k}{55} \right\rfloor - 1 & \text{for } k = 34, 43, 49, 52, 54 \\ \left\lfloor \frac{146k}{55} \right\rfloor & \text{otherwise} \end{cases}$$

For $k > 62$, assume that the result holds for $k - 55$. Since $\omega_k(\mu(C_{11})) \ge \omega_{k-55}(\mu(C_{11})) + \omega_{55}(\mu(C_{11}))$, the result holds for $k > 62$. □

Additionally, computational results have shown that $\mu(C_{13})$ is cliquishly unstable at $k = 7, 9,$ and 18.

	k												
nodes	1	2	9	10	11	16	19	20	25	28	37	46	55
x_1	1	1	1	1	2	1	2	3	2	3	4	5	6
x_2	0	0	0	1	1	2	2	2	3	3	4	5	6
x_3	0	0	1	1	1	2	2	2	3	3	4	5	6
x_4	0	0	1	1	1	2	2	2	2	3	4	5	6
x_5	0	0	1	1	1	2	2	2	3	3	4	5	6
x_6	0	0	1	1	1	1	2	2	3	3	4	5	6
x_7	0	0	1	2	1	2	2	2	2	3	4	5	6
x_8	0	0	2	2	1	2	2	2	3	3	4	5	6
x_9	0	0	1	1	1	1	2	2	3	3	4	5	6
x_{10}	0	0	0	1	1	2	2	2	3	3	4	5	6
x_{11}	0	1	1	1	2	2	2	3	3	3	4	5	6
y_1	0	1	0	1	0	2	2	1	3	3	4	4	5
y_2	0	1	1	1	1	2	2	2	2	3	3	4	5
y_3	0	0	1	1	1	2	2	2	2	2	4	4	5
y_4	0	0	1	1	1	1	3	2	3	2	4	4	5
y_5	0	0	0	1	1	1	2	2	2	2	3	4	5
y_6	0	0	1	1	1	2	2	2	2	3	3	4	5
y_7	0	0	1	0	1	1	2	2	3	4	3	4	5
y_8	0	0	1	0	1	1	1	2	2	3	3	5	5
y_9	0	0	2	1	1	2	1	2	2	2	3	5	5
y_{10}	0	0	1	1	2	1	1	2	2	2	3	4	5
y_{11}	0	0	0	1	1	1	1	1	2	2	4	4	5
z	1	1	4	4	5	7	9	9	11	13	17	21	25
$\omega_k(\mu(C_{11}))$	2	5	24	26	29	42	50	53	66	74	98	122	146

Table 9: The weight on each node in a k-clique of $\mu(C_{11})$.

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